<table>
<thead>
<tr>
<th>Title</th>
<th>NORMAL SUBSPACES OF $\kappa^2$ (General Topology and Related Problems)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kemoto, Nobuyuki; Nogura, Tsugunori; Smith, Kerry D.; Yajima, Yukinobu</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 953: 61-72</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-06</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60395">http://hdl.handle.net/2433/60395</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
NORMAL SUBSPACES OF $\kappa^2$

大分大教育 家本 宣幸 (Nobuyuki Kemoto)
愛媛大理 野倉 希紀 (Tsugunori Nogura)
Auburn 大院  Kerry D. Smith
神奈川大工 矢島 幸信 (Yukinobu Yajima)

Let $F$ and $H$ be subsets of a space $X$. $F$ and $H$ are separated if there are disjoint open sets $U$ and $V$ with $F \subset U$ and $H \subset V$. Moreover let $\mathcal{U}$ be an open cover of a space $X$. A collection $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ of subsets of $X$ is shrinking of $\mathcal{U}$ if $F(U) \subset U$ for each $U \in \mathcal{U}$. Here we do not require $\mathcal{F}$ covers $X$. A space is normal if each pair of disjoint closed sets are separated. A space $X$ is shrinking if each open cover of $X$ has a closed shrinking, i.e. a shrinking by closed sets, which covers $X$. By these definitions, shrinking spaces are normal, and collectionwise normal spaces are normal. It is well known that all subspaces of an ordinal space, more generally all GO-spaces, are shrinking, (collectionwise) normal and countably paracompact. It is also well known the product space $\omega_1 \times (\omega_1 + 1)$ is not normal, but it is countably paracompact. In [KOT], the normality of $A \times B$, where $A$ and $B$ are subspaces of an ordinal, was characterized and it was shown that normality, shrinking and collectionwise normality of $A \times B$ are equivalent. In particular:

**Theorem 1.** [KOT] Let $A$ and $B$ be subspaces of $\omega_1$. Then the following are equivalent:

1. $A \times B$ is (collectionwise) normal.
2. $A \times B$ is shrinking.
(3) $A$ is not stationary in $\omega_1$, $B$ is not stationary in $\omega_1$ or $A \cap B$ is stationary.

(4) $A \times B$ is countably paracompact.

Take disjoint stationary sets $A$ and $B$ in $\omega_1$. Then by this theorem, $A \times B$ is neither normal nor countably paracompact.

Question in [KOT].

(a) If $A$ and $B$ are subspaces of an ordinal, then is $A \times B$ countably meta-compact?

(b) For any subspace $X$ of the square of an ordinal, are normality, collectionwise normality and shrinking property equivalent?

Recently an affirmative answer of (a) is given by N. Kemoto and K. D. Smith as follows.

**Theorem 2.** [KS] All subspaces of the square of an ordinal are countably meta-compact.

In the proof of Theorem 2, they used a set-theoretical technic "the diagonal intersection". We thought this technic would be applicable for solving (b). We have gotten a complete affirmative answer of (b). For brevity, we will show the equivalence of normality and shrinking property of subspaces of $\omega_1^2$.

Note that, if $A$ is a countable subspace of $\omega_1$, then, since $A$ is non-stationary, by Theorem 1, $A \times B$ is normal for each $B \subset \omega_1$. In particular, as is well known, $(\omega + 1) \times \omega_1$ is normal. But as is shown in the next example, there is a non-normal subspace of $(\omega + 1) \times \omega_1$.

**Example 1.** Put $X = \omega \times \omega_1 \cup \{\omega\} \times (\omega_1 \setminus \text{Lim}(\omega_1))$, where $\text{Lim}(A) = \{\alpha < \omega_1 : \sup(A \cap \alpha) = \alpha\}$. Note that $\text{Lim}(A)$ is the set of all cluster point of $A$ in $\omega_1$, hence
it is closed in \( \omega_1 \). Put \( F = \omega \times \text{Lim}(\omega_1) \) and \( H = \{ \omega \} \times (\omega_1 \setminus \text{Lim}(\omega_1)) \). Then they are disjoint closed sets in \( X \). Let \( U \) be an open set containing \( H \). For each \( \alpha \in \omega_1 \setminus \text{Lim}(\omega_1) \), pick \( n(\alpha) \in \omega \) such that \( [n(\alpha), \omega] \times \{ \alpha \} \subset U \). Since \( \omega_1 \setminus \text{Lim}(\omega_1) \) is uncountable, there are uncountable subset \( C \subset \omega_1 \setminus \text{Lim}(\omega_1) \) and \( n \in \omega \) such that \( n(\alpha) = n \) for each \( \alpha \in C \). Observe that \( [n, \omega] \times C \subset U \). Pick \( \alpha \in \text{Lim}(C) \). Noting \( \text{Lim}(C) \subset \text{Lim}(\omega_1) \), we have \( \langle n, \alpha \rangle \in [n, \omega] \times \text{Lim}(C) \cap F \subset \text{Cl} \cap U \cap F \). This argument shows \( X \) is not normal.

We use the following notation: Let \( X \subset \omega_1^2 \), \( \alpha < \omega_1 \) and \( \beta < \omega_1 \). Put \( V_\alpha(X) = \{ \beta < \omega_1 : \langle \alpha, \beta \rangle \in X \} \), \( H_\beta(X) = \{ \alpha < \omega_1 : \langle \alpha, \beta \rangle \in X \} \) and \( \triangle(X) = \{ \alpha < \omega_1 : \langle \alpha, \alpha \rangle \in X \} \). Moreover put \( A = \{ \alpha < \omega_1 : V_\alpha(X) \text{ is stationary in } \omega_1 \} \) and \( B = \{ \beta < \omega_1 : H_\beta(X) \text{ is stationary in } \omega_1 \} \). Finally, for subsets \( C \) and \( D \) of \( \omega_1 \), put \( X_C = X \cap C \times \omega_1 \), \( X^D = X \cap \omega_1 \times D \) and \( X_C^D = X \cap C \times D \).

We will show:

**Theorem.** Let \( X \subset \omega_1^2 \). Then the following are equivalent.

1. \( X \) is shrinking.
2. \( X \) is normal.
3. 3-1a) If \( \alpha \) is a limit ordinal in \( \omega_1 \) and \( V_\alpha(X) \) is not stationary in \( \omega_1 \), then there is a cub (=closed unbounded) set \( D \subset \omega_1 \) such that \( X_{\{ \alpha \}} \) and \( X^D \) are separated.

3-1b) If \( \beta \) is a limit ordinal in \( \omega_1 \) and \( H_\beta(X) \) is not stationary in \( \omega_1 \), then there is a cub set \( C \subset \omega_1 \) such that \( X^{\{ \beta \}} \) and \( X_C \) are separated.

3-2) If \( \triangle(X) \) is not stationary in \( \omega_1 \), then there is a cub set \( C \subset \omega_1 \) such that \( X_C \) and \( X^C \) are separated.
Intuitively, we may consider (3-1a) is a condition which guarantees the normality (shrinking) of $X_{\alpha+1}$ for each $\alpha < \omega_1$, and (3-1b) the normality (shrinking) of $X^{\beta+1}$ for each $\beta < \omega_1$. After knowing $X_{\alpha+1}$ and $X^{\beta+1}$ are normal (shrinking) for each $\alpha, \beta < \omega_1$, (3-2) is a condition which guarantees the normality (shrinking) of $X$.

Before proving this theorem, we prepare some lemmas.

**Lemma 0.**

1. If $C$ is a cub set in $\omega_1$, then $\omega_1 \setminus C$ is represented as a free union of bounded open intervals of $\omega_1$, and covered by a disjoint collection of bounded closed and open intervals in $\omega_1$.

2. If $X \subseteq \omega_1^2$, $C$ is a cub set in $\omega_1$ and $X_{\alpha+1}$ is normal (shrinking) for each $\alpha < \omega_1$, then $X_{\omega_1 \setminus C}$ is normal (shrinking).

**Proof.** (1): Put $h(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Then

$$\omega_1 \setminus C = \bigoplus_{\alpha \in C \setminus \text{Lim}(C)} (h(\alpha), \alpha) \subset \bigoplus_{\alpha \in C \setminus \text{Lim}(C)} (h(\alpha), \alpha].$$

(2): Assume $X_{\alpha+1}$ is normal (shrinking) for each $\alpha < \omega_1$. Let $\alpha < \omega_1$ be a limit ordinal. Take a strictly increasing cofinal sequence $\{\alpha_n : n \in \omega\}$ in $\alpha$. Then $X_\alpha = \bigoplus_{n \in \omega} X_{(\alpha_{n-1}, \alpha_n]}$, where $\alpha_{-1} = -1$, is normal (shrinking), because $X_{(\alpha_{n-1}, \alpha_n]}$ is a closed and open subspace of $X_{\alpha_{n+1}}$. Therefore $X_\alpha$ is normal (shrinking) for each $\alpha < \omega_1$. Since, by (1), $X_{\omega_1 \setminus C} = \bigoplus_{\alpha \in C \setminus \text{Lim}(C)} X_{(h(\alpha), \alpha]}$, it is normal (shrinking).

It is easy to show:

**Lemma 1.** Assume $X$ is the finite union of closed subspaces $X_i$'s, $i \in \mathbb{N}$. If $\mathcal{U}$ is an open cover such that, for each $i \in \mathbb{N}$, $\mathcal{U}$ has a closed shrinking $\mathcal{F}_i$ which covers $X_i$, then $\mathcal{U}$ has a closed shrinking which covers $X$. 


This shows the following:

**Lemma 2.** Assume $X$ is the union of two shrinking open subspaces $Y$ and $Z$. If $X \setminus Y$ and $X \setminus Z$ are separated, then $X$ is shrinking.

**Lemma 3.** If $X$ is a normal subspace of $\omega_1^2$ such that $\triangle(X)$ is not stationary in $\omega_1$, then there is a cub set $C$ in $\omega_1$ such that $X \cap C^2 = \emptyset$.

**Proof.** First we show the following claim.

**Claim.** $A = \{\alpha < \omega_1 : V_\alpha(X) \text{ is stationary in } \omega_1\}$ is not stationary in $\omega_1$.

**Proof of Claim.** Assume $A$ is stationary in $\omega_1$. For each $\alpha \in A$, fix $h(\alpha) < \omega_1$ with $\alpha < h(\alpha) \in V_\alpha(X) \cap \bigcap_{\alpha' \in A \cap \alpha} \text{Lim}(V_{\alpha'}(X))$. For each $\alpha \in \omega_1 \setminus A$, define $h(\alpha) = 0$. Take a cub set $C'$ in $\omega_1$ disjoint from $\triangle(X)$ and put $C = \{\alpha < \omega_1 : \forall \alpha' < \alpha (h(\alpha') < \alpha)\} \cap C'$. Then $C$ is cub in $\omega_1$, therefore $A' = A \cap C$ is stationary in $\omega_1$. For each $\alpha \in A'$, put $x_\alpha = (\alpha, h(\alpha))$, then, by $h(\alpha) \in V_\alpha(X)$, we have $x_\alpha \in X$. We shall show $F = \{x_\alpha : \alpha \in A'\}$ is closed discrete in $X$. To show this, let $\langle \gamma, \delta \rangle \in X$. First assume $\gamma \in \omega_1 \setminus C$. Then, by the closedness of $C$, there is $\gamma' < \gamma$ such that $\langle \gamma', \gamma \rangle \cap C = \emptyset$. Then $U = (\gamma', \gamma) \times \omega_1 \cap X$ is an neighborhood of $\langle \gamma, \delta \rangle$ missing $F$. Next assume $\gamma \in C$. If $\gamma > \delta$, then $U = (\delta, \gamma] \times [0, \delta] \cap X$ is also a neighborhood of $\langle \gamma, \delta \rangle$ missing $F$. So assume $\gamma \leq \delta$. Since $C'$ is disjoint from $\triangle(X)$ and $\gamma \in C \subset C'$, we have $\gamma \neq \delta$. Then $U = [0, \gamma] \times (\gamma, \delta] \cap X$ is an neighborhood of $\langle \gamma, \delta \rangle$ which intersects $F$ with at most one point. This argument shows $F$ is closed discrete in $X$.

Since $A'$ is stationary in $\omega_1$, we can decompose $A'$ into two disjoint stationary sets $T_0$ and $T_1$ in $\omega_1$. Put $F_i = \{x_\alpha : \alpha \in T_i\}$ for each $i \in 2 = \{0, 1\}$. Let $U_i$ be an open set containing $F_i$ for each $i \in 2$. For each $\alpha \in T_i$, by $x_\alpha = (\alpha, h(\alpha)) \in F_i \subset U_i$, we
can fix \( f(\alpha) < \alpha \) and \( g(\alpha) < h(\alpha) \) such that \( (f(\alpha), \alpha] \times (g(\alpha), h(\alpha)] \cap X \subset U_i \). By the PDL (=Pressing Down Lemma), we find \( \gamma_i < \omega_1 \) and a stationary set \( T'_i \subset T_i \) such that \( f(\alpha) = \gamma_i \) for each \( \alpha \in T'_i \). Put \( \gamma = \max\{\gamma_0, \gamma_1\} \). Then we have \( (\gamma, \alpha] \times (g(\alpha), h(\alpha)] \cap X \subset U_i \) for each \( \alpha \in T'_i \) with \( i \in 2 \). Fix \( \alpha_0 \in A \) with \( \gamma < \alpha_0 \). Moreover fix \( \beta_0 \in \cap_i \text{Lim}(T'_i) \cap V_{\alpha_0}(X) \) with \( \alpha_0 < \beta_0 \). We shall show \( \langle \alpha_0, \beta_0 \rangle \in \text{Cl}U_0 \cap \text{Cl}U_1 \). To show this, let \( V \) be an open neighborhood of \( \langle \alpha_0, \beta_0 \rangle \). Then we can find \( \beta < \beta_0 \) such that \( \alpha_0 \leq \beta \) and \( \{\alpha_0\} \times (\beta, \beta_0] \cap X \subset V \). By \( \beta_0 \in \text{Lim}(T'_0) \), we can find \( \delta, \delta' \in T'_0 \) with \( \beta < \delta < \delta' < \beta_0 \). Since \( T'_0 \subset C \) and \( \delta < \delta' \), we have \( h(\delta) < \delta' \). On the other hand, by \( \beta < h(\delta), g(\delta) < h(\delta) \) and \( h(\delta) \in \text{Lim}(V_{\alpha_0}(X)) \), there is \( \nu_0 \in V_{\alpha_0}(X) \) such that \( \max\{\beta, g(\delta)\} < \nu_0 < h(\delta) \). Then

\[
\langle \alpha_0, \nu_0 \rangle \in \{\alpha_0\} \times (\beta, \beta_0] \cap (\gamma, \delta] \times (g(\delta), h(\delta)] \cap X \subset V \cap U_0.
\]

Thus \( \langle \alpha_0, \beta_0 \rangle \in \text{Cl}U_0 \). Similarly we have \( \langle \alpha_0, \beta_0 \rangle \in \text{Cl}U_1 \). But this contradicts the normality of \( X \). This completes the proof of the Claim.

Similarly we can prove \( B \) is not stationary in \( \omega_1 \).

Take a cub set \( D \) in \( \omega_1 \) disjoint from \( A \cup B \cup \Delta(X) \). For each \( \gamma \in D \), by \( \gamma \notin A \cup B \), we can fix a cub set \( C_\gamma \) in \( \omega_1 \) disjoint from \( V_\gamma(X) \cup H_\gamma(X) \). Then, by a similar argument of [Ku, II, Lemma 6.14], \( E = \{\alpha \in D : \forall \gamma \in D \cap \alpha(\alpha \in C_\gamma)\} \) is a cub set in \( \omega_1 \). Assume \( \langle \gamma, \alpha \rangle \in X \cap E^2 \). Since \( D \) is disjoint from \( \Delta(X) \) and \( E \subset D \), we have \( \gamma \neq \alpha \). We may assume \( \gamma < \alpha \). Since \( \alpha \in E \) and \( \gamma \in E \cap \alpha \subset D \cap \alpha \), we have \( \alpha \in C_\gamma \). Thus \( \alpha \notin V_\gamma(X) \). This shows \( \langle \gamma, \alpha \rangle \notin X \), a contradiction. This completes the proof of Lemma 3.

*Proof of the Theorem.* (1) \( \rightarrow \) (2) is evident.
(2) $\rightarrow$ (3): Let $X$ be a normal subspace of $\omega_1^2$.

(3-1a): Assume $\alpha$ is a limit ordinal in $\omega_1$ and $V_\alpha(X)$ is not stationary in $\omega_1$. Take a cub set $D$ in $\omega_1$ disjoint from $V_\alpha(X)$. Since $X_\{\alpha\}$ and $X^D$ are disjoint closed sets of the normal space $X$, they are separated.

(3-1b): Similar.

(3-2): By Lemma 3.

(3) $\rightarrow$ (1): Assume the clause (3). First we show the following Lemma.

**Lemma 4.** $X_{\alpha+1}$ is shrinking for each $\alpha < \omega_1$.

**Proof.** We prove this Lemma by induction. The cases of $\alpha = 0$ and $\alpha = \alpha' + 1$ are almost trivial. So assume $\alpha$ is a limit ordinal in $\omega_1$ and $X_{\alpha+1}$ is shrinking for each $\alpha' < \alpha$.

First assume $V_\alpha(X)$ is not stationary in $\omega_1$. By (3-1a), take a cub set $D$ in $\omega_1$ such that $X_\{\alpha\}$ and $X^D$ are separated, therefore $X_\{\alpha\}$ and $X^D_{\alpha+1}$ are separated. The argument in the proof of (2) of Lemma 0 shows $X_\alpha$ is shrinking open subspace of $X_{\alpha+1}$. Since $X^D_{\alpha+1}$ is a free union of countable subspaces, it is shrinking open subspace of $X_{\alpha+1}$. Since $X_{\alpha+1} = X_\alpha \cup X^D_{\alpha+1}$, $X_{\alpha+1} \setminus X_\alpha = X_\{\alpha\}$ and $X_{\alpha+1} \setminus X^D_{\alpha+1} = X^D_{\alpha+1}$, by Lemma 2, $X_{\alpha+1}$ is shrinking.

Next assume $V_\alpha(X)$ is stationary in $\omega_1$. Let $U$ be an open cover of $X_{\alpha+1}$. For each $\beta \in V_\alpha(X)$, fix $f(\beta) < \alpha$, $g(\beta) < \beta$ and $U(\beta) \in U$ such that $(f(\beta), \alpha] \times (g(\beta), \beta] \cap X \subset U(\beta)$. By the PDL and $|\alpha| < \omega_1$, we find $\alpha_0 < \alpha$, $\beta_0 < \beta$ and a stationary set $S \subset V_\alpha(X)$ such that $f(\beta) = \alpha_0$ and $g(\beta) = \beta_0$ for each $\beta \in S$. Put $Z = (\alpha_0, \alpha] \times (\beta_0, \omega_1) \cap X$. We show:

**Claim.** There is a closed shrinking which covers $Z$
Proof of the Claim. For each pair $\beta$ and $\beta'$ in $S$, define $\beta \sim \beta'$ by $U(\beta) = U(\beta')$.

Then $\sim$ is an equivalence relation on $S$. For each $E \in S/\sim$, put $U_E = U(\beta)$ for some (equivalently, any) $\beta \in E$.

Case 1. There is $E \in S/\sim$ which is unbounded in $\omega_1$.

In this case, for each $U \in \mathcal{U}$, put
\[
F(U) = \begin{cases} 
Z, & \text{if } U = U_E, \\
\emptyset, & \text{otherwise.}
\end{cases}
\]
Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is a desired one.

Case 2. Each $E \in S/\sim$ is bounded in $\omega_1$.

In this case, by induction, take a strictly increasing sequence $\{\beta(\delta) : \delta < \omega_1\}$ in $\omega_1$ and a sequence $\{E(\delta) : \delta < \omega_1\} \subset S/\sim$ satisfying $\sup(\bigcup_{\delta' < \delta} E(\delta')) < \beta(\delta) \in E(\delta)$ for each $\delta < \omega_1$. Note that elements of $\{E(\delta) : \delta < \omega_1\}$ are all distinct. For each $U \in \mathcal{U}$, put
\[
F(U) = \begin{cases} 
(\alpha_0, \alpha] \times (\beta_0, \beta(\delta)] \cap X, & \text{if } U = U_{E(\delta)} \text{ for some } \delta < \omega_1, \\
\emptyset, & \text{otherwise.}
\end{cases}
\]
Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is a desired one. This completes the proof of the Claim.

By the inductive assumption, $X_{\alpha+1}$ is shrinking. Moreover, by countability of $X_{\alpha+1}$, it is shrinking. Since $X_{\alpha+1}$ is the union of closed subspaces $X_{\alpha+1}$ and $Z$, by the above claim and Lemma 1, we can find a closed shrinking of $\mathcal{U}$ which covers $X_{\alpha+1}$. This completes the proof of Lemma 4.

In a similar way, using (3-1b), we can show $X^{\beta+1}$ is shrinking for each $\beta < \omega_1$.

To show $X$ is shrinking, first assume $\Delta(X)$ is not stationary. By (3-2), there is a cub set $C$ in $\omega_1$ such that $X_C$ and $X^C$ are separated. Therefore, by Lemma 0 and Lemma 2, $X = X_{\omega_1 \setminus C} \cup X^{\omega_1 \setminus C}$ is shrinking.
Finally assume $\triangle(X)$ is stationary. Let $\mathcal{U}$ be an open cover of $X$. For each $\alpha \in \triangle(X)$, fix $f(\alpha) < \alpha$ and $U(\alpha) \in \mathcal{U}$ such that $(f(\alpha), \alpha]^2 \cap X \subset U(\alpha)$. Then, by the PDL, we find $\alpha_0 < \omega_1$ and a stationary set $S \subset \triangle(X)$ such that $f(\alpha_0) = \alpha_0$ for each $\alpha \in S$. Put $Z = (\alpha_0, \omega_1)^2 \cap X$. Then, by a similar argument of Lemma 4, we can get a closed shrinking of $\mathcal{U}$ which covers $X = X_{\alpha_0+1} \cup X^{\alpha_0+1} \cup Z$. Thus $X$ is shrinking. This completes the proof of the Theorem.

Hereafter we give some examples and related problems.

Consider $X = \omega_1^2$. Since $V_\alpha(X)$ and $H_\beta(X)$ are the stationary set $\omega_1$ for each $\alpha, \beta < \omega_1$ and $\triangle(X)$ is also the stationary set $\omega_1$, the clause (3) of the Theorem is satisfied. So $X$ is normal.

**Example 2.** Let $A$ and $B$ be disjoint stationary sets in $\omega_1$ and put $X = A \times B$. Let $\alpha$ be a limit ordinal in $\omega_1$. Then we have

$$V_\alpha(X) = \begin{cases} B, & \text{if } \alpha \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Therefore, if $V_\alpha(X)$ is not stationary, it must be $\alpha \notin A$ and $V_\alpha(X) = \emptyset$, so $X_{\{\alpha\}} = \emptyset$. Therefore $X_{\{\alpha\}}$ and $X^{\omega_1}$ are separated. This argument witnesses (3-1a). Similarly we have (3-1b). Therefore $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$.

Note that $\triangle(X) = \emptyset$. Let $C$ be a cub set in $\omega_1$. Then $X \cap C^2 = (A \cap C) \times (B \cap C) \neq \emptyset$, equivalently $X_C \cap X^C \neq \emptyset$. Thus $X_C$ and $X^C$ can not be separated. Therefore $X$ is not normal, because the clause (3-2) is not satisfied.

**Example 3.** Let $X = \{(\alpha, \beta) \in \omega_1^2 : \alpha \leq \beta\}$ and $Y = \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$. Checking (3-1a) and (3-1b), we can show $X_{\alpha+1}$, $X^{\beta+1}$, $Y_{\alpha+1}$ and $Y^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$. 
Since $\Delta(X) = \omega_1$ is stationary, (3-2) for $X$ is satisfied. Thus $X$ is normal (but this is obvious, because $X$ is a closed subspace of $\omega_1^2$). On the other hand, note that $\Delta(Y) = \emptyset$. For each cub set $C$ in $\omega_1$, pick $\alpha$ and $\beta$ in $C$ with $\alpha < \beta$. Then $(\alpha, \beta) \in Y \cap C^2$. Therefore (3-2) for $Y$ is not satisfied. Thus $Y$ is not normal.

Let $X = \omega_1 \times (\omega_1 + 1)$. Observe that $X \cap \omega_1^2 = \omega_1^2$ is normal, and $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$. Since $\{(\alpha, \alpha) : \alpha \in \omega_1\}$ and $X^{(\omega_1)}$ can not be separated, $X$ is not normal. Note that both $\Delta(X)$ and $H_{\omega_1}(X)$ are the stationary set $\omega_1$. Next we give such an example $X \subset \omega_1 \times (\omega_1 + 1)$, but $\Delta(X)$ and $H_{\omega_1}(X)$ are not stationary.

**Example 4.** Let

$$X = [\omega_1 \setminus \mathrm{Lim}(\omega_1)] \times [(\omega_1 + 1) \setminus \mathrm{Lim}(\omega_1)] \cup \{(\alpha, \alpha + 1) : \alpha \in \mathrm{Lim}(\omega_1)\}.$$

Observe that $X \cap \omega_1^2$ is normal, $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$ and both $\Delta(X)$ and $H_{\omega_1}(X)$ are the non-stationary set $\omega_1 \setminus \mathrm{Lim}(\omega_1)$. By a similar argument in Lemma 3, we can see $F = \{(\alpha, \alpha + 1) : \alpha \in \mathrm{Lim}(\omega_1)\}$ is closed (discrete). We shall show $F$ and $X^{(\omega_1)}$ can not be separated. To show this, let $U$ be an open set containing $F$. For each $\alpha \in \mathrm{Lim}(\omega_1)$, by $(\alpha, \alpha + 1) \in F \subset U$, take $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \times \{\alpha + 1\} \cap X \subset U$. By the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \mathrm{Lim}(\omega_1)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Take $\beta \in \omega_1 \setminus \mathrm{Lim}(\omega_1)$ with $\alpha_0 < \beta$. Noting $(\beta, \alpha + 1) \in X$ for each $\alpha \in S$ with $\alpha > \beta$, we have

$$(\beta, \omega_1) \in \mathrm{Cl}\{(\beta, \alpha + 1) : \alpha \in S, \alpha > \beta\} \cap X^{(\omega_1)} \subset \mathrm{Cl}U \cap X^{(\omega_1)}.$$

Thus $F$ and $X^{(\omega_1)}$ can not be separated.
In these connections, we have the next question.

**Question 1.** Does there exist a non-normal subspace $X$ of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X_{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$.

In this connection, we show:

**Proposition.** If $X = A \times B$ is a subspace of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X_{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$, then $X$ is normal.

**Proof.** If $A$ is not stationary in $\omega_1$, then take a cub set $C$ in $\omega_1$ disjoint from $A$. Then, by (2) of Lemma 0, $X = X_{\omega_1 \setminus C}$ is normal. Similarly $X$ is normal if $B$ is not stationary in $\omega_2$. So we may assume $A$ and $B$ are stationary in respectively $\omega_1$ and $\omega_2$. Let $U = \{U_i : i \in 2\}$ be an open cover of $X$. Fix $\alpha \in A$. For each $\beta \in B$, fix $f(\alpha, \beta) < \alpha$, $g(\alpha, \beta) < \beta$ and $i(\alpha, \beta) \in 2$ such that $(f(\alpha, \beta), \alpha] \times (g(\alpha, \beta), \beta] \cap X \subset U_{i(\alpha, \beta)}$. Applying the PDL to $B$, we find $f(\alpha) < \alpha$, $g(\alpha) < \omega_2$, $i(\alpha) \in 2$ and a stationary set $B(\alpha) \subset B$ in $\omega_2$ such that $f(\alpha, \beta) = f(\alpha)$, $g(\alpha, \beta) = g(\alpha)$ and $i(\alpha, \beta) = i(\alpha)$ for each $\beta \in B(\alpha)$. Then, applying the PDL to $A$, we find $\alpha_0 < \omega_1$, $i_0 \in 2$ and a stationary set $A' \subset A$ in $\omega_1$ such that $f(\alpha) = \alpha_0$ and $i(\alpha) = i_0$ for each $\alpha \in A'$. Put $\beta_0 = \sup\{g(\alpha) : \alpha \in A'\}$. Then we have $Z = (\alpha_0, \omega_1) \times (\beta_0, \omega_2) \cap X \subset U_{i_0}$. Since $X$ is the union of closed subspaces, $X_{\alpha_0+1}$, $X_{\beta_0+1}$ and $Z$, $U$ has a closed shrinking which covers $X$. Therefore $X = A \times B$ is normal.

By Theorem 1, normality and countable paracompactness of $A \times B \subset \omega_1^2$ are equivalent. In this connection, it is natural to ask:

**Question 2.** For any $X \subset \omega_1^2$, are normality and countable paracompactness equivalent?
Finally we restate a question from [KOT]

*Question 3.* For any subspace $X$ of the square of an ordinal, are countable paracompactness, expandability, strong $D$-property and weak $D(\omega)$-property equivalent?

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, OITA UNIVERSITY, DANNOHARU, OITA, 870-11, JAPAN**

*E-mail address:* nkemoto@cc.oita-u.ac.jp

**DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, EHIME UNIVERSITY, MATSUYAMA, JAPAN**

*E-mail address:* nogura@dpcs4370.dpc.ehime-u.ac.jp

**DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, ALABAMA, 36849-5310, USA**

*E-mail address:* smithk8@mail.auburn.edu

**DEPARTMENT OF MATHEMATICS, KANAGAWA UNIVERSITY, YOKOHAMA 221, JAPAN**

*E-mail address:* yuki@kani.cc.kanagawa-u.ac.jp