SOME REMARKS ON THE DUGUNDJI EXTENSION THEOREMS

岛根大学総合理工 服部泰直 (Yasunao Hattori)
静岡大学教育学部 大田春外（Haruto Ohta）

1. Results that are known or easily proved

Let $X$ be a space, $A$ a closed subspace of $X$ and $Z$ a locally convex linear topological space. Let $C(X, Z)$ be the linear space of all continuous mappings from $X$ to $Z$. A linear transformation $u : C(A, Z) \to C(X, Z)$ is said to be a Dugundji extender if $u$ satisfies the following conditions: For each $f \in C(A, Z)$,

(a) $u(f)$ is an extension of $f$, and

(b) the range of $u(f)$ is contained in the closed convex hull of the range of $f$.

The study of this area is initiated by Dugundji [2]. He proved that for every closed subspace $A$ of a metrizable space $X$ there exists a Dugundji extender $u : C(A, \mathbb{R}) \to C(X, \mathbb{R})$. Michael ([8]) noticed that the Dugundji extender constructed by Dugundji is continuous with respect to the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

We shall consider the Dugundji extention theorems on product spaces.

**Definition 1.1.** Let $X$ be a space, $A$ a closed subspace of $X$ and $Z$ a locally convex linear topological space. Then we say that $A$ is $D(Z)$-embedded in $X$ if there is a Dugundji extender $u : C(A, Z) \to C(X, Z)$. Furthermore, we say that $A$ is $D$-embedded in $X$ if $A$ is $D(Z)$-embedded in $X$ for every locally convex linear topological space $Z$.

**Definition 1.2.** Let $X$ be a space, $A$ a closed subspace of $X$ and $Z$ a locally convex linear topological space. Then we say that $A$ is $\pi_D(Z)$-embedded in $X$ if for every space $Y$ there is a Dugundji extender $u : C(A \times Y, Z) \to C(X \times Y, Z)$. Furthermore, $A$ is said to be $\pi_D$-embedded in $X$ if $A$ is $\pi_D(Z)$-embedded in $X$ for every locally convex linear topological space $Z$.

**Definition 1.3.** Let $X$ be a space, $A$ a closed subspace of $X$ and $Z$ a locally convex linear topological space. Then we say that $A$ is continuously $\pi_D(Z)$-embedded (resp. $\pi_D$-embedded) in $X$ if we can choose the Dugundji extender $u$ as is continuous with respect the pointwise convergence topology, the compact-open topology and the uniform convergence topology.

For a space $X$ and a locally convex linear topological space $Z$ we denote $C_u(X, Z)$ the linear topological space of all continuous mappings from $X$ to $Z$ with the uniform convergence topology, i.e., the sets of the form $V(f) = \{g \in C(X, Z) : g(x) - f(x) \in V\}$, where $V$ is a neighborhood of the origin of $Z$ consists a basic neighborhoods of $f \in C_u(X, Z)$. Let $C_{co}(X, Z)$ be the linear topological space of all continuous mappings from $X$ to $Z$ with the compact-open topology.
A mapping \( f : X \to Y \) is called a \( Z\)-map if \( f(Z) \) is closed for every zero-set \( Z \) of \( X \). Then we have the following.

**Theorem 1.1.** Let \( X \) and \( Y \) be spaces and \( A \) a \( D\)-embedded subspace of \( X \). Let \( p_A : A \times Y \to A \) and \( p_Y : A \times Y \to Y \) be the projections. If either of the following conditions is satisfied, then \( A \times Y \) is \( D\)-embedded in \( X \times Y \):

1. \( p_A \) is a \( Z\)-map.
2. \( p_Y \) is a \( Z\)-map and there is a continuous Dugundji extender \( u : C_u(A, Z) \to C_u(X, Z) \) for every locally convex linear topological space \( Z \).

**Theorem 1.2.** ([4]) Let \( X \) and \( Y \) be spaces, \( A \) a closed subspace of \( X \) and \( Z \) a locally convex linear topological space. Suppose that \( X \) is locally compact or \( X \times Y \) is a \( k\)-space. If there exists a continuous Dugundji extender \( u : C_{\omega}(A, Z) \to C_{\omega}(X, Z) \), then \( A \times Y \) is \( D(Z)\)-embedded in \( X \times Y \).

**Remark.** In Theorem 1.2, the continuity of the Dugundji extender \( u \) can not be dropped. In fact, let \( X = [0, \omega_1] \times [0, \omega) - \{(\omega_1, \omega)\} \) and \( A = [0, \omega_1] \times \{\omega\} \) be the closed subspace of \( X \). It is clear that \( A \) is \( D(\mathbb{R})\)-embedded in \( X \). Let \( Y = [0, \omega_1] \) be the space with the following topology: For each \( y < \omega_1 \) \( y \) is an isolated point of \( Y \) and \( \omega_1 \) has a neighborhood base of the usual order topology. It follows that \( A \times Y \) is not \( C\)-embedded in \( X \times Y \), and hence \( A \times Y \) is not \( D(\mathbb{R})\)-embedded in \( X \times Y \).

In [9] and [10], Stares proved that every closed subspace of spaces satisfying the decreasing \( (G) \) is \( \pi\)-embedded and every such space has the Dugundji extension property. Before stating the theorem, we recall the definition of spaces satisfying the decreasing \( (G) \) from [1]. Let \( \mathcal{W} = \{W(x) : x \in X\} \) be a collection of subsets of \( X \), where \( \mathcal{W}(x) = \{W(x, n) : n \in \omega\} \) such that \( x \in W(x, n) \) for every \( x \in X \) and \( n \in \omega \). Then we say that \( \mathcal{W} \) is decresing if \( W(x, n + 1) \subset W(x, n) \) for every \( n \in \omega \), and \( \mathcal{W} \) satisfies \( (G) \) if

\( (G) \) for each \( x \in X \) and each open set \( U \) with \( x \in U \) there is an open neighborhood \( V = V(x, U) \) of \( x \) such that \( y \in V \) implies \( x \in W(y, s) \subset U \) for some \( s \in \omega \).

We say that a space \( X \) satisfies the decreasing \( (G) \) if there is a collection \( \mathcal{W} = \{W(x) : x \in X\} \) satisfying decreasing \( (G) \). We notice that every stratifiable space satisfies the decreasing \( (G) \) ([10]). Now, we have the following.

**Theorem 1.3.** Let \( X \) be a regular space satisfying the decreasing \( (G) \) and \( A \) a closed subspace of \( X \). Then \( A \) is continuously \( \pi_D\)-embedded in \( X \).

2. **Results about GO-spaces**

In [7], we proved that for a perfectly normal GO-space \( X \) with \( E(X) \) is \( \sigma\)-discrete in \( X \), a closed subspace \( A \) of \( X \) and \( Z \) a locally convex linear topological space \( Z \), there is a Dugundji extender \( u \) from \( C(A, Z) \) to \( C(X, Z) \), where \( E(X) = \{x \in X : (\leftarrow, x] \) or \([x, \rightarrow) \) is open in \( X \} \). We extend the theorem above as follows.

**Theorem 2.1.** Let \( X \) be a perfectly normal GO-space such that \( E(X) \) is \( \sigma\)-discrete in \( X \). Then every closed subspace \( A \) of \( X \) is continuously \( \pi_D\)-embedded in \( X \).
Theorem. Let $A$ be a closed subspace of $X$. Then $X - A$ is the union of a disjoint family $\mathcal{U}$ of convex components of $X - A$. Since $X$ is perfectly normal, it follows from [3, Theorem 2.4.5] that $\mathcal{U}$ is $\sigma$-discrete in $X$. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$, where $\mathcal{U}_n$ is discrete in $X$. Similarly, let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ be a disjoint and $\sigma$-discrete family of convex components of $\text{Int } A$. For each $U \in \mathcal{U}$ we choose $x(U) \in U$. We put $M_U = \{x(U) : U \in \mathcal{U}\}$. For each convex open set $C$ in $X$, we put
\[
\mathcal{L}(C) = \{a \in A : a < x\text{ for all } x \in C\},
\]
and
\[
\mathcal{R}(C) = \{a \in A : a > x\text{ for all } x \in C\},
\]
if the right-hand of the above equations exist.

Then for each $n$, we put $\mathcal{U}_n^L = \{U \in \mathcal{U}_n : l(U) \text{ exists}\}$ and $\mathcal{U}_n^R = \{U \in \mathcal{U}_n : r(U) \text{ exists}\}$.

Similarly, we define $\mathcal{V}_n^L$ and $\mathcal{V}_n^R$. Furthermore, we put
\[
\mathcal{L}_n = \{l(U) : U \in \mathcal{U}_n^L\},
\]
\[
\mathcal{R}_n = \{r(U) : U \in \mathcal{U}_n^R\},
\]
\[
\mathcal{L}'_n = \{l(V) : V \in \mathcal{V}_n^L\},
\]
and
\[
\mathcal{R}'_n = \{r(V) : V \in \mathcal{V}_n^R\}.
\]

It is easy to see that all of $\mathcal{L}_n, \mathcal{R}_n, \mathcal{L}'_n$ and $\mathcal{R}'_n$ are closed discrete in $X$. Let $L = \bigcup_{n=1}^{\infty} \mathcal{L}_n$, $R = \bigcup_{n=1}^{\infty} \mathcal{R}_n$, $L' = \bigcup_{n=1}^{\infty} \mathcal{L}'_n$ and $R' = \bigcup_{n=1}^{\infty} \mathcal{R}'_n$. Furthermore, we put
\[
B = \{a \in A : (L \cup R) : a \in \bigcup U^{-}(a) \cup U^{+}(a)\} \times
\]
where $U^{-}(a) = \{U \in \mathcal{U} : x(U) < a\}$ and $U^{+}(a) = \{U \in \mathcal{U} : x(U) > a\}$. Let
\[
M = M_U \cup L \cup R \cup L' \cup R' \cup (E(X) \cap A) \cup B.
\]

Then $M$ is a GO-space and $D = M - B$ is $\sigma$-discrete in $M$. Since $E(M) \subset D$ and $D$ is dense in $M$, it follows from [3, Theorem 3.1] that $M$ is metrizable. Then there exists a compatible metric $\rho$ on $M$ bounded by 1.

We shall define a mapping $\varphi : X \rightarrow 2^A$. Let $x \in X$. If $x \in A$, then we put $\varphi(x) = \{x\}$. Let $x \in X - A$. Then there is $U \in \mathcal{U}_n$ such that $x \in U$.

Case 1. Suppose that $U \in \mathcal{U}_n \cap \mathcal{U}_m$. If $U = \{x\}$, we put $\varphi(x) = \{\ell(U)\}$. If $U$ contains at least two points, we choose points $s(U)$ and $t(U)$ of $U$ such that $s(U) < t(U)$. We put
\[
\varphi(x) = \begin{cases} 
\{\ell(U)\}, & \text{if } x < s(U), \\
\{\ell(U), r(U)\}, & \text{if } s(U) \leq x \leq t(U), \\
\{r(U)\}, & \text{if } x > t(U).
\end{cases}
\]

Case 2. If $U \in \mathcal{U}_n$ and $U \notin \mathcal{U}_n$, then we put $\varphi(x) = \{\ell(U)\}$.

Case 3. If $U \notin \mathcal{U}_n$ and $U \in \mathcal{U}_n$, then we put $\varphi(x) = \{r(U)\}$.

Case 4. Finally, we suppose that $U \notin \mathcal{U}_n \cup \mathcal{U}_m$. Then we put $\varphi(x) = \{a(U)\}$, where $a(U)$ is defined in the proof of Theorem 2.1 in [7]. Then we can see that $\varphi : X \rightarrow 2^A$ is upper semicontinuous.

To define an extender $u : C(A \times Y, Z) \rightarrow C(X \times Y, Z)$, let $f \in C(A \times Y, Z)$. First, for each $n$ and each $U \in \mathcal{U}_n$ we shall define a continuous function $f_U : U \times Y \rightarrow Z$. We consider the following four cases.
Case 1. Suppose that $U \in \mathcal{U}_{n}^k \cap \mathcal{U}_{n}^r$. If $U = \{x\}$, we define $f_{U}(x, y) = f(l(U), y)$ for each $y \in Y$. If $U$ contains at least two points, we define

$$
f_{U}(x, y) = \begin{cases} 
(1 - \psi_{U})(x) \cdot f(l(U), y) + \psi_{U}(x) \cdot f(r(U), y), & \text{if } x < s(U), \\
f(l(U), y), & \text{if } s(U) \leq x \leq t(U), \\
f(r(U), y), & \text{if } x > t(U),
\end{cases}
$$

for each $(x, y) \in U \times Y$, where $\psi_{U} : X \rightarrow I$ is a continuous mapping such that $(\leftarrow, l(U)] \subset \psi_{U}^{-1}(0)$ and $[r(U), \rightarrow) \subset \psi_{U}^{-1}(1)$.

Case 2. If $U \in \mathcal{U}_{n}^k$ and $U \notin \mathcal{U}_{n}^r$, then we put $f_{U}(x, y) = f(l(U), y)$ for each $(x, y) \in U \times Y$.

Case 3. If $U \notin \mathcal{U}_{n}^k$ and $U \in \mathcal{U}_{n}^r$, then we put $f_{U}(x, y) = f(r(U), y)$ for each $(x, y) \in U \times Y$.

Case 4. If $U \notin \mathcal{U}_{n}^k \cup \mathcal{U}_{n}^r$, $f_{U}(x, y) = f(a(U), y)$ for each $(x, y) \in U \times Y$.

We define a function $u(f) : X \times Y \rightarrow Z$ as follows:

$$u(f)(x, y) = \begin{cases} 
f(x, y), & \text{if } x \in A, \\
f_{U}(x, y), & \text{if } x \in U \text{ for some } U \in \mathcal{U}.
\end{cases}
$$

In a similar fashion to [7], we can see that $u(f)$ is a continuous extension of $f$ and the range of $u(f)$ is contained in the closed convex hull of the range of $f$.

By use of the upper semicontinuity of $\varphi$, we can show that the extender $u$ above is continuous with respect to the point convergence topology, compact-open topology and uniform convergence topology (cf. [8]).

In a similar fashion as the proof of Theorem 2.1, we obtain the following (in fact, the proof of this case is more simple than Theorem 2.1).

Theorem 2.2. Let $X$ be a GO-space, $A$ a closed subspace of $X$ and $X - A = \bigcup \mathcal{U}$, where $\mathcal{U}$ is a disjoint family of convex components of $X - A$. If $\mathcal{U}' = \{ U \in \mathcal{U} : U \text{ has neither } l(U) \text{ nor } r(U) \}$ is discrete in $X$, then $A$ is continuously $\pi_{D}$-embedded in $X$.

Corollary 2.1. Let $X$ be a locally compact GO-space. Then every closed subspace $A$ of $X$ is continuously $\pi_{D}$-embedded in $X$.

Corollary 2.2. Every closed subspace of the Sorgenfrey line $S$ is continuously $\pi_{D}$-embedded.

Corollary 2.3. Let $X$ be a GO-space such that the underlining ordered set is well-ordered. Then every closed subspace $A$ of $X$ is continuously $\pi_{D}$-embedded.

Now, we have the following corollaries.

Corollary 2.4. Let $X_{i}(i = 1, 2, \cdots, n)$ be perfectly normal GO-spaces with $E(X_{i})$ $\sigma$-discrete in $X_{i}$ and $A_{i}$ are closed subsets in $X_{i}$. Then, $\prod_{i=1}^{n} A_{i}$ is $D$-embedded in $\prod_{i=1}^{n} X_{i}$.

Corollary 2.5. Let $\kappa$ be an ordinal and $A_{i}(i = 1, 2, \cdots, n)$ are closed subsets of $\kappa$. Then $\prod_{i=1}^{n} A_{i}$ is $D$-embedded in $\kappa^{n}$. 
Remark. In [5], Heath and Lutzer proved that for every closed subspace $A$ of a GO-space $X$ there is a simultaneous extender $u : C^*(A) \to C^*(X)$. However, Heath, Lutzer and Zenor [6] proved that there is no Dugundji extender $u : C^*(\mathbb{Q}) \to C^*(\mathbb{M})$ which is continuous when both function spaces are equipped with the compact-open topology nor the pointwise convergence topology, where $\mathbb{M}$ is the Michael line and $\mathbb{Q}$ is the subspace of $\mathbb{M}$ consisting of all rationals.

REFERENCES

7. Y. Hattori, $\pi$-embeddings and Dugundji extension theorems for generalized ordered spaces, preprint.