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BRANCH LOCI AND MONODROMY OF NORMAL SINGULARITIES

Makoto Namba

(This is a joint work with Mr. Ryoichi Ueno, a D.C. student of RIMS.)

1. Introduction

We denote by $\Delta^n(O, \varepsilon)$ the $n$-dimensional polydisc in $\mathbb{C}^n$ with the center the origin $O$ and the (multi)radius $\varepsilon = (\varepsilon', \varepsilon_n)$, where $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_{n-1})$.

In this talk, we prove the following theorem:

**Theorem 1.** Let $(X, x)$ be an $n$-dimensional normal singular point. Then there exists a surjective proper finite holomorphic mapping

$$\mu : (X, x) \longrightarrow (\Delta^n(O, \varepsilon), O)$$

for a sufficiently small $\varepsilon$, whose branch locus is contained in the hypersurface

$$B = \{(x', x_n) \in \Delta^n(O, \varepsilon) \mid (x_n - g_1(x')) \ldots (x_n - g_N(x')) = 0\},$$

where $g_j(x')$ are holomorphic functions of $x' = (x_1, \ldots, x_{n-1})$ such that $g_j(O) = 0$.

Let $B$ be a hypersurface of $\Delta^n(O, \varepsilon)$ defined as in the theorem. From the theorem, we can construct a lot of normal singular points if we compute the fundamental group $\pi_1(\Delta^n(O, \varepsilon) - B)$ and construct homomorphisms

$$\varphi : \pi_1(\Delta(O, \varepsilon) - B) \longrightarrow S_d$$

($S_d$ is the $d$-th symmetric group), whose images are transitive. In fact, by the theorem of Grauert and Remmert ([1]), there exists a (unique up to isomorphisms) normal singular point $(X, x)$ and a surjective proper holomorphic mapping

$$\mu : (X, x) \longrightarrow (\Delta^n(O, \varepsilon), O)$$

of degree $d$ whose branch locus is contained in the hypersurface $B$ and the monodromy representation is $\varphi$.

We carry this program out for two dimensional normal singular points.
2. Proof of Theorem 1

Let \((X, x)\) be an \(n\)-dimensional normal singular point. It is known (see Gunning-Rossi ([2])) that there exists a surjective proper finite holomorphic mapping

\[
\pi : (X, x) \longrightarrow (\Delta^n(O, \epsilon), O)
\]

for a sufficiently small \(\epsilon\), whose branch locus \(B_\pi\) is given by

\[
B_\pi = \{(x', x_n) \in \Delta^n(O, \epsilon) \mid f(x', x_n) = 0\},
\]

where

\[
x' = (x_1, \ldots, x_{n-1}),
\]

\[
f(x', x_n) = x_n^N + c_{N-1}(x')x_n^{N-1} + \cdots + c_0(x'),
\]

\(c_j(x')\) are holomorphic functions on \(\Delta^n(O, \epsilon)\) with \(c_j(O) = 0\).

(That is, \(f(x', x_n)\) is a Weierstrass polynomial.)

The holomorphi mapping \(\mu\) in the theorem is defined to be the composition

\[
\mu = G_{N-1} \circ \cdots \circ G_1 \circ \pi
\]

where \(G_j\) are polynomial type mappings and \(N\) is the degree of the above Weierstrass polynomial \(f(x', x_n)\).

We assume for simplicity

\[N = 4.\]

(The proof for general \(N\) is similar.)

(i) Put

\[
G_1 : (x', x_n) \mapsto (z', z_n) = (x', f(x', x_n)).
\]

This is a surjective proper finite holomorphic mapping from an open neighborhood of \(O\) onto an open neighborhood of \(O\). The properness follows from the fact that the roots of an algebraic equation are (multi-valued) continuous functions of the coefficients.

The branch locus of \(\pi\) \((= B_\pi)\) is mapped by \(G_1\) to \(\{z_n = 0\}\). Consider the mapping \(G_1 \circ \pi\). The branch locus of this mapping is contained in the union of \(\{z_n = 0\}\) and the branch locus of \(G_1\), which is contained in the hypersurface \(\{R = 0\}\), where \(R\) is the resultant of

\[
f(z', x_n) - z_n \quad \text{and} \quad \frac{\partial f}{\partial x_n}(z', x_n)
\]

as polynomials of \(x_n\). Note that \(R\) can be written as

\[R = -4^4 z_n^3 + (\text{the lower terms}).\]
Put 
\[ f_1 = \frac{R}{-4^4} = z_n^3 + d_2(z')z_n^2 + d_1(z')z_n + d_0(z'). \]

Note that \( f_1 \) is again a Weierstrass polynomial, that is \( d_j(O) = 0 \).

Thus the branch locus of \( G_1 \circ \pi \) is contained in the union of the hypersurfaces \( \{ z_n = 0 \} \) and \( \{ f_1 = 0 \} \).

(ii) Next put 
\[ G_2 : (z', z_n) \mapsto (w', w_n) = (z', f_1(z', z_n)). \]

A similar argument to (i) shows that the branch locus of the composition \( G_2 \circ G_1 \circ \pi \) is contained in the union of the hypersurfaces 
\[ w_n = 0, \quad w_n = f_1(w', 0) = d_0(w') \quad \text{and} \quad f_2 = 0, \]
where \( f_2 \) is the resultant of 
\[ f_1(w', z_n) - w_n \quad \text{and} \quad \frac{\partial f_1}{\partial z_n}(w', z_n) \]
(as polynomials of \( z_n \)) divided by \( 3^3 \). This is again a Weierstrass polynomial:
\[ f_2 = w_n^2 + c_1(w')w_n + e_0(w'). \]

Note that \( \{ w_n = 0 \} \) and \( \{ w_n = d_0(w') \} \) contain \( G_2(\{ f_1 = 0 \}) \) and \( G_2(\{ z_n = 0 \}) \), respectively.

(iii) Finally put 
\[ G_3 : (w', w_n) \mapsto (v', v_n) = (w', f_2(w', w_n)). \]

A similar argument to (i) shows that the branch locus of the composition 
\[ \mu = G_3 \circ G_2 \circ G_1 \circ \pi \]
is contained in the union of the hypersurfaces 
\[ \{ v_n = 0 \}, \quad \{ v_n = e_0(v') \}, \quad \{ v_n = f_2(v', d_0(v')) := h_0(v') \} \quad \text{and} \quad \{ v_n = h_1(v') \}, \]
where \( v_n - h_1(v') \) is the resultant of 
\[ f_2(v', w_n) - v_n \quad \text{and} \quad \frac{\partial f_2}{\partial w_n}(v', w_n) \]
(as polynomials of \( w_n \)) divided by \( -2^2 \):
\[ h_1(v') = e_0(v') - \frac{c_1^2(v')}{4}. \]

Note that \( \{ v_n = 0 \} \) and \( \{ v_n = e_0(v') \} \) contain \( G_3(\{ f_2 = 0 \}) \) and \( G_3(\{ w_n = 0 \}) \), respectively. Note also that the equation \( v_n = h_0(v') \) is obtained by eliminating \( w_n \) from the equations 
\[ v_n = f_2(v', w_n) \quad \text{and} \quad w_n = d_0(v'). \]

This proves Theorem 1.

A similar method to the proof of Theorem 1 shows the following theorem:
Theorem 2.

Let $V$ be an $n$ dimensional algebraic variety. Then there exists a projective normal algebraic variety $W$ which is birational to $V$, and a surjective proper finite morphism $F$ of $W$ to the complex projective space $\mathbb{P}^n$ such that the branch locus of $F$ is contained in the union of the hyperplane $H_{\infty}$ at infinity and hypersurfaces whose defining equations in the affine coordinate system are

$$x_n = f_j(x_1, \ldots, x_{n-1}), \quad (j = 1, \ldots, N),$$

where $f_j$ are polynomials of $n-1$ variables.

3. Fundamental Groups

In the rest of this talk, we assume

$$n = 2.$$

Let $B$ be the curve in $\Delta^2(O, \epsilon)$ defined by

$$B = \{(y - g_1(x)) \ldots (y - g_N(x)) = 0\},$$

where $(x, y)$ is the coordinate system and $g_j(x)$ are holomorphic functions with $g_j(0) = 0$.

We can compute the fundamental group $\pi_1(\Delta^2(O, \epsilon) - B)$ by the method of Zariski-van Kampen. That is, we take a sufficiently small positive number $r$, which is smaller than $\epsilon$ and we consider the line $x = r$. The line meets with the curve $B$ at $N$ points $q_j = (r, y_j), \quad 1 \leq j \leq N$. Taking a reference point $o$ on the line with $o \neq q_j$, we consider the lassos (meridians) $\gamma_j, \quad 1 \leq j \leq N$, which start from the point $o$ and round the points $q_j$. Next consider the circle $\{re^{it} \mid 0 \leq t \leq 2\pi\}$. When a point moves on the circle counterclockwisely, the $N$ intersection points of the curve $B$ and the line $x = re^{it}$ induces a braid, which induces the braid monodromy on the lassos $\gamma_j$, which gives the generating relations between them. The fundamental group $\pi_1(\Delta^2(O, \epsilon) - B)$ is the group generated by $\gamma_j, \quad 1 \leq j \leq N$, with the generating relations.

We describe the fundamental group dividing into several cases depending on the forms of the power series expansions at $x = 0$ of the holomorphic functions $g_j$.

Case 1. $g_j(x) = a_j x + \text{higher terms}, \quad (a_j \neq a_k \quad \text{for} \ j \neq k).$

In this case,

$$\pi_1(\Delta^2(O, \epsilon) - B) = \langle \gamma_1, \ldots, \gamma_N \mid \gamma_j \gamma_0 = \gamma_0 \gamma_j, \quad \text{for} \ 1 \leq j \leq N \rangle,$$

where

$$\gamma_0 = \gamma_N \ldots \gamma_1.$$
Case 2. $g_j(x) = a_0 x + a_j x^2 + \text{higher terms}, \quad (a_j \neq a_k \quad \text{for} \; j \neq k)$.

In this case,

$$\pi_1(\Delta^2(O, \epsilon) - B) = \langle \gamma_1, \ldots, \gamma_N \mid \gamma_j \gamma_0^2 = \gamma_0^2 \gamma_j \quad \text{for} \; 1 \leq j \leq N \rangle,$$

where

$$\gamma_0 = \gamma_N \cdots \gamma_1.$$

Case 3.

$$g_1(x) = a_1 x + b_1 x^2 + \text{higher terms},$$

$$g_2(x) = a_1 x + b_2 x^2 + \text{higher terms},$$

$$g_3(x) = a_1 x + b_3 x^2 + \text{higher terms},$$

$$g_4(x) = a_2 x + c_1 x^2 + \text{higher terms},$$

$$g_5(x) = a_2 x + c_2 x^2 + \text{higher terms},$$

$$(a_1 \neq a_2, \quad c_1 \neq c_2, \quad b_j \text{ are distinct}).$$

In this case,

$$\pi_1(\Delta^2(O, \epsilon) - B) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid \gamma_j \delta_1 \gamma_0 = \delta_1 \gamma_0 \gamma_j \quad (j = 1, 2, 3), \quad \gamma_j \delta_2 \gamma_0 = \delta_2 \gamma_0 \gamma_j \quad (j = 4, 5) \rangle,$$

where

$$\gamma_0 = \gamma_5 \gamma_4 \gamma_3 \gamma_2 \gamma_1, \quad \delta_1 = \gamma_3 \gamma_2 \gamma_1, \quad \delta_2 = \gamma_5 \gamma_4.$$

Case 4.

$$g_1(x) = a_1 x + b_1 x^2 + c_1 x^3 + \text{higher terms},$$

$$g_2(x) = a_1 x + b_1 x^2 + c_2 x^3 + \text{higher terms},$$

$$g_3(x) = a_1 x + b_2 x^2 + c'_1 x^3 + \text{higher terms},$$

$$g_4(x) = a_1 x + b_2 x^2 + c'_2 x^3 + \text{higher terms},$$

$$g_5(x) = a_2 x + b'_1 x^2 + d_1 x^3 + \text{higher terms},$$

$$g_6(x) = a_2 x + b'_2 x^2 + d_2 x^3 + \text{higher terms},$$

$$g_7(x) = a_2 x + b'_2 x^2 + d'_1 x^3 + \text{higher terms},$$

$$g_8(x) = a_2 x + b'_2 x^2 + d'_2 x^3 + \text{higher terms},$$

$$(a_1 \neq a_2, \quad b_1 \neq b_2, \quad b'_1 \neq b'_2, \quad c_1 \neq c_2, \quad c'_1 \neq c'_2, \quad d_1 \neq d_2, \quad d'_1 \neq d'_2).$$

In this case,
\[ \pi_1(\Delta^2(O, \epsilon) - B) = \langle \gamma_1, \ldots, \gamma_8 \mid \gamma_j \epsilon_1 \delta_1 \gamma_0 = \epsilon_1 \delta_1 \gamma_0 \gamma_j \quad (j = 1, 2), \quad \gamma_j \epsilon_2 \delta_1 \gamma_0 = \epsilon_2 \delta_1 \gamma_0 \gamma_j \quad (j = 3, 4), \]
\[ \gamma_j \epsilon_3 \delta_2 \gamma_0 = \epsilon_3 \delta_2 \gamma_0 \gamma_j \quad (j = 5, 6), \quad \gamma_j \epsilon_4 \delta_2 \gamma_0 = \epsilon_4 \delta_2 \gamma_0 \gamma_j \quad (j = 7, 8) \]

where

\[ \epsilon_1 = \gamma_2 \gamma_1, \quad \epsilon_2 = \gamma_4 \gamma_3, \quad \epsilon_3 = \gamma_6 \gamma_5, \quad \epsilon_4 = \gamma_8 \gamma_7, \]
\[ \delta_1 = \gamma_4 \gamma_3 \gamma_2 \gamma_1, \quad \delta_2 = \gamma_8 \gamma_7 \gamma_6 \gamma_5, \quad \gamma_0 = \gamma_8 \cdots \gamma_1. \]

The fundamental group in the general case can be written in a similar way.

4. Construction of Monodromy

We want to find homomorphisms

\[ \varphi: \pi_1(\Delta - B) \to S_d \]

such that the image is transitive, where

\[ \Delta = \Delta^2(O, \epsilon) \]

and \( S_d \) is the \( d \)-th symmetric group. We discuss our method only for \( B \) in Case 1 in the last section. (As for \( B \) in the general case, our method can be discussed in a similar way.)

The fundamental group \( \pi_1(\Delta - B) \) in Case 1 is generated by

\[ \gamma_1, \ldots, \gamma_N \]

with the generating relations

\[ \gamma_0 \gamma_j = \gamma_j \gamma_0, \quad (j = 1, \ldots, N), \]

where

\[ \gamma_0 = \gamma_N \cdots \gamma_1. \]

The homomorphism \( \varphi \) is constructed if we find permutations \( B_1, \ldots, B_N \) and \( A \) of \( d \)- letters such that

\[ AB_j = B_j A, \quad (j = 1, \ldots, N) \]
and 

\[ A = B_N \ldots B_1. \]

In fact, we define \( \varphi \) by

\[ \varphi(\gamma_j) = B_j, \quad (j = 1, \ldots, N). \]

We can find such permutations as follows: Let \( A \) be any permutation of \( d \)-letters. Let \( B_1, \ldots, B_{N-1} \) be any permutations in \( Z_A(S_d) \), the centralizer of \( A \) in \( S_d \). Put

\[ B_N = A(B_{N-1} \ldots B_1)^{-1}. \]

However the subgroup \( c_\tau \) of \( S_d \) generated by \( B_1, \ldots, B_N \) and \( A \) is not transitive in general. We can easily show the following lemma, whose proof is omitted:

**Lemma 1.** Let \( G \) be a subgroup of \( Z_A(S_d) \) which contains \( A \). If \( A \) is expressed as the product of cyclic permutations without common letters which are not of all equal length, then \( G \) is not transitive.

Let

\[ A = (a_1 \ldots a_s)(b_1 \ldots b_s)\ldots(c_1 \ldots c_s) \]

be the decomposition into the product of cyclic permutations of equal length \( s \) without common letters. Consider the \( t \) sets

\[ a = \{a_1, \ldots, a_s\}, \quad b = \{b_1, \ldots, b_s\}, \ldots, \quad c = \{c_1, \ldots, c_s\}, \quad (d = st). \]

Then we can easily show the following two lemmas, whose proofs are omitted.

**Lemma 2.** Every permutation \( B \) in \( Z_A(S_d) \) induces naturally a permutation \( \Psi(B) \) of \( t \) letters \( a, b, \ldots, c \). The mapping \( \Psi \) is a homomorphism of \( Z_A(S_d) \) onto \( S_t \) whose kernel is isomorphic to the abelian group \( (\mathbb{Z}/s\mathbb{Z})^t \).

**Lemma 3.** Let \( G \) be a subgroup of \( Z_A(S_d) \) which contains \( A \). Then \( G \) is a transitive subgroup of \( S_d \) if and only if \( \Psi(G) \) is a transitive subgroup of \( S_t \).

Using these lemmas, we can construct a lot of homomorphisms

\[ \varphi : \pi_1(\Delta(O, \epsilon), O) \to S_d \]

and consequently a lot of two dimensional normal singularities \((X, x)\) and covering mappings

\[ \mu : (X, x) \to (\Delta(O, \epsilon), O), \]

whose branch loci are contained in the curve \( B \) and the monodromies are \( \varphi \).
Example. Put \[ A = (12)(34)(56). \] Then \( Z_A(S_6) \) consists of 48 permutations. Among them, we choose \[ B_1 = (146235), B_2 = (135246), B_3 = (145236), \quad (d = 6, N = 3). \] Note that \[ A = B_3B_2B_1. \] Let \[ \varphi : \pi_1(\Delta, O) \to S_6 \] be the homomorphism defined by \[ \varphi(\gamma_j) = B_j \quad (j = 1, 2, 3). \] Then the corresponding covering mapping \[ \mu : (X, x) \to (\Delta(O, \epsilon), O) \] is a non-Galois covering of mapping degree 6 which branches at 3 lines passing through \( O \) and the ramification indices are all 6.

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