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A Survey of [AJS]

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This is a survey of the work [AJS] by H.H. Andersen, J.C. Jantzen and W. Soergel. There are also excellent expositions by the authors [A2], [S1], [S2], of which [A2] includes the entire aspect of Lusztig's program.

During the AMS Summer Institute 1986 at Arcata I had an opportunity to ask G. Lusztig how he had come to his conjectural formula [L1] that should describe the irreducible characters of simple $\mathbb{F}_p$-groups in terms of the Kazhdan-Lusztig polynomials. He kindly explained me the idea, that is in [H], and said it would be easier to relate the conjecture to his analogous conjecture for affine Kac-Moody Lie algebras than to derive the exact formula in the category of modules for the $\mathbb{F}_p$-groups or for their infinitesimal subgroups.

Meanwhile, quantized enveloping algebras were discovered by V. G. Drinfeld and Jimbo M. Their representation theory at roots of 1 has subsequently been related to that of affine Kac-Moody Lie algebras by D. Kazhdan and Lusztig [KL1, 2] and [L4], to the former [AJS] has related the representation theory of simple $\mathbb{F}_p$-groups, and Lusztig's conjectural formula for affine Kac-Moody Lie algebras has been verified by Kashiwara M. and Tanisaki T. [KT]. Altogether Lusztig's conjectural modular irreducible character formula is now proved to hold for large p and in type A, D, and E.

The morphism spaces of modules for simple $\mathbb{F}_p$-groups are $\mathbb{F}_p$-linear whereas those for quantized enveloping algebras over cyclotomic fields $\mathbb{Q}(\zeta)$ are $\mathbb{Q}(\zeta)$-linear, hence one cannot hope to have an equivalence between these categories. Neither is $\mathbb{F}_p$ flat over $\mathbb{Z}$. In order to overcome the difficulties,

[AJS] works not over $\mathbb{F}_p$, $\mathbb{Q}(\zeta)$ or $\mathbb{Z}$, but over various localizations of the completions of the Cartan part of the universal enveloping algebra of the Lie algebra of the $\mathbb{F}_p$-group and of the quantized enveloping algebra over $\mathbb{Q}(\zeta)$, introduces certain combinatorial categories over these algebras and finally over the symmetric algebra of the root lattice, then applies some standard techniques of finite dimensional algebras.

a° The problem

(a1) Let us first fix the notations.

R an irreducible root system with the set of coroots $R^\vee$

R^+ a positive system of R

Σ the simple system of R^+

X the weight lattice of R

X^+ the set of dominant weights of X

$\geq$ the standard partial order on X such that $\lambda \geq \mu$ iff $\lambda - \mu \in \sum_{\alpha \in R^+} \mathbb{N}\alpha$

W the Weyl group of R

$W_a = W \ltimes \mathbb{Z}$ the affine group of W

$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$

α_0 the dominant short root of R^+

$h = \langle \rho, \alpha_0^\vee \rangle + 1$ the Coxeter number of R

$(d_\alpha)_{\alpha \in \Sigma} \in \{1, 2, 3\}^\Sigma$ minimal such that $[(d_\alpha \langle \beta, \alpha^\vee \rangle)]_{\alpha, \beta \in \Sigma}$ is symmetric

(a2) Let $k = \mathbb{F}_p$ the prime field of characteristic $p > 0$, and $\mathfrak{G}_k$ the simply connected simple k-group with a maximal torus $\mathfrak{T}_k$ split over $\mathbb{Z}$ and the associated root system R. We will identify X with the weight group $\mathbf{Grp}_k(\mathfrak{T}_k, \mathfrak{G}\mathfrak{L}_1)$ of $\mathfrak{T}_k$.

If M is a $\mathfrak{T}_k$-module, M admits a weight space decomposition $M = \coprod_{\lambda \in X} M_\lambda$ with $M_\lambda = \{m \in M \mid t(m \otimes 1) = m \otimes \lambda(t)$ in $M \otimes A\ \forall A \in \mathbf{Alg}_k$ and $t \in \mathfrak{T}_k(A)\}$, where $\mathbf{Alg}_k$ denotes the category of commutative k-algebras. One calls $\lambda \in X$ a weight of M iff $M_\lambda \neq 0$. Set $\mathrm{ch}M = \Sigma_{\lambda\in X}(\dim M_\lambda)e(\lambda)$, called the character of M, in the group algebra $\mathbb{Z}[X]$ of X with the natural basis $e(\lambda)$, $\lambda \in X$.

There is a bijection, due to C. Chevalley [J], (II.2.4), between X^+ and the set of the isomorphism classes of the simple $\mathfrak{G}_k$-modules such that

$$(1) \qquad \lambda \longmapsto L(\lambda)_k \quad \text{simple of highest weight } \lambda.$$

The fundamental problem in the representation theory of $\mathfrak{G}_k$ has been to find all $\mathrm{ch}L(\lambda)_k$.

(a3) Let $\mathfrak{F}_{\mathfrak{G}} : \mathfrak{G}_k \to \mathfrak{G}_k$ be the Frobenius endomorphism of $\mathfrak{G}_k$. Let $X_k = \{\mu \in X^+ \mid \langle \mu, \alpha^\vee\rangle \leq p-1 \ \forall \alpha \in \Sigma\}$. If $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in X_k$ and $\lambda^1 \in X^+$, Steinberg's tensor product theorem says

$$L(\lambda)_k \simeq L(\lambda^0)_k \otimes_k L(\lambda^1)_k^{[1]} \quad \text{in} \quad \mathfrak{G}_k\mathbf{Mod},$$

where $L(\lambda^1)_k^{[1]}$ is the composite of the representation $L(\lambda^1)_k$ with $\mathfrak{F}_{\mathfrak{G}}$. Hence we have only to find all $\mathrm{ch}L(\lambda)_k$, $\lambda \in X_k$.

(a4) Let $\mathfrak{G}_1 = \ker \mathfrak{F}_{\mathfrak{G}}$ the Frobenius kernel of $\mathfrak{G}_k$. It is an infinitesimal subgroup of $\mathfrak{G}_k$ defined by the Hopf algebra $k[\mathfrak{G}]/\mathfrak{m}_k^p$, where $k[\mathfrak{G}]$ is the Hopf algebra of $\mathfrak{G}_k$ with the augmentation ideal $\mathfrak{m}_k$. Due to C.W. Curtis [J], (II.3.15),

$$(1) \qquad L(\lambda)_k,\ \lambda \in X_k,\ \textit{remains simple as } \mathfrak{G}_1\textit{-module.}$$

In order to keep track of the weights, however, we will work in the category of $\mathfrak{G}_1\mathfrak{T}_k$-modules.

In $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ the simples are still parametrized by their highest weights, varying though over the entire X. We will denote the simple of highest weight $\mu \in X$ in $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ by $L_k(\mu)$. Then

$$(2) \qquad L_k(\mu) \simeq L(\mu^0)_k \otimes_k p\mu^1 \quad \text{with} \quad p\mu^1 = (\mu^1)^{[1]}.$$

(a5) Let $\lambda \in X$. If $\mathfrak{B}_k$ is the Borel subgroup of $\mathfrak{G}_k$ whose roots are $-R^+$, regard λ as a $\mathfrak{B}_k$-module via the projection $\mathfrak{B}_k \to \mathfrak{T}_k$, and let

$$\hat{Z}_k(\lambda) = \{f \in \mathbf{Sch}_k(\mathfrak{G}_1\mathfrak{T}_k, \mathbb{A}^1) \mid f(A)(xb) = (\lambda(A)(b))^{-1}f(A)(x) \quad \forall x \in \mathfrak{G}_1\mathfrak{T}_k(A), b \in \mathfrak{B}_1\mathfrak{T}_k(A), A \in \mathbf{Alg}_k\},$$

that is just the $\mathfrak{G}_1\mathfrak{T}_k$-module of global sections of the invertible sheaf on the quotient $\mathfrak{G}_1\mathfrak{T}_k/\mathfrak{B}_1\mathfrak{T}_k$ induced by the $\mathfrak{B}_1\mathfrak{T}_k$-module λ, where $\mathfrak{B}_1$ is the Frobenius kernel of $\mathfrak{B}_k$ and $\mathbf{Sch}_k$ denotes the category of k-schemes. The $\mathfrak{G}_1\mathfrak{T}_k$-module structure is given by $xf = f(x^{-1}?)$. Regarded as a functor $\mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \to \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ $\hat{Z}_k$ is exact, that makes the representation theory of $\mathfrak{G}_1\mathfrak{T}_k$ more algebraic than that of $\mathfrak{G}_k$. One has

$$\mathrm{ch}\hat{Z}_k(\lambda) = e(\lambda) \prod_{\alpha\in R^+} \frac{1-e(-p\alpha)}{1-e(-\alpha)},$$

hence the composition factor multiplicity $[\hat{Z}_k(\lambda) : L_k(\lambda)] = 1$, and all the other composition factors of $\hat{Z}_k(\lambda)$ have highest weights $< \lambda$. It follws that the determination of $\mathrm{ch}L_k(\lambda)$ is now reduced to counting the decomposition numbers $[\hat{Z}_k(\lambda) : L_k(\mu)]$ for all $\lambda, \mu \in X$.

(a6) Define a partition of X into disjoint subsets, called the blocks of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$, to be the finest partition such that λ and μ belong to the same block if $\mathbf{Ext}^1_{\mathfrak{G}_1\mathfrak{T}_k}(L_k(\lambda), L_k(\mu)) \neq 0$. The linkage principle [J], (II.6.17) says

(1) *each block is contained in a W_a-orbit,*

where we let W_a act on X by $\gamma w \cdot_k \lambda = w(\lambda+\rho) - \rho + p\gamma$, $\gamma \in \mathbb{Z}R$, $w \in W$, and $\lambda \in X$.

If b is a block of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$, denote by $\mathfrak{G}_1\mathfrak{T}_k(b)$ the full subcategory of $\mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ consisting of all modules whose composition factors are of the form $L_k(\lambda)$, $\lambda \in b$. If Ω and Γ are two W_a-orbits in X, one has an exact functor

$$T_\Omega^\Gamma : \coprod_{b\subseteq\Omega} \mathfrak{G}_1\mathfrak{T}_k(b) \longrightarrow \coprod_{b\subseteq\Gamma} \mathfrak{G}_1\mathfrak{T}_k(b),$$

called the translation functor from Ω to Γ, that is both left and right adjoint to the translation functor T_Γ^Ω [J], (II.7).

(a7) Let $\mathfrak{A}_k = \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle x+\rho, \alpha_0^\vee\rangle < p \ \forall \alpha \in R^+\}$. The W_a-translates of $\mathfrak{A}_k$ are called alcoves. In particular, $\mathfrak{A}_k$ is called the bottom dominant alcove. One has

$$\mathfrak{A}_k \cap X \neq \emptyset \quad \text{iff} \quad 0 \in \mathfrak{A}_k \quad \text{iff} \quad p \geq h.$$

Assume from now on that $0 \in \mathfrak{A}_k$ throughout the rest of the survey.

Let $W_a^+ = \{w \in W_a \mid w \cdot_k 0 \in X^+\}$ and $W_1 = \{w \in W_a \mid w \cdot_k 0 \in X_k\}$. Note that both W_a^+ and W_1 are independent of k.

As $\hat{Z}_k(\lambda)$ is indecomposable, one can write by the linkage principle

$$\mathrm{ch}L_k(\lambda) = \sum_{w \in W_a} a_{\lambda w} \mathrm{ch}\hat{Z}_k(w \cdot_k \lambda), \quad a_{\lambda w} \in \mathbb{Z}.$$

If μ belong to the "upper closure" of the alcove of λ, then the translation principle [J], (II.7.17)(b) yields

$$\mathrm{ch}L_k(\mu) = \sum_{w \in W_a} a_{\lambda w} \mathrm{ch}\hat{Z}_k(w \cdot_k \mu). \tag{1}$$

Also $\hat{Z}_k(\lambda + p\mu) = \hat{Z}_k(\lambda) \otimes_k p\nu \ \forall \nu \in X$, hence together with (a4)(2)

$$[\hat{Z}_k(\lambda + p\nu) : L_k(\eta + p\nu)] = [\hat{Z}_k(\lambda) : L_k(\eta)]. \tag{2}$$

As any weight belongs to the upper closure of an alcove, for $p \geq h$ the problem is now reduced to counting all

$$[\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)], \quad w \in W_a, w' \in W_1. \tag{3}$$

(a8) One says a $\mathfrak{G}_1\mathfrak{T}_k$-module admits a $\hat{Z}_k$-filtration if it has a filtration in $\mathfrak{G}_1\mathfrak{T}_k$**Mod** with the factors of the form $\hat{Z}_k(\nu)$, $\nu \in X$.

Let $Q_k(\lambda)$ be the projective cover of $L_k(\lambda)$, $\lambda \in X$, in $\mathfrak{G}_1\mathfrak{T}_k$**Mod**. The Brauer-Humphreys reciprocity [J], (II.11.4) says

(1) $Q_k(\lambda)$ *admits a* $\hat{Z}_k$*-filtration*

and that the multiplicities in the $\hat{Z}_k$-filtration are given by

$$[Q_k(\lambda) : \hat{Z}_k(w \cdot_k \lambda)] = [\hat{Z}_k(w \cdot_k \lambda) : L_k(\lambda)], \tag{2}$$

where the factors of the filtration must be of the form $\hat{Z}_k(w \cdot_k \lambda)$, $w \in W_a$, by the linkage principle. Hence the problem is further reduced to finding the multiplicities in $\hat{Z}_k$-filtrations

$$[Q_k(w \cdot_k \lambda) : \hat{Z}_k(w' \cdot_k \lambda)] \quad \forall w \in W_1, w' \in W_a. \tag{3}$$

(a9) Let $\Omega_0 = W_a \cdot_k 0$ and $\lambda \in \Omega_0$. In one case the $\hat{Z}_k$-filtration of $Q_k(\lambda)$

is well-understood. The Steinberg module $\hat{Z}_k((p-1)\rho) = L_k((p-1)\rho) = L((p-1)\rho)_k$ is a projective indecomposable [J], (II.10.2), hence also

$$\hat{Z}_k((p-1)\rho + p\nu) \simeq \hat{Z}_k((p-1)\rho) \otimes_k p\nu \quad \forall \nu \in X.$$

If λ lies in the top alcove of the box $p\lambda^1 + X_k$, then [J], (II.11.10)

(1) $$Q_k(\lambda) = T^{\Omega_o}_{W_a \cdot_k ((p-1)\rho + p\lambda^1)} \hat{Z}_k((p-1)\rho + p\lambda^1),$$

in a $\hat{Z}_k$-filtration of which all $\hat{Z}_k(w \cdot_k \lambda^0 + p(\rho - w\rho + \lambda^1))$, $w \in W$, appear exactly once. More generally [J], (II.9.19),

(a10) **Lemma**. *Let $\lambda, \mu \in X$ belonging to the closure of an alcove. Then $T^{W_a \cdot_k \mu}_{W_a \cdot_k \lambda} \hat{Z}_k(\lambda)$ has a $\hat{Z}_k$-filtration with the factors*

$$\hat{Z}_k(w \cdot_k \mu), \quad w \in C_{W_a}(\lambda)/C_{W_a}(\lambda) \cap C_{W_a}(\mu),$$

each appearing exactly once.

(a11) Let Σ_a be the set of reflexions of W_a in a wall of $\mathfrak{A}_k$, that is independent of k. If $s \in \Sigma_a$, choose $\mu_s \in X \cap \overline{\mathfrak{A}_k}$ with $C_{W_a}(\mu_s) = \{1, s\}$, and set $T_s = T^{W_a \cdot_k \mu_s}_{\Omega_0}$, $T'_s = T^{\Omega_0}_{W_a \cdot_k \mu_s}$, and $\Theta_s = T_s \circ T'_s$.

For $\lambda \in \Omega_0$ define a sequence $I = (s_1, \ldots, s_r)$ of elements of Σ_a inductively as follows. If λ lies in the top alcove of the box $p\lambda^1 + X_k$, take $I = \emptyset$. Otherwise choose $s_1 \in \Sigma_a$ such that $\lambda < ws_1 \cdot_k 0$ if $\lambda = w \cdot_k 0$, $w \in W_a$, and that $ws_1 \cdot_k 0 \in p\lambda^1 + X_k$. Now set

$$Q^I_k(\lambda) = \Theta_{s_1} \circ \ldots \circ \Theta_{s_r} Q^{\emptyset}_k(\lambda)$$

with $Q^{\emptyset}_k(\lambda) = T^{\Omega_0}_{W_a \cdot_k ((p-1)\rho + p\lambda^1)} \hat{Z}_k((p-1)\rho + p\lambda^1)$. From (a10) we know the $\hat{Z}_k$-filtration of $Q^I_k(\lambda)$. On the other hand, if $\hat{\lambda} = w_0 \cdot_k \lambda^0 + p(\lambda^1 + 2\rho)$,

(1) $$Q^I_k(\lambda) = \coprod_{\substack{\nu \in \Omega_0 \\ \lambda \uparrow \nu \uparrow \hat{\nu} \uparrow \hat{\lambda}}} Q_k(\nu)^{m_k(\lambda,\nu)} \quad \text{with} \quad m_k(\lambda, \lambda) = 1,$$

where $\uparrow$ is a partial order on X such that $\nu \uparrow \nu'$ if $\nu' = s_\beta \cdot_k \nu + pm\beta \geq \nu$ for some $\beta \in R^+$ and $m \in \mathbb{Z}$ [J], (II.11.6).

As the $\operatorname{ch} Q(\nu)$ are linearly independent, the $m_k(\lambda, \nu)$ are uniquely determined. Then by induction on $\hat{\lambda} - \lambda$ finding all $m_k(\lambda, \nu)$ will determine the

$\hat{Z}_k$-filtration of each $Q_k(\nu)$, $\nu \in \Omega_0$.

(a12) The set of $w \in W_a$ with

$$0 \uparrow w \cdot_k 0 \uparrow \widehat{w \cdot_k 0} \uparrow \hat{0} = 2(p-1)\rho$$

is finite and independent of k. Enumerate those w_1, ..., w_{n_0} such that if $w_i \cdot_k 0 \uparrow w_j \cdot_k 0 \uparrow \widehat{w_j \cdot_k 0} \uparrow \widehat{w_i \cdot_k 0}$, then $j \leq i$. Note that

$$W_1 \subseteq \{w_1, \dots, w_{n_0}\}.$$

For each $w_i \cdot_k 0$, $i \in [1, n_0]$, choose a sequence $I(i)$ as in (a11) and set $Q^{[i]}(k) = Q_k^{I(i)}(w_i \cdot_k 0)$. Then

(1) $$Q^{[i]}(k) = \coprod_{j=1}^{i} Q_k(w_j \cdot_k 0)^{m_k(j,i)} \quad \text{with} \quad m_k(i,i) = 1.$$

Set $Q(k) = \coprod_{i=1}^{n_0} Q^{[i]}(k)$ and let

$$\mathcal{E}_{[i],[j]}(k) = \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}(Q^{[i]}(k), Q^{[j]}(k)), \quad \mathcal{E}(k) = \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}(Q(k), Q(k)).$$

Then $\mathcal{E}(k) = \coprod_{i,j \in [1,n_0]} \mathcal{E}_{[i],[j]}(k)$. Under the composition each $\mathcal{E}(k)_{[i],[i]}$ and $\mathcal{E}(k)$ form finite dimensional k-algebras.

Let $1 = \sum_{n \in E_k(i)} e_k^n(i)$ be a decomposition into orthogonal primitive idempotents in $\mathcal{E}(k)_{[i],[i]}$, where $E_k(i)$ is an indexing set with $e_k^0(i)$ corresponding to $Q_k(w_i \cdot_k 0)$, i.e., $Q_k(w_i \cdot_k 0) \simeq e_k^0(i) Q^{[i]}(k)$. Then $1 = \sum_{i=1}^{n_0} \sum_{n \in E_k(i)} e_k^n(i)$ is a decomposition into orthogonal primitive idempotents in $\mathcal{E}(k)$. Now

(2) *$e_k^n(i)$ is conjugate to $e_k^m(j)$ in $\mathcal{E}(k)$, i.e., there is some $u \in \mathcal{E}(k)^\times$ with $e_k^n(i) = u e_k^n(j) u^{-1}$, iff $\mathcal{E}(k) e_k^n(i) \simeq \mathcal{E}(k) e_k^n(j)$ in $\mathcal{E}(k)$***Mod** *iff $e_k^n(i) Q(k) \simeq e_k^m(j) Q(k)$ in $\mathfrak{G}_1\mathfrak{T}_k$***Mod**.

Hence if $n \neq 0$, $e_k^n(i)$ is conjugate to some $e_k^0(j)$ for $j < i$ while $e_k^0(i)$ is not conjugate to any of $e_k^m(j)$, $m \in E_k(j)$ with $j < i$. It follows that

(3) $$m_k(j,i) = \#\{s \in E_k(i) \mid E_k^s(i) \text{ is conjugate to } e_k^0(j) \text{ in } \mathcal{E}(k)\}.$$

(a13) By transferring from $\mathfrak{G}_k$**Mod** to $\mathfrak{G}_1\mathfrak{T}_k$**Mod** one has obtained finite dimensional projectives (in $\mathfrak{G}_k\mathbf{Mod}$ there are no finite dimensional injectives nor projectives), and the translations in W_a have been reflected in a simple manner: for each λ and $\nu \in X$,

$$L_k(\lambda + p\nu) \simeq L_k(\lambda) \otimes_k p\nu, \quad \hat{Z}_k(\lambda + p\nu) \simeq \hat{Z}_k(\lambda) \otimes_k p\nu; \quad \text{and} \quad Q_k(\lambda + p\nu) \simeq Q_k(\lambda) \otimes_k p\nu.$$

In characteristic 0 similar phenomenon occurs with the quantized enveloping algebra.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ with v an indeterminate and $U(\mathcal{A})$ Lusztig's $\mathcal{A}$-form of the Drinfeld-Jimbo quantized enveloping algebra over $\mathbb{Q}(v)$ [L3]. Let $\ell \in \mathbb{N}^+$ prime to the nonzero entries of the Cartan matrix of R, ζ a primitive ℓ-th root of 1 in $\mathbb{C}$, $\kappa = \mathbb{Q}(\zeta)$, and $U(\kappa) = U(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$. Lusztig has discovered a characteristic 0 analogue of the Frobenius kernel in $U(\kappa)$, that is an $\ell^{|R|}(2\ell)^{|\Sigma|}$-dimensional subalgebra $\mathfrak{u}(\kappa)$ of $U(\kappa)$ generated by $E_{\pm\alpha}$, K_α, $\alpha \in \Sigma$. Let $\mathcal{C}_{U(\kappa)}$ be the category of finite dimensional $U(\kappa)$-modules with K_α^ℓ acting by 1 for each $\alpha \in \Sigma$. One has $K_\alpha^{2\ell} = 1$ in $U(\kappa)$. Then (cf. [APW1], (9.12); if ℓ is not a prime power, one argues as in [AW]) each $M \in \mathcal{C}_{U(\kappa)}$ admits a weight space decomposition with respect to the Cartan subalgebra $U^0(\kappa) = U^0(\mathcal{A}) \otimes_{\mathcal{A}} \kappa$ with $U^0(\mathcal{A})$ the $\mathcal{A}$-subalgebra of $U(\mathcal{A})$ generated by $K_\alpha^{\pm 1}$ and $\left[\begin{smallmatrix} K_\alpha \\ m \end{smallmatrix}\right] = \prod_{i=1}^{m} \frac{K_\alpha v^{d_\alpha(-i+1)} - K_\alpha^{-1} v^{-d_\alpha(-i+1)}}{v^{d_\alpha i} - v^{-d_\alpha i}} \otimes 1$, $\alpha \in \Sigma$, $m \in \mathbb{N}$:

$$(1) \quad M = \coprod_{\lambda \in X} M_\lambda \quad \text{with} \quad M_\lambda = \{m \in M \mid um = \lambda(u)m \ \forall u \in U^0(\kappa)\},$$

where $\lambda(K_\alpha) = \zeta^{d_\alpha\langle\lambda,\alpha^\vee\rangle}$ and $\lambda(\left[\begin{smallmatrix} K_\alpha \\ m \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \langle\lambda,\alpha^\vee\rangle \\ m \end{smallmatrix}\right]_{d_\alpha}$ with $\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right]_{d_\alpha} = \prod_{i=1}^{m} \frac{v^{d_\alpha(r-i+1)} - v^{-d_\alpha(r-i+1)}}{v^{d_\alpha i} - v^{-d_\alpha i}} \otimes 1$.

The simples of $\mathcal{C}_{U(\kappa)}$ are parametrized by their highest weights in X^+ as in $\mathfrak{G}_k\mathbf{Mod}$. Let $X_\kappa = \{\mu \in X^+ \mid \langle\mu, \alpha^\vee\rangle \leq \ell - 1 \ \forall \alpha \in \Sigma\}$. If $L(\lambda)_\kappa$ denotes the simple of $\mathcal{C}_{U(\kappa)}$ of highest weight $\lambda \in X^+$ and if $\lambda = \lambda^0 + \ell\lambda^1$ with $\lambda^0 \in X_\kappa$ and $\lambda^1 \in X$, then Lusztig's tensor product theorem [LMR], (7.4) asserts

$$(2) \qquad L(\lambda)_\kappa \simeq L(\lambda^0)_\kappa \otimes_\kappa \bar{L}(\lambda^1)_\kappa^{[1]} \quad \text{in } \mathcal{C}_{U(\kappa)},$$

where $\bar{L}(\lambda^1)_\kappa^{[1]}$ is the composite of the simple representation $\bar{L}(\lambda^1)_\kappa$ of $\mathfrak{G}_\kappa$, i.e., of the universal enveloping algebra $U(\mathrm{Lie}(\mathfrak{G}_\kappa))$ of the Lie algebra $\mathrm{Lie}(\mathfrak{G}_\kappa)$

of $\mathfrak{G}_\kappa$, with Lusztig's lift $U(\kappa) \to U(\mathrm{Lie}(\mathfrak{G}_\kappa))$ of the Frobenius morphism [L3], (8.16) such that for each $\alpha \in \Sigma$ and $n \in \mathbb{N}$

$$E_{\pm\alpha}^{(n)} \mapsto \begin{cases} \bar{E}_{\pm\alpha}^{(\frac{n}{\ell})} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases} \qquad K_\alpha^{\pm 1} \mapsto K_\alpha^{\pm 1}, \qquad \begin{bmatrix} K_\alpha \\ n \end{bmatrix} \mapsto \begin{cases} \begin{bmatrix} H_\alpha \\ \frac{n}{\ell} \end{bmatrix} & \text{if } \ell \mid n \\ 0 & \text{otherwise,} \end{cases}$$

where $(\bar{E}_{\pm\beta}, H_\alpha)_{\alpha\in\Sigma,\beta\in R}$ is a basis of $\mathrm{Lie}(\mathfrak{G}_\kappa)$ obtained from a Chevalley basis, and $E_{\pm\alpha}^{(r)} = \frac{E_{\pm\alpha}^r}{[r]_{d_\alpha}^!}$ in $U(\kappa)$ with $[r]_{d_\alpha}^! = \prod_{i=1}^r \frac{v^{d_\alpha i} - v^{-d_\alpha i}}{v^{d_\alpha} - v^{-d_\alpha}} \otimes 1$ while $\bar{E}_{\pm\alpha}^{(r)} = \frac{\bar{E}_{\pm\alpha}^r}{r!}$ in $U(\mathrm{Lie}(\mathfrak{G}_k))$. By [AW], (1.9)

(3) $\quad L(\lambda^0)_\kappa$ *remains simple as* $\mathfrak{u}(\kappa)$*-module.*

Again in order to keep track of the weights, we will consider $\tilde{\mathfrak{u}}(\kappa) = U^0(\kappa)\mathfrak{u}(\kappa)$ and the category $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ of all finite dimensional $\tilde{\mathfrak{u}}(\kappa)$-modules admitting weight space decompositions (1) with K_α^ℓ acting by 1 for each $\alpha \in \Sigma$. The category $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ resembles much the categoty $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ of finite dimensional $\mathfrak{G}_1\mathfrak{T}_k$-modules [APW2], (4.7/4.10) (again if ℓ is not a prime power, refer to [AW]). In particular, finding the irreducible characters of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ is reduced for $\ell \geq h$ to the determination of the multiplicity $m_\kappa(j,i)$ of the projective cover $Q_\kappa(w_j \cdot_\kappa 0)$ of $L_\kappa(w_j \cdot_\kappa 0)$ in the projective $Q^{[i]}(\kappa)$:

(4) $$Q^{[i]}(\kappa) = \coprod_{j\leq i} Q_\kappa(w_j \cdot_\kappa 0)^{m_\kappa(j,i)},$$

using the notations of (a12) to define $Q^{[i]}(\kappa)$, where $\cdot_\kappa$ is the $(\cdot_k)$-action of W_a on X with p replaced by ℓ. Define $\mathcal{E}_{[i],[j]}(\kappa)$, $\mathcal{E}(\kappa)$, and the idempotents as in (a12) with k replaced by κ. Then

(5) $$m_\kappa(j,i) = \#\{s \in E_\kappa(i) \mid e_\kappa^s(i) \text{ is conjugate to } e_\kappa^0(j) \text{ in } \mathcal{E}(\kappa)\}.$$

(a14) We are not to ask for an equivalence of categories between $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ and $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$, but to expect for p and $\ell \geq h$

(1) $$m_k(i,j) = m_\kappa(i,j) \quad \forall i,j.$$

Indeed, a morphism space in $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ is finite dimensional over $\mathbb{F}_p$ while that in $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ is finite dimensional over $\mathbb{Q}(\zeta)$.

If $p = \ell < h$, however, Andersen and Jantzen have found an example [A1], (7.9) that $\mathrm{ch}\, L_k(\lambda) \neq \mathrm{ch}\, L_\kappa(\lambda)$ for some $\lambda \in X_k = X_\kappa$.

b° The theorem

(b1) Retain the notations of (a12/13).

Theorem (*cf.* **[AJS], Corollary 16.8)** *There is a $\mathbb{Z}$-algebra $\mathcal{E}$ of finite type as $\mathbb{Z}$-module with isomorphisms*

$$\mathcal{E} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}(k) \quad \text{in } k\mathbf{Alg} \quad \text{and} \quad \mathcal{E} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}(\kappa) \quad \text{in } \kappa\mathbf{Alg}.$$

Moreover, $\mathcal{E}$ admits a decomposition $\mathcal{E} = \coprod_{i,j \in [1,n_0]} \mathcal{E}_{[i],[j]}$ such that $\mathcal{E}_{[i],[j]}\mathcal{E}_{[n],[m]} \subseteq \delta_{jn}\mathcal{E}_{[i],[m]}$ for each i, j, m and n, and that the above isomorphisms restrict to isomorphisms

$$\mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} k \simeq \mathcal{E}_{[i],[j]}(k) \quad \text{and} \quad \mathcal{E}_{[i],[j]} \otimes_{\mathbb{Z}} \kappa \simeq \mathcal{E}_{[i],[j]}(\kappa),$$

respectively.

(b2) **Remark** (*cf.* **[AJS], Corollary 16.11)** *One can realize $\mathcal{E}$ such that $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{d}]$ is free of finite type over $\mathbb{Z}[\frac{1}{d}]$ with $d = (h-1)!$.*

(b3) For a commutative ring A let us write $\mathcal{E}_A = \mathcal{E} \otimes_{\mathbb{Z}} A$. There is a finite extension field F of $\mathbb{Q}$ that is a splitting field of $\mathcal{E}_{\mathbb{Q}}$ [NT], Theorem 2.3.11. Let $\mathfrak{o}_F$ be the ring of algebraic integers in F and let $1 = \sum_{n \in E_F(i)} e_F^n(i)$, $1 \leq i \leq n_0$, and $1 = \sum_{i=1}^{n_0} \sum_{n \in E_F(i)} e_F^n(i)$ be decompositions into orthogonal primitive idempotents in $(\mathcal{E}_{[i],[i]})_F$ and $\mathcal{E}_F$, respectively. One can find $N \in \mathbb{N}^+$ such that if $\mathfrak{o} = \mathfrak{o}_F[\frac{1}{N}]$, then (cf. [NT], Lemma 1.13.14)

(2) $$\mathfrak{o} \text{ is of finite type as } \mathbb{Z}[\frac{1}{N}]\text{-module,}$$

(3) $$\mathcal{E}_{\mathfrak{o}} \text{ is } \mathfrak{o}\text{-free of finite type,}$$

(4) $$\text{all } e_F^n(i) \text{ live in } \mathcal{E}_{\mathfrak{o}},$$

i.e., one can write $e_F^n(i) = e^n(i) \otimes 1$ with idempotents $e^n(i)$ in $\mathcal{E}_{\mathfrak{o}}$, and

(5) *$e^n(i)$ and $e^m(j)$ are conjugate in $\mathcal{E}_F$ iff they are so in $\mathcal{E}_{\mathfrak{o}}$ $\forall i, j, n, m$.*

If $\mathfrak{m} \in \operatorname{Max}(\mathfrak{o})$, $\mathfrak{o}_{\mathfrak{m}}$ is a DVR as $\mathfrak{o}$ is a Dedekind domain [AM], (9.5). Put $\mathfrak{o}' = \mathfrak{o}_{\mathfrak{m}}$, $\mathfrak{m}' = \mathfrak{m}\mathfrak{o}'$, and let $\hat{\mathfrak{o}}'$ be the completion of $\mathfrak{o}'$ in the $\mathfrak{m}'$-adic topology.

Then $\hat{\mathfrak{o}}'$ is a complete DVR with the maximal ideal $\hat{\mathfrak{m}}' = \mathfrak{m}'\hat{\mathfrak{o}}'$ (cf. [B1], (VI.5.3), Proposition 5) and with $\hat{\mathfrak{o}}'/\hat{\mathfrak{m}}' \simeq \mathfrak{o}'/\mathfrak{m}' \simeq \mathfrak{o}/\mathfrak{m}$ [AM], (10.16). In fact, if $\hat{\mathfrak{o}}$ is the completion of $\mathfrak{o}$ in the $\mathfrak{m}$-adic topology, then $\hat{\mathfrak{o}} \simeq \hat{\mathfrak{o}}'$ [B1], Exercise III.2.27(a). As F is a splitting field of $\mathcal{E}_{\mathbb{Q}}$, the $e^n(i)$ remain primitive in $\mathcal{E}_{\mathrm{Frac}(\hat{\mathfrak{o}}')}$, hence in $\mathcal{E}_{\hat{\mathfrak{o}}'}$. Also $e^n(i)$ and $e^m(j)$ are conjugate in $\mathcal{E}_{\hat{\mathfrak{o}}'}$ iff they are so in $\mathcal{E}_{\mathfrak{o}'}$. Hence (cf. [NT], Theorem 1.14.2(ii))

(6) *the $e^n(i)$ remain primitive in $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}}$,*

and (cf. [NT], Theorem 1.14.2(iii))

(7) *$e^n(i)$ and $e^m(j)$ are conjugate in $\mathcal{E}_{\mathfrak{o}/\mathfrak{m}}$ iff they are so in $\mathcal{E}_{\mathfrak{o}}$.*

Rearrange the index sets $E(i)$ of the primitive idempotents in $\mathcal{E}_{\mathfrak{o}}$ so that $e^0(i)$ is not conjugate in $\mathcal{E}_{\mathfrak{o}}$ to any of $e^m(j)$, $m \in E(j), j < i$.

(b4) As the simples of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ are absolutely simple, any indecomposable projective of $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ remains indecomposable projective under field extensions. Hence

(1) $m_\kappa(j,i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$

Also if $p >> 0$ so that $p \notin \mathfrak{o}^\times$, then considering $\mathfrak{m} \in \mathrm{Max}(\mathfrak{o})$ with $p \in \mathfrak{m}$ yields

(2) $m_k(j,i) = \#\{s \in E(i) \mid e^s(i) \text{ is conjugate to } e^0(j) \text{ in } \mathcal{E}_{\mathfrak{o}}\}.$

Hence for $p >> 0$

(3) $$m_k(j,i) = m_\kappa(j,i).$$

(b5) Let $\mathfrak{u}^-(\kappa)$ be the κ-subalgebra of $\mathfrak{u}(\kappa)$ generated by $E_{-\alpha}$, $\alpha \in \Sigma$, and let $\tilde{\mathfrak{u}}^\flat(\kappa) = \mathfrak{u}^-(\kappa)U^0(\kappa)$. Define a category $\mathcal{C}_{\tilde{\mathfrak{u}}^\flat(\kappa)}$ of finite dimensional $\tilde{\mathfrak{u}}^\flat(\kappa)$-modules just like $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$. In analogy to the functor $\hat{Z}_k : \mathfrak{B}_1\mathfrak{T}_k\mathbf{Mod} \to \mathfrak{G}_1\mathfrak{T}_k\mathbf{Mod}$ one has an induction functor $\tilde{Z}_\kappa : \mathcal{C}_{\tilde{\mathfrak{u}}^\flat(\kappa)} \to \mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$ defined by $\tilde{Z}_\kappa(M) = \tilde{\mathfrak{u}}^\flat(\kappa)\mathbf{Mod}(\tilde{\mathfrak{u}}(\kappa), M)$ [APW2], (1.2). Then

(1) $$\mathrm{ch}\,\tilde{Z}_\kappa(\lambda) = e(\lambda)\prod_{\alpha\in R^+}\frac{1-e(-\ell\alpha)}{1-e(-\alpha)} \quad \forall\lambda \in X,$$

and [APW2], (4.10)

(2) *the Brauer-Humphreys reciprocity carries over to $\mathcal{C}_{\tilde{\mathfrak{u}}(\kappa)}$.*

Corollary (*cf.* **[AJS], Corollary 16.23**) *Assume $\ell \geq h$ and $p >> 0$ relative to R. Then for each $w, w' \in W_a$ there is $d(w, w') \in \mathbb{N}$ independent of ℓ and p such that*

$$[\hat{Z}_k(w \cdot_k 0) : L_k(w' \cdot_k 0)] = d(w, w') = [\tilde{Z}_\kappa(w \cdot_\kappa 0) : L_\kappa(w' \cdot_\kappa 0)].$$

In particular, if $p = \ell$, then

$$ch\, L_k(w \cdot_k 0) = ch\, L_\kappa(w \cdot_\kappa 0) \quad \forall w \in W,$$

hence together with the translation principle

$$ch\, L(\lambda)_k = ch\, L(\lambda)_\kappa \quad \forall \lambda \in X_k = X_\kappa.$$

(b6) It follows that the irreducible characters of $\mathfrak{G}_k$**Mod** are obtained from that of $\mathcal{C}_{U(\kappa)}$ if $p >> 0$. Hence from [KL1, 2], [L4] and [KT] Lusztig's conjectural irreducible character formula in $\mathfrak{G}_k$**Mod** holds if $p >> 0$ and if R is of type A, D or E.

c° Reformulation of categories

(c1) In order to treat much alike categories $\mathfrak{G}_1\mathfrak{T}_k$**mod** and $\mathcal{C}_{\tilde{\mathrm{u}}(\kappa)}$ simultaneously, we will reformulate these categories as follows.

Case 1. Let $k[\mathfrak{G}]$ be the Hopf algebra defining $\mathfrak{G}_k$ and $\mathfrak{m}_k$ the augmentation ideal of $k[\mathfrak{G}]$. Let $\mathrm{Dist}(\mathfrak{G}_k) = \varinjlim_{n \geq 0} \mathbf{Mod}_k(k[\mathfrak{G}]/\mathfrak{m}_k^{n+1}, k)$ the algebra of distributions of $\mathfrak{G}_k$, that inherits the structure of Hopf algebra from $k[\mathfrak{G}]$.

Any $\mathfrak{G}_k$-module M is a $k[\mathfrak{G}]$-comodule, hence a $\mathrm{Dist}(\mathfrak{G}_k)$-module : if $\Delta_M = id_{k[\mathfrak{G}]} \in \mathfrak{G}_k(k[\mathfrak{G}]) : M \to M \otimes_k k[\mathfrak{G}]$ is the comodule map, then each $x \in \mathrm{Dist}(\mathfrak{G}_k)$ acts on M by $(M \otimes_k x) \circ \Delta_M$. Conversely, any finite dimensional $\mathrm{Dist}(\mathfrak{G}_k)$-module carries a structure of $\mathfrak{G}_k$-module [J], (II.1.20).

The Hopf algebra of $\mathfrak{G}_1$ is $k[\mathfrak{G}]/\mathfrak{m}_k^p$, hence $\mathrm{Dist}(\mathfrak{G}_1) = (k[\mathfrak{G}]/\mathfrak{m}_k^p)^*$. Then $\mathfrak{G}_1\mathbf{Mod} = \mathrm{Dist}(\mathfrak{G}_1)\mathbf{Mod}$: if M is a $\mathrm{Dist}(\mathfrak{G}_1)$-mod, one gets the comodule map by the commutative diagram

$$\begin{array}{ccccc} m & & M & \dashrightarrow & M \otimes_k k[\mathfrak{G}]/\mathfrak{m}_k^p \\ \downarrow & & \downarrow & & \| \\ x \mapsto xm & & \mathbf{Mod}_k(\mathrm{Dist}(\mathfrak{G}_1), M) & \xrightarrow{\sim} & M \otimes_k \mathrm{Dist}(\mathfrak{G}_1)^*. \end{array}$$

Let $\mathfrak{g} = \mathrm{Lie}(\mathfrak{G}_k) = \mathrm{Mod}_k(\mathfrak{m}_k/\mathfrak{m}_k^2, k) \leq \mathrm{Dist}(\mathfrak{G}_k)$, and $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ the triangular decomposition with $\mathfrak{h} = \mathrm{Lie}(\mathfrak{T}_k)$. For each $x \in \mathfrak{g}$ one has $x^p \in \mathfrak{g}$ in $\mathrm{Dist}(\mathfrak{G}_k)$ [DG], (II.7.2.3), which we will denote by $x^{[p]}$. In particular [DG], (II.7.2.2), if $x \in \mathfrak{n}^\pm$, then $x^{[p]} = 0$ while if $x \in \mathfrak{h}$, then $x^{[p]} = x$.

If $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, then

(1) $$x^p - x^{[p]} \in Z(U(\mathfrak{g})),$$

where x^p is the p-th power of x in $U(\mathfrak{g})$. One calls $U^{[p]}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} \mid x \in \mathfrak{g})$ the restricted enveloping algebra of $\mathfrak{g}$. There is a commutative diagram of k-algebras

(2) $$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\text{natural}} & \mathrm{Dist}(\mathfrak{G}_k) \\ \downarrow & & \uparrow \\ U^{[p]}(\mathfrak{g}) & \underset{\sim}{\longrightarrow} & \mathrm{Dist}(\mathfrak{G}_1). \end{array}$$

Fix a k-basis $(H_\alpha, \bar{E}_\beta \mid \alpha \in \Sigma, \beta \in R)$ of $\mathfrak{g}$ with $H_\alpha = [\bar{E}_\alpha, \bar{E}_{-\alpha}]$ obtained from a Chevalley basis. Let $I = (\bar{E}_\beta^p \mid \beta \in R) \trianglelefteq U(\mathfrak{g})$ and set $\bar{U}(\mathfrak{g}) = U(\mathfrak{g})/I$. The adjoint action of $\mathfrak{T}_k$ on $U(\mathfrak{g})$ stabilizes I, hence $\bar{U}(\mathfrak{g})$ comes equipped with an X-gradation given by the $\mathfrak{T}_k$- action. As $\bar{E}_\beta^p \in Z(U(\mathfrak{g}))$, $\bar{U}(\mathfrak{g})$ retains a PBW-type basis $(\bar{E}^m H^r \bar{F}^n \mid m, n \in [0, p-1]^{R^+}, r \in \mathbb{N}^\Sigma)$ with

$$\bar{E}^m = \prod_{\beta \in R^+} \bar{E}_\beta^{m_\beta}, H_r = \prod_{\alpha \in \Sigma} H_\alpha^{r_\alpha} \text{ and } \bar{F}^n = \prod_{\beta \in R^+} \bar{E}_{-\beta}^{n_\beta}.$$

The degree of $\bar{E}^m H^r \bar{F}^n$ is $\sum_{\beta \in R^+} (m_\beta - n_\beta)\beta$.

Case 2. Let U_2 be the De Concini-Kac version [DCK], (1.5) of the quantized enveloping algebra over κ, i.e., the κ-algebra with the generators $E_{\pm\alpha}$, $K_\alpha^{\pm 1}$, $\alpha \in \Sigma$, and the same relations as the Drinfeld-Jimbo algebra over $\mathbb{Q}(v)$ with v replaced by ζ. Let $U_2^\pm$ (resp. U_2^0) be the κ-subalgebra of U_2 generated by $E_{\pm\alpha}$ (resp. $K_\alpha^{\pm 1}$), $\alpha \in \Sigma$. For each $w \in W$ let T_w be the endomorphism of U_2 carried over from [LQG]. If $\beta \in R^+$, choose $w \in W$ with $w^{-1}\beta \in \Sigma$, and set $E_\beta = T_w(E_{w^{-1}\beta})$ and $E_{-\beta} = T_w(E_{-w^{-1}\beta})$. In case $\beta \in \Sigma$, the $E_{\pm\beta}$ so defined coincide with the old ones. One can then make U_2 into an X-graded algebra by giving $E_\beta, \beta \in R$ (resp. $K_\alpha, \alpha \in \Sigma$), degree β (resp. 0). By [DCK], Corollary 3.1

$$E_\beta^\ell, K_\alpha^\ell \in Z(U_2) \quad \forall \beta \in R \text{ and } \alpha \in \Sigma.$$

Let $I^\pm = (E_\beta^\ell \mid \beta \in \pm R^+) \trianglelefteq U_2^\pm$ and $I = (I^\pm) \trianglelefteq U_2$. If $f \in \kappa\mathbf{Alg}(U_2, U(\kappa))$ with $E_{\pm\alpha} \mapsto E_{\pm\alpha}$ and $K_\alpha \mapsto K_\alpha$ for each $\alpha \in \Sigma$, then f induces an isomorphism of κ-algebras

$$U_2/(I, K_\alpha^{2\ell} - 1 \mid \alpha \in \Sigma) \simeq \mathfrak{u}(\kappa).$$

Moreover, $I^\pm = \ker(f \mid_{U_2^\pm})$, hence $I^\pm$ are defined independent of the choice of the T_w's.

Under a suitable choice of the T_w's and orderings in the products U_2/I retains a PBW-type κ-basis $(E^m K^r F^n \mid m, n \in [0, \ell-1]^{R^+}, r \in \mathbb{Z}^\Sigma)$ with

$$E^m = \prod_{\beta \in R^+} E_\beta^{m_\beta}, K_r = \prod_{\alpha \in \Sigma} K_\alpha^{r_\alpha} \text{ and } F^n = \prod_{\beta \in R^+} E_{-\beta}^{n_\beta}.$$

(c2) In order to treat the two cases simultaneously, we will denote (κ, ℓ) also by (k, p) and set

$$(U, U^\pm, U^0) = \begin{cases} (\bar{U}(\mathfrak{g}), U(\mathfrak{n}^\pm) + I/I, U(\mathfrak{h}) + I/I) & \text{in Case 1} \\ (U_2/I, U_2^\pm + I/I, U_2^0 + I/I) & \text{in Case 2.} \end{cases}$$

Hence as k-algebras

$$U^0 \simeq \begin{cases} k[H_\alpha \mid \alpha \in \Sigma] & \text{the polynomial algebra in } H_\alpha \text{ in Case 1} \\ k[K_\alpha^{\pm 1} \mid \alpha \in \Sigma] & \text{the Laurent polynomial algebra in } K_\alpha \text{ in Case 2,} \end{cases}$$

and U has

(3) *a structure of k-Hopf algebra (nontrivial in Case 2),*

(4) *a triangular decomposition, i.e.,*
a k-linear bijection $U^- \otimes_k U^0 \otimes_k U^+ \to U$ *under the multiplication,*

and

(5) *an X-gradation, indicated by subscripts, such that*

$$U^0 \subseteq U_0, U^+ \subseteq \coprod_{\nu \geq 0} U_\nu, U^- \subseteq \coprod_{\nu \leq 0} U_\nu, \text{ and } (U^+)_0 = k \cdot 1 = (U^-)_0.$$

Define a group homomorphism $\tilde{\ } : X \to \mathbf{Alg}_k(U^0, U^0)^\times$ by

$$\tilde{\lambda}(H) = H + \lambda(H) \quad \forall H \in \mathfrak{h} \quad \text{in Case 1}$$
$$\tilde{\lambda}(K_\alpha) = \zeta^{d_\alpha \langle \lambda, \alpha^\vee \rangle} K_\alpha \quad \forall \alpha \in \Sigma \quad \text{in Case 2.}$$

Then for each $s \in U^0$ and $u \in U_\lambda$ one has $su = u\tilde{\lambda}(s)$.

(c3) Let A be a noetherian domain over U^0 with a structure homomorphism $\pi : U^0 \to A$ (the assumption that A be a domain is only for convenience in the present survey). We define a category $\mathcal{C}_A$ as follows. An object of $\mathcal{C}_A$ is a $U \otimes_k A$-module M, which is as A-module of finite type and X-graded. We regard U and A imbedded in $U \otimes_k A$ as $U \otimes 1$ and $1 \otimes A$, respectively, and write $(u \otimes a)m = uma$. We require

(1) $$U_\nu M_\lambda \subseteq M_{\lambda+\nu} \quad \forall \nu \in X$$

and

(2) $$sm = m\pi(\tilde{\lambda}(s)) \quad \forall s \in U^0 \text{ and } m \in M_\lambda.$$

A morphism of $\mathcal{C}_A$ is a morphism of $U \otimes_k A$-modules that preserves the X-gradings.

The category $\mathcal{C}_A$ is equipped with a duality operation. There is an involutory antiautomorphism τ of U [AJS], (1.6) such that

$$E_\alpha \longmapsto E_{-\alpha} \quad \forall \alpha \in \Sigma \quad \text{and} \quad s \longmapsto s \quad \forall s \in U^0.$$

If $M \in \mathcal{C}_A$, define M^τ to be $\mathbf{Mod}A(M, A)$ with U acting by $(uf)(m) = f(\tau(u)m)$ and with the X-gradation given by

$$(M^\tau)_\lambda = \{f \in M^\tau \mid f(M_\mu) = 0 \quad \forall \mu \neq \lambda\} \simeq \mathbf{Mod}A(M_\lambda, A).$$

If M is A-projective, $(M^\tau)^\tau \simeq M$ in $\mathcal{C}_A$.

Replacing U by U^0U^+ (resp. U^0) one defines likewise the categories $\mathcal{C}_A^{\geq 0}$ and $\mathcal{C}_A^0$.

If $M \in \mathcal{C}_A^0$ is projective in the category of right A-modules $\mathbf{Mod}A$, define the character of M by

$$\operatorname{ch} M = \sum_{\lambda \in X} \operatorname{rk}_A(M_\lambda)e(\lambda) \quad \text{in } \mathbb{Z}[X].$$

(c4) **Case 1.** Take $A = k$ with the structure homomorphism $\pi : U^0 \to k$ annihilating $\mathfrak{h}$. Then for each $\lambda \in X$ and $u \in \mathfrak{h}$

$$\tilde{\lambda}(u^p) = \tilde{\lambda}(u)^p = \lambda(u)^p = \lambda(u) = \tilde{\lambda}(u).$$

Hence the U-module structure on $M \in \mathcal{C}_k$ factors through $U^{[p]}(\mathfrak{g})$. Consequently, M comes equipped with a structure of $\mathrm{Dist}(\mathfrak{G}_1)$-module. Moreover, the X-gradation on M makes M into a $\mathfrak{T}_k$-module such that

$$t(xm \otimes 1) = (\mathrm{Ad}(t)(x \otimes 1))t(m \otimes 1) \text{ in } M \otimes_k A' \quad \forall t \in \mathfrak{T}_k(A'), A' \in \mathbf{Alg}_k,$$

hence into a $\mathfrak{G}_1\mathfrak{T}_k$-module. One can thus identify $\mathcal{C}_k$ with $\mathfrak{G}_1\mathfrak{T}_k\mathbf{mod}$ the category of finite dimensional $\mathfrak{G}_1\mathfrak{T}_k$-modules.

Case 2. Take $A = k$ with $\pi : U^0 \to k$ such that $K_\alpha \mapsto 1 \ \forall \alpha \in \Sigma$. Then for each $\lambda \in X$ and $\alpha \in \Sigma$

$$\tilde{\lambda}(K_\alpha^p) = \tilde{\lambda}(K_\alpha)^p = \zeta^{pd_\alpha \langle \lambda, \alpha^\vee \rangle} = 1.$$

Hence together with the X-gradation one can identify $\mathcal{C}_k$ with $\mathcal{C}_{\tilde{\mathrm{u}}(k)}$.

(c5) The forgetful functor gives an equivalence of categories from $\mathcal{C}_A^0$ to the category of X-graded A-modules of finite type, hence

(1) $\mathcal{C}_A^0$ *has enough projectives.*

Define a functor $\Phi_A : \mathcal{C}_A^0 \to \mathcal{C}_A$ by setting $\Phi_A(M) = U \otimes_{U^0} M$, $M \in \mathcal{C}_A^0$, with U acting by the left multiplication on U while A acting as given on M. The X-gradation on $\Phi_A(M)$ is defined by $\Phi_A(M)_\lambda = \sum_{\nu \in X} U_\nu \otimes_{U^0} M_{\lambda-\nu}$.

Define likewise a functor $\Phi_A^{\geq 0} : \mathcal{C}_A^0 \to \mathcal{C}_A^{\geq 0}$ by $\Phi_A^{\geq 0}(M) = U^0U^+ \otimes_{U^0} M$. Then

(2) Φ_A *(resp.* $\Phi_A^{\geq 0}$*) is exact and*
left adjoint to the forgetful functor from $\mathcal{C}_A$ *(resp.* $\mathcal{C}_A^{\geq 0}$*) to* $\mathcal{C}_A^0$.

Hence from (1)

(3) *both* $\mathcal{C}_A$ *and* $\mathcal{C}_A^{\geq 0}$ *have enough projectives.*

(c6) Define likewise a functor $Z_A : \mathcal{C}_A^{\geq 0} \to \mathcal{C}_A$ by setting

$$Z_A(M) = U \otimes_{U^0U^+} M, \quad M \in \mathcal{C}_A^{\geq 0},$$

with the X-gradation on $Z_A(M)$ defined by $Z_A(M)_\lambda = \coprod_{\nu \in X} (U^-)_\nu \otimes_k M_{\lambda-\nu}$, using an A-linear isomorphism $Z_A(M) \simeq U^- \otimes_k M$. Then

(1) Z_A *is exact and left adjoint to the forgetful functor* $\mathcal{C}_A \to \mathcal{C}_A^{\geq 0}$

and

(2) $$\Phi_A = Z_A \circ \Phi_A^{\geq 0}.$$

An object of $\mathcal{C}_A^0$ can be made into an object of $\mathcal{C}_A^{\geq 0}$ through an isomorphism $U^0U^+/\coprod_{\nu>0}(U^0U^+)_\nu \simeq U^0$. In particular, if $\lambda \in X$, define $A^\lambda \in \mathcal{C}_A^0$ by

$$(A^\lambda)_\nu = \begin{cases} A & \text{if } \nu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Regarding A^λ as an object of $\mathcal{C}_A^{\geq 0}$, set $Z_A(\lambda) = Z_A(A^\lambda)$. Then

(3) $$\operatorname{ch} Z_A(\lambda) = e(\lambda) \prod_{\beta \in R^+} \frac{1 - e(-p\beta)}{1 - e(-\beta)},$$

that coincides with $\operatorname{ch} \hat{Z}_k(\lambda)$ of §a.

In case $A = F$ is a field

(4) *$Z_F(\lambda)$ has a simple head of highest weight λ,*

which we will denote by $L_F(\lambda)$. All simples of $\mathcal{C}_F$ arise in this way.

(c7) A Z-filtration of $M \in \mathcal{C}_A$ is a chain in $\mathcal{C}_A$ with the successive subquotients isomorphic to some $Z_A(\lambda)$, $\lambda \in X$. By (c6)(3)

(1) *the multiplicity of $Z_A(\lambda)$ in a Z-filtration is independent of the choice of the Z-filtrations.*

As $\Phi_A = Z_A \circ \Phi_A^{\geq 0}$ and as both Z_A and $\Phi_A^{\geq 0}$ are exact,

(2) *any $M \in \mathcal{C}_A$ admits an epi $Q \to M$ in $\mathcal{C}$ with Q projective having a Z-filtration.*

Moreover,

(c8) **Lemma (*cf.* [AJS], Lemma 2.16)** *If A is local, any direct summand of an object of $\mathcal{C}_A$ with a Z-filtration admits a Z-filtration. In particular, any projective of $\mathcal{C}_A$ has a Z-filtration.*

Proof. One has [AJS], (2.14)

(1) $$Ext^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0,\ \lambda, \mu \in X, \text{ then } \mu > \lambda.$$

Let $M = M' \oplus M''$ in $\mathcal{C}_A$ with M having a Z-filtration. If $A = F$ is a field, the standard argument applies: if λ is a maximal weight of M with $r = \dim_F M_\lambda$, then by (1) there is $V \leq M$ with $V \simeq Z_F(\lambda)^{\oplus_r}$ such that M/V has a Z-filtration with $[M/V : Z_F(\lambda)] = 0$. If $m \in M'_\lambda \setminus 0$, let $\hat{m} \in \mathcal{C}_F(Z_F(\lambda), M')$ induced by the adjunction from a morphism $F^\lambda \to M'$ in $\mathcal{C}^{\geq 0}$ such that $1 \mapsto m$. Then $\mathrm{im}(\hat{m}) \leq V$. Denote by $\hat{m}'$ the morphism $Z_F(\lambda) \to V$ induced from $\hat{m}$. As $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) \simeq \mathcal{C}_F^{\geq 0}(Z_F(\lambda), \lambda) \simeq F$, $\mathcal{C}_F(Z_F(\lambda), Z_F(\lambda)) = F\mathrm{id}_{Z_F(\lambda)}$. Hence

(2) $\qquad$ *$\hat{m}'$ is a split mono with* $\mathrm{coker}(\hat{m}') \simeq Z_F(\lambda)^{\oplus_{r-1}}$.

Then $M'/\mathrm{im}(\hat{m}) \oplus M'' \simeq M/\mathrm{im}(\hat{m}')$ retains a Z_F-filtration. The assertion follows by induction on the length of a Z-filtration on M.

In general, let $\mathfrak{p} \in \mathrm{Spec}A$ with $\kappa(\mathfrak{p})$ the residue field of $A_\mathfrak{p}$. As $Z_A(\lambda) \otimes_A \kappa(\mathfrak{p}) \simeq Z_{\kappa(\mathfrak{p})}(\lambda)$ in $\mathcal{C}_{\kappa(\mathfrak{p})}$, it suffices to check by above that

(3) *if $L \in \mathcal{C}_A$ is A-free with $L_{\kappa(\mathfrak{p})} = L \otimes_A \kappa(\mathfrak{p})$ admitting a Z-filtration in $\mathcal{C}_{\kappa(\mathfrak{p})}$ for each $\mathfrak{p} \in \mathrm{Spec}A$, then L admits a Z-filtration in $\mathcal{C}_A$.*

Let λ be a maximal weight of L. By (1) again if $s = \dim_{\kappa(\mathfrak{p})}(L_{\kappa(\mathfrak{p})})_\lambda$, there is $L' \leq L_{\kappa(\mathfrak{p})}$ with $L' \simeq Z_{\kappa(\mathfrak{p})}(\lambda)^{\oplus_s}$ and such that $L_{\kappa(\mathfrak{p})}$ has a Z-filtration with $[L_{\kappa(\mathfrak{p})}/L' : Z_{\kappa(\mathfrak{p})}(\lambda)] = 0$. As A is local, L_λ remains A-free, say $L_\lambda = Ae_1 \oplus \ldots \oplus Ae_s$. If $\hat{e}_1 \in \mathcal{C}_A(Z_A(\lambda), L)$ with $1 \otimes 1 \mapsto e_1$, then as in (2)

(4) *$\hat{e}_1 \otimes_A \kappa(\mathfrak{p})$ is injective and* $\mathrm{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))$ *admits a Z-filtration.*

On the other hand, using the duality operator τ of (c3) one has a commutative diagram

$$\begin{array}{ccc}
\mathbf{Mod}A(L, A) \otimes_A \kappa(\mathfrak{p}) & \xrightarrow{(\hat{e}_1)^\tau \otimes_A \kappa(\mathfrak{p})} & \mathbf{Mod}A(Z_A(\lambda), A) \otimes_A \kappa(\mathfrak{p}) \\
\wr\downarrow & & \downarrow\wr \\
\mathbf{Mod}\kappa(\mathfrak{p})(L_{\kappa(\mathfrak{p})}, \kappa(\mathfrak{p})) & \xrightarrow[(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau]{} & \mathbf{Mod}\kappa(\mathfrak{p})(Z_{\kappa(\mathfrak{p})}(\lambda), \kappa(\mathfrak{p})).
\end{array}$$

By (4) $(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))^\tau$ is surjective, hence $(\hat{e}_1)^\tau \otimes_A A_\mathfrak{p}$ is surjective for each $\mathfrak{p} \in \mathrm{Spec}A$ by NAK. Then $(\hat{e}_1)^\tau$ is surjective [AM], (3.9). As $Z_A(\lambda)$ is A-free, the short exact sequence in $\mathcal{C}_A$

$$0 \to \ker((\hat{e}_1)^\tau) \to L^\tau \xrightarrow{(\hat{e}_1)^\tau} Z_A(\lambda)^\tau \to 0$$

splits in **Mod**A. Then $\ker((\hat{e}_1)^\tau)$ is A-free as A is local.. Hence $(\ker((\hat{e}_1)^\tau))^\tau$ is A-free in the short exact sequence of $\mathcal{C}_A$

$$0 \longrightarrow Z_A(\lambda) \xrightarrow{\hat{e}_1} L \longrightarrow (\ker((\hat{e}_1)^\tau))^\tau \longrightarrow 0.$$

By (4) $(\ker((\hat{e}_1)^\tau))^\tau \otimes_A \kappa(\mathfrak{p}) \simeq \operatorname{coker}(\hat{e}_1 \otimes_A \kappa(\mathfrak{p}))$ has a Z-filtration in $\mathcal{C}_{\kappa(\mathfrak{p})}$ for each $\mathfrak{p} \in \operatorname{Spec}A$. Then by induction on $\operatorname{rk}_A L$ $(\ker((\hat{e}_1)^\tau))^\tau$ admits a Z-filtration, and (3) follows.

The second assertion follows from (c7)(2).

(c9) Define a partitin of X into disjoint subsets, called the blocks over A, by taking a finest partition such that λ and μ belong to the same block if either $\mathcal{C}_A(Z_A(\lambda), Z_A(\mu)) \neq 0$ or $\operatorname{Ext}^1_{\mathcal{C}_A}(Z_A(\lambda), Z_A(\mu)) \neq 0$. Let $\mathcal{B}_A$ be the set of blocks over A. Let $\mathcal{D}_A$ be the full subcategory of $\mathcal{C}_A$ consisting of all objects with a Z-filtration. If b is a block over A, let $\mathcal{D}_A(b)$ be the full subcategory of $\mathcal{D}_A$ consisting of all objects such that the subquotients of a Z-filtration are $Z_A(\lambda)$, $\lambda \in b$. Let $\mathcal{C}_A(b)$ be the full subcategory of $\mathcal{C}_A$ consisting of all that are the images of objects of $\mathcal{D}_A(b)$.

(c10) **Theorem (*cf.* [AJS], Theorem 6.10)** (i) *If b, b' are disjoint blocks over A, then*

$$\operatorname{Ext}^{\cdot}_{\mathcal{C}_A}(M, M') = 0 \quad \forall M \in \mathcal{C}_A(b) \text{ and } M' \in \mathcal{C}_A(b').$$

(ii) *Each $M \in \mathcal{C}_A$ admits a block decomposition $M = \coprod_{b \in \mathcal{B}_A} M_b$ with M_b the largest subobject of M belonging to $\mathcal{C}_A(b)$.*

(iii) *For each block b over A the category $\mathcal{C}_A(b)$ is closed under taking homomorphic images, submodules, extensions, and finite direct sums.*

(c11) Relative to the structure homomorphism $\pi : U^0 \longrightarrow A$, let

$$R_\pi = \begin{cases} \{\beta \in R \mid \Pi_{j=1}^{p}(\pi(H_\beta) + j) \notin A^\times\} & \text{in Case 1} \\ \{\beta \in R \mid \Pi_{j=1}^{p}(\pi([K_\beta : j]) \notin A^\times\} & \text{in Case 2,} \end{cases}$$

where $[K_\beta : j] = \begin{bmatrix} K_\beta : j \\ 1 \end{bmatrix} = \frac{K_\beta \zeta^{jd_\beta} - K_\beta^{-1}\zeta^{-jd_\beta}}{\zeta^{d_\beta} - \zeta^{-d_\beta}}$ $(\neq \begin{bmatrix} K_\beta \\ j \end{bmatrix})$ and $d_\beta = d_\alpha$ if $\alpha \in \Sigma$ with $\beta \in W\alpha$. Then R_π forms a root system with the Weyl group

$W_\pi = \langle s_\beta \mid \beta \in R_\pi \rangle$ and a positive system of roots $R_\pi^+ = R_\pi \cap R^+$. Let $W_{\pi,a} = W_\pi \ltimes \mathbb{Z}R_\pi \leq W_a$. It will be convenient to introduce

$$B = \begin{cases} U^0[\frac{1}{\prod_{j=1}^{p-1}(H_\beta + j)} \mid \beta \in R^+] & \text{in Case 1} \\ U^0[\frac{1}{\prod_{j=1}^{p-1}([K_\beta ; j])} \mid \beta \in R^+] & \text{in Case 2.} \end{cases}$$

Proposition (*cf.* **[AJS], Proposition 6.13)** *Suppose A is a B-algebra. If $b \in \mathcal{B}_A$ and $\lambda \in b$, then $b \subseteq W_{\pi,a} \cdot_k \lambda$.*

(c12) Regard k as a U^0-algebra via the augmentation. For each $E \in \mathcal{C}_k$ and $M \in \mathcal{C}_A$ one can make $E \otimes_k M$ into an object of $\mathcal{C}_A$ by letting U (resp. A) act via the comultiplication (resp. only on M). The gradation is defined by $(E \otimes_k M)_\lambda = \coprod_{\nu \in X} E_\nu \otimes_k M_{\lambda - \nu}$.

Assume A is a B-algebra. Let W' be a reflexion subgroup of W_a with $W_{\pi,a} \leq W'$. An alcove for W' is a connected component of $X \otimes_{\mathbb{Z}} \mathbb{R}$ with the hyperplanes in W' deleted. Let Ω and Γ be two W'-orbits in X. The closure of an alcove for W' contains exactly one element $\lambda \in \Omega$ and $\mu \in \Gamma$. Then $W(\mu - \lambda)$ is independent of the choice of the alcove. Let ν be the unique dominant weight of $W(\mu - \lambda)$. Choose a simple E of highest weight ν in $\mathfrak{G}_k$**Mod** (resp. $\mathcal{C}_{U(k)}$) in Case 1 (resp. Case 2). Let $\mathcal{C}_A(\Omega) = \coprod_{b \subseteq \Omega} \mathcal{C}_A(b)$ and $\mathcal{C}_A(\Gamma) = \coprod_{b \subseteq \Gamma} \mathcal{C}_A(b)$. If $\mathrm{pr}_\Gamma : \mathcal{C}_A \to \mathcal{C}_A(\Gamma)$ is the functor such that $\mathrm{pr}_\Gamma M = \coprod_{b \subseteq \Gamma} M_b$, one gets an exact functor

$$T_\Omega^\Gamma = pr_\Gamma \circ (E \otimes_k ?) : \mathcal{C}_A(\Omega) \longrightarrow^{C}_{A} (\Gamma),$$

called the translation functor from Ω to Γ. In case $A = k$ the functor recovers the translation functor in $\mathfrak{G}_k$**Mod** and $\mathcal{C}_k$. As usual [AJS], (7.6),

(1) $\quad T_\Omega^\Gamma$ *is both left and right adjoint to* T_Γ^Ω.

Denote the adjunctions by $\mathrm{adj}_1 : \mathcal{C}_A(\Omega)(?, T_\Gamma^\Omega ?') \to \mathcal{C}_A(\Gamma)(T_\Omega^\Gamma ?, ?')$ and $\mathrm{adj}_2 : \mathcal{C}_A(\Gamma)(?, T_\Omega^\Gamma ?') \to \mathcal{C}_A(\Omega)(T_\Gamma^\Omega ?, ?')$.

(c13) **Lemma** (*cf.* **[AJS], Lemma 7.5)** *Assume A is a B-algebra. Let $\lambda, \mu \in X$ in the closure of an alcove for W' and $\Omega = W' \cdot_k \lambda$, $\Gamma = W' \cdot_k \mu$. Then $T_\Omega^\Gamma Z_A(\lambda)$ has a Z-filtration with factors $Z_A(w \cdot_k \mu)$, $w \in$*

$C_{W'}(\lambda)/C_{W'}(\lambda) \cap C_{W'}(\mu)$, *each occuring exactly once.*

d° Deformations

(d1) Recall that we are after a characteristic free description of $\mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$. By (c6)(3) and (c8) we may replace $\hat{Z}_k(?)$ of §a by $Z_k(?)$ in $\mathcal{C}_k$. We will study $\mathcal{C}_k$ by deformations.

Let $\mathfrak{m} \in \mathrm{Spec}(U^0)$ be the annihilator of the trivial 1-dimensional representation:

$$\mathfrak{m} = \begin{cases} (H_\alpha \mid \alpha \in \Sigma) & \text{in Case 1} \\ (K_\alpha - 1 \mid \alpha \in \Sigma) & \text{in Case 2.} \end{cases}$$

Let $\hat{A} = \hat{U}^0$ be the completion of U^0 at $\mathfrak{m}$, denoted by $A(k)$ in [AJS]. Then $\hat{A}$ is a noetherian complete local domain, flat over U^0, with maximal ideal $\mathfrak{m}\hat{A}$ and the residue field k. One may regard Spec$\hat{A}$ as a formal neighbourhood of $\mathfrak{m}$ in $\mathrm{Spec}(U^0)$ (cf. [K], pp. 315-316). Note (cf. [B1], Exercise III.2.27(a)) that $\hat{A}$ is also the completion of B in the $\mathfrak{m}B$-adic topology.

(d2) **Lemma** (*cf.* **[AJS], Lemma 14.2)** *If A is a noetherian complete local domain, the Krull-Schmidt theorem holds in $\mathcal{C}_A$.*

(d3) Let $\mathcal{P}_A$ be the full subcategory of $\mathcal{C}_A$ consisting of all its projectives.

Theorem (*cf.* **[AJS], Proposition 3.3/Theorem 4.19))** (i) *If $P, Q \in \mathcal{P}_A$, then $\mathcal{C}_A(P, Q)$ is projective of finite type in* $\mathbf{Mod}_A$. *If A' is a noetherian domain over A, then in* $\mathbf{Mod}_{A'}$

$$\mathcal{C}_A(P, Q) \otimes_A A' \simeq \mathcal{C}_{A'}(P \otimes_A A', Q \otimes_A A').$$

(ii) *If A is local with the residue field F, then $? \otimes_A F : \mathcal{P}_A \to \mathcal{P}_F$ gives a bijection between the isomorphism classes.*

(d4) In particular, $Q^{[i]}(k) \in \mathcal{P}_k$ lifts to

$$Q^{[i]}(\hat{A}) = \Theta_{i_1} \circ \ldots \circ \Theta_{i_r} \circ T_{\Delta_i}^{\Omega_0} Z_{\hat{A}}(\nu_i)$$

of $\mathcal{P}_{\hat{A}}$, where $\nu_i = (p-1)\rho + p(w_i \cdot_k 0)^1$, $\Omega_0 = W_a \cdot_k 0$, $\Delta_i = W_a \cdot_k \nu_i$, $\Theta_{i_j} = \Theta_{s_{i_j}} = T_{\Gamma_{i_j}}^{\Omega_0} \circ T_{\Omega_0}^{\Gamma_{i_j}}$ with $\Gamma_{i_j} = W_a \cdot_k \mu_{s_{i_j}}$ (cf. (a11/12)). We will see the projectivity of $Z_{\hat{A}}(\nu_i)$ in (d14). Hence we want now a characteristic free description of $\mathcal{C}_{\hat{A}}(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$.

Let $\hat{A}^{\emptyset} = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+]$ and $\hat{A}^{\beta} = \hat{A}[\frac{1}{H_\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$, $\beta \in R^+$, with $H_\alpha = [K_\alpha : 0]$ in Case 2. Note that $\hat{A}^{\emptyset}$ and all $\hat{A}^{\beta}$ are naturally B-algebras. Put for simplicity $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}}$, $\mathcal{C}_\emptyset = \mathcal{C}_{\hat{A}^\emptyset}$, $\mathcal{C}_\beta = \mathcal{C}_{\hat{A}^\beta}$, and $M^\emptyset = M \otimes_{\hat{A}} \hat{A}^\emptyset$, $M^\beta = M \otimes_{\hat{A}} \hat{A}^\beta$ if $M \in \mathcal{C}_\wedge$. Let also $Z_\wedge(\lambda) = Z_{\hat{A}}(\lambda)$, $Z_\emptyset(\lambda) = Z_{\hat{A}^\emptyset}(\lambda) \simeq Z_\wedge(\lambda)^\emptyset$, and $Z_\beta(\lambda) = Z_{\hat{A}^\beta}(\lambda) \simeq Z_\wedge(\lambda)^\beta$ for each $\lambda \in X$.

(d5) By our standing hypothesis that $p = \operatorname{ch} k \geq h$ in Case 1, we have

Lemma (*cf.* **[AJS], Lemma 9.1)** $\hat{A} = \bigcap_{\beta \in R^+} \hat{A}^\emptyset$.

(d6) Let $P, Q \in \mathcal{P}_{\hat{A}}$. As Q is $\hat{A}$-flat, one may regard $Q \leq Q^\beta \leq Q^\emptyset$ for each $\beta \in R^+$. Then

$$\begin{aligned} &\mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset)\\ &\simeq \mathcal{C}_\wedge(P, Q^\emptyset) \simeq \mathcal{C}_\wedge(P,Q) \otimes_{\hat{A}} \hat{A}^\emptyset \quad \text{as } \hat{A}^\emptyset \text{ is flat over } \hat{A} \text{ (cf. [AJS], Lemma 3.2)}\\ &\geq \mathcal{C}_\wedge(P, Q^\beta) \simeq \mathcal{C}_\wedge(P,Q) \otimes_{\hat{A}} \hat{A}^\beta \quad \text{as } \hat{A}^\beta \text{ is flat over } \hat{A}\\ &\geq \mathcal{C}_\wedge(P,Q). \end{aligned}$$

As $\mathcal{C}_\wedge(P,Q)$ is $\hat{A}$-flat, one gets from (d5)

(1) $$\mathcal{C}_\wedge(P,Q) = \cap_{\beta \in R^+} \mathcal{C}_\beta(P^\beta, Q^\beta) \quad \text{inside } \mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset).$$

(d7) Now $\mathcal{C}_\emptyset$ has a simple structure. If $\operatorname{Frac}(\hat{A})$ is the fractional field of $\hat{A}$, $\mathcal{C}_{\operatorname{Frac}(\hat{A})}$ is semisimple. To explain that, let us resume the general set-up of $\mathcal{C}_A$.

Let $w \in W$. Twist $\pi : U^0 \to A$ by T_w^{-1} to define another U^0-algebra $A[w]$ with the structure homomorphism $\pi \circ T_w^{-1}$. If $M \in \mathcal{C}_A$, define $M[w] \in \mathcal{C}_{A[w]}$ to be the A-module M with each $u \in U$ acting by $T_w^{-1}(u)$ and the gradation given by $M[w]_\nu = M_{w^{-1}\nu}$. Then the functor $M \mapsto M[w]$ is an equivalence of categories from $\mathcal{C}_A$ to $\mathcal{C}_{A[w]}$. If M is A-projective, then

$$\operatorname{ch}(M[w]) = w(\operatorname{ch} M).$$

Working with the positive system $w(R^+)$ instead of R^+, define

$$Z_A^w(\lambda) = U \otimes_{U^0 T_w(U^+)} A^\lambda \quad \in \mathcal{C}_A \quad \forall \lambda \in X.$$

Then (cf. [AJS], (4.4)(2)) for each $x \in W$

(1) $$Z_A^x(\lambda)[w] \simeq Z_{A[w]}^{wx}(w\lambda) \quad \text{in } \mathcal{C}_{A[w]},$$

and (cf. [AJS], Lemma 4.10)

(2) $$Z_A(\lambda)^\tau \simeq Z_A^{w_0}(\lambda - 2(p-1)\rho).$$

In particular (cf. [J], (9.2)),

(3) $$Z_k^{w_0}(\lambda) \simeq \hat{Z}_k(\lambda + 2(p-1)\rho) \text{ of §a.}$$

(d8) Fix $\alpha \in \Sigma$ and put $s = s_\alpha \in \Sigma_a$. Let $U(-\alpha)$ be the subalgebra of U generated by $E_{-\alpha}$, and let $P(\alpha) = U(-\alpha)U^0U^+ \leq U$. Define a full subcategory $\mathcal{C}_A^\alpha$ of $(P(\alpha) \otimes_k A)$**Mod** just like $\mathcal{C}_A$. Define likewise $Z_A^\alpha(\lambda) = P(\alpha) \otimes_{U^0U^+} A^\lambda$ and $(Z_A^\alpha)^s(\lambda) = P(\alpha) \otimes_{U^0T_s(U^+)} A^\lambda \in \mathcal{C}_A^\alpha$ for each $\lambda \in X$. As the multiplication $U(-\alpha) \otimes_k U^0U^+ \to P(\alpha)$ is bijective,

(1) *$Z_A^\alpha(\lambda)$ (resp. $(Z_A^\alpha)^s(\lambda)$) is A-free of basis*

$$v_i = E_{-\alpha}^{(i)} \otimes 1 \ (\textit{resp. } v_i' = E_\alpha^{(i)} \otimes 1\ ),$$

where $E_{-\alpha}^{(i)} = \frac{E_{-\alpha}^i}{i!} \otimes 1$ (resp. $E_\alpha^{(i)} = \frac{E_\alpha^i}{[i]_{d_\alpha}^!} \otimes 1$) in Case 1 (resp. Case 2).

One has (cf. [AJS], (5.4))

(2) $$P(\alpha) = U(-\alpha)U^0U(\alpha) \oplus Q(\alpha) \text{ with } Q(\alpha) = \coprod_{\nu \notin \mathbb{Z}\alpha} P(\alpha)_\nu,$$

(3) *T_s stabilizes all $P(\alpha)$, $U(-\alpha)U^0U(\alpha)$ and $Q(\alpha)$,*

and that

(4) *$Q(\alpha)$ annihilates both $Z_A^\alpha(\lambda)$ and $(Z_A^\alpha)^s(\lambda)$.*

Hence one can describe the $P(\alpha)$-action on both $Z_A^\alpha(\lambda)$ and $(Z_A^\alpha)^s(\lambda)$ explicitly (cf. [AJS], (5.5)). In particular, there is unique

(5) $$\phi_\alpha \in \mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha)) \quad \text{such that} \quad v_0 \mapsto v_{p-1}'.$$

Then ϕ_α forms an A-basis of $\mathcal{C}_A^\alpha(Z_A^\alpha(\lambda), (Z_A^\alpha)^s(\lambda - (p-1)\alpha))$ and one has (cf. [AJS], (5.6))

(6) $$\phi_\alpha(v_i) = \begin{cases} (-1)^i v_{p-1-i}' \binom{\pi(H_\alpha) + \langle \lambda, \alpha^\vee \rangle}{i} & \text{in Case 1} \\ (-1)^i v_{p-1-i}' \pi\left(\left[\begin{smallmatrix} K_\alpha; \langle \lambda, \alpha^\vee \rangle \\ i \end{smallmatrix}\right]\right) & \text{in Case 2.} \end{cases}$$

It follows that

(7) $\qquad$ *if $\alpha \notin R_\pi$, then ϕ_α is bijective.*

If $\alpha \in R_\pi$, let $n_\alpha(\lambda) \in [1, p]$ such that $\pi(H_\alpha) + \langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha(\lambda) \cdot 1$ in Case 1 (resp. $\pi(K_\alpha)^2 \zeta^{2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} = \zeta^{2d_\alpha n_\alpha(\lambda)}$ in Case 2). One has (cf. [AJS], (5.9)) that

(8) $\qquad$ *if $n_\alpha(\lambda) = p$, then ϕ_α is still bijective.*

(d9) If $w \in W$, from ϕ_α over $A[w^{-1}]$ one gets

(1) $$\phi \in \mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$$

such that the diagram

$$\begin{array}{ccc}
Z_A^w(w\lambda) & \xrightarrow{\phi} & Z_A^{ws}(w\lambda - (p-1)w\alpha) \\
\wr\downarrow & & \downarrow\wr \\
Z_{A[w^{-1}]}(\lambda)[w] & & Z^s_{A[w^{-1}]}(\lambda - (p-1)\alpha)[w] \\
\wr\downarrow & & \downarrow\wr \\
(U \otimes_{P(\alpha)} Z^\alpha_{A[w^{-1}]}(\lambda))[w] & \xrightarrow[(U \otimes_{P(\alpha)} \phi_\alpha)[w]]{} & \{U \otimes_{P(\alpha)} (Z^\alpha_{A[w^{-1}]})^s(\lambda - (p-1)\alpha)\}[w].
\end{array}$$

commutes. As ϕ sends the standard generator of $Z_A^w(w\lambda)$ to an A-basis element of $Z_A^{ws}(w\lambda - (p-1)w\alpha)_{w\lambda}$,

(2) $\qquad$ *ϕ is an A-basis of $\mathcal{C}_A(Z_A^w(w\lambda), Z_A^{ws}(w\lambda - (p-1)w\alpha))$.*

One may compare the construction of ϕ with the intertwining homomorphism

$$H^i(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(s_\alpha \cdot_k \nu)) \longrightarrow H^{i-1}(\mathfrak{G}_k/\mathfrak{B}_k, \mathcal{L}(\nu))$$

for $\alpha \in \Sigma$ and $\nu \in X$ with $\langle \nu + \rho, \alpha^\vee \rangle \geq 0$ in $\mathfrak{G}_k$**Mod** [J], (II.5/6).

Choose a reduced expression $w_0 = s_1 s_2 \dots s_N$ of w_0. If $w_i = s_1 s_2 \dots s_{i-1}$, $1 \leq i \leq N+1$, with $w_1 = 1$, and if $\lambda\langle w_i \rangle = \lambda + (p-1)(w_i\rho - \rho)$, one gets an A-basis ϕ_i of $\mathcal{C}_A(Z_A^{w_i}(\lambda\langle w_i \rangle), Z_A^{w_{i+1}}(\lambda\langle w_{i+1} \rangle))$ like ϕ of (1). One gets from (d8)(7)

(3) $\qquad$ *if $R_\pi = \emptyset$, then $Z_A(\lambda) \simeq Z_A^w(\lambda\langle w \rangle)$ $\forall w \in W$,*

i.e., the "Borel-Weil-Bott" theorem holds in $\mathcal{C}_A$ if $R_\pi = \emptyset$.

(d10) Let $\Phi = \phi_N \circ \ldots \circ \phi_1 \in \mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$.

Lemma (*cf.* **[AJS], Lemma 5.13)** *The morphism Φ is nonzero and forms an A-basis of $\mathcal{C}_A(Z_A(\lambda), Z_A^{w_0}(\lambda - 2(p-1)\rho))$.*

(d11) **Lemma** (*cf.* **[AJS], Lemma 4.9)** *If $A = F$ is a field, then*

$$L_F(\lambda) = \operatorname{im}\Phi = \operatorname{soc}_{\mathcal{C}_F} Z_F^{w_0}(\lambda - 2(p-1)\rho).$$

(d12) For each $\beta \in R_\pi$ define $n_\beta \in [1, p]$ as in (d8). One now obtains

Lemma (*cf.* **[AJS], Lemma 6.3)** *Assume $A = F$ is a field with the structure homomorphism π.*

(i) *If $\lambda \in X$ with $n_\beta(\lambda) = p$ for each $\beta \in R_\pi^+$, then $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$ in $\mathcal{C}_F$.*

(ii) *If $R_\pi^+ = \phi$, then $Z_F(\lambda) \simeq L_F(\lambda) \simeq Q_F(\lambda)$ for each $\lambda \in X$, i.e., $\mathcal{C}_F$ is a semisimple category.*

Proof. As ϕ is bijective, $L_F(\lambda) \simeq Z_F(\lambda)$ for each $\lambda \in X$ by (d11). If $\mu \in X$, then (cf. [AJS], Proposotion 4.6)

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{C}_F}(L_F(\lambda), L_F(\mu)) &\simeq \operatorname{Ext}^1_{\mathcal{C}_F}(L_F(\mu), L_F(\lambda)) \quad \text{using the duality } \tau \\ &\simeq \mathcal{C}_F(\operatorname{rad}_{\mathcal{C}_F} Z_F(\lambda), L_F(\mu)) \quad \text{if } \mu \not> \lambda \\ &= 0.\end{aligned}$$

Hence $L_F(\lambda)$ is both projective and injective in $\mathcal{C}_F$.

(d13) **Proposition** (*cf.* **[AJS], Corollary 3.5)** *Let $M \in \mathcal{C}_A$ with a Z-filtration. Then M is projective in $\mathcal{C}_A$ iff $M \otimes_A (A/\mathfrak{m})$ is projective in $\mathcal{C}_{A/\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of A.*

(d14) We conclude from (d12/13) that for each $\lambda \in X$

(1) $Z_A((p-1)\rho + p\lambda)$ *is projective in* $\mathcal{C}_A$,

(2) *the block of* λ *over* $\hat{A}^\emptyset$ *is a singleton* $\{\lambda\}$,

and that

(3) $\quad Z_\emptyset(\lambda)$ *is a progenerator of* $\mathcal{C}_\emptyset(\{\lambda\})$.

Back to $P, Q \in \mathcal{P}_{\hat{A}}$, one can write $P^\emptyset = \coprod_{\lambda\in X} Z_\emptyset(\lambda)^{p_\lambda}$ and $Q^\emptyset = \coprod_{\lambda\in X} Z_\emptyset(\lambda)^{q_\lambda}$ with $p_\lambda, q_\lambda \in \mathbb{N}$. Then

$$\mathcal{C}_\emptyset(P^\emptyset, Q^\emptyset) \simeq (\hat{A}^\emptyset)^{\sum_{\lambda\in X} p_\lambda q_\lambda}.$$

In particular, if $P^\emptyset = Q^{[i]}(\hat{A}^\emptyset) = Q^{[i]}(\hat{A})^\emptyset$ and $Q^\emptyset = Q^{[j]}(\hat{A}^\emptyset) = Q^{[j]}(\hat{A})^\emptyset$, p_λ (resp. q_λ) are determined independent of k, hence

(4) $\quad \mathcal{C}_\emptyset(Q^{[i]}(\hat{A}^\emptyset), Q^{[j]}(\hat{A}^\emptyset))$ *is described independent of* k.

(d15) More generally,

Lemma (*cf.* **[AJS], E.4)** *Let* $\lambda \in X$. *For each* $M, N \in \mathcal{C}_\emptyset(\{\lambda\})$ *one has an isomorphism of* $\hat{A}^\emptyset$*- modules*

$$\mathcal{C}_\emptyset(M,N) \longrightarrow \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N)) \quad \text{via} \quad f \longmapsto f \circ ?.$$

Proof. Put $P = Z_\emptyset(\lambda)$ and $M(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M)$, likewise $N(\lambda)$. Consider first the case $M = P^m$ and $N = P^n$ for $m, n \in \mathbb{N}^+$. If $\pi_s : P^m \to P$ (resp. $i_r : P \to P^n$) is the projection onto the s-th (resp. injection from the r-th) component, one has a commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\
{\scriptstyle \mathcal{C}_\emptyset(P^m, i_r)}\uparrow & & \uparrow{\scriptstyle \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), \mathcal{C}_\emptyset(P, i_r))} \\
\mathcal{C}_\emptyset(P^m, P) & & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P(\lambda)) \\
{\scriptstyle \mathcal{C}_\emptyset(\pi_s, P)}\uparrow & & \uparrow{\scriptstyle \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(P, \pi_s), P(\lambda))} \\
\mathcal{C}_\emptyset(P, P) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P(\lambda), P(\lambda)) \\
f & \longmapsto & f \circ ?
\end{array}$$

with the bottom horizontal map bijective as $P(\lambda) = \mathcal{C}_\emptyset(P, P) \simeq \hat{A}^\emptyset$. Hence

(1) $\quad$ *the assertion holds with* $M = P^m$ *and* $N = P^n$.

If N is arbitrary, as P is a generator of $\mathcal{C}_\emptyset(\{\lambda\})$, N admits a finite presentation in $\mathcal{C}_\emptyset(\{\lambda\})$: $P^{n'} \longrightarrow P^n \to N \to 0$ exact. Then $P^{n'}(\lambda) \to P^n(\lambda) \to N(\lambda) \to 0$ remains exact as $?(\lambda)$ is exact, hence one gets a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_\emptyset(P^m, P^{n'}) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^{n'}(\lambda)) \\ \downarrow & & \downarrow \\ \mathcal{C}_\emptyset(P^m, P^n) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), P^n(\lambda)) \\ \downarrow & & \downarrow \\ \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

As P^m (resp. $P^m(\lambda) \simeq (\hat{A}^\emptyset)^m$) is a projective of $\mathcal{C}_\emptyset$ (resp. $\mathbf{Mod}_{\hat{A}^\emptyset}$), the left and the right vertical sequences are both exact. By (1) the top and the middle horizontal maps are bijective, hence also the bottom by the 5-lemma, i.e.,

(2) *the assertion holds if $M = P^m$.*

Finally, write $P^{m'} \to P^m \to M \to 0$ exact in $\mathcal{C}_\emptyset$. One then gets a commutative diagram of exact columns

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{C}_\emptyset(M, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(M(\lambda), N(\lambda)) \\ \downarrow & & \downarrow \\ \mathcal{C}_\emptyset(P^m, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^m(\lambda), N(\lambda)) \\ \downarrow & & \downarrow \\ \mathcal{C}_\emptyset(P^{m'}, N) & \longrightarrow & \mathbf{Mod}_{\hat{A}^\emptyset}(P^{m'}(\lambda), N(\lambda)). \end{array}$$

As the middle and the bottom horizontal maps are bijective by (2), the top horizontal map is bijective by the 5-lemma again, hence the assertion.

(d16) More detailed examination of the ϕ_i and Φ shows that $\mathcal{C}_\beta$ behaves like $\mathfrak{sl}_2$- category. If $\lambda \in X$, $\beta \in R^+$, and if $n \in \mathbb{N}$ minimal with $\langle \lambda+\rho, \beta^\vee \rangle \equiv -n \mod p$, put $\beta \uparrow \lambda = \lambda + n\beta$.

Theorem (*cf.* **[AJS], Proposition 8.6/Corollary 8.7)** *Let* $\lambda \in X$ *and* $\beta \in R^+$.

(i) *If* $\beta \uparrow \lambda = \lambda$, *then* $Z_\beta(\lambda)$ *is a projective of* $\mathcal{C}_\beta$.

(ii) *Suppose* $\beta \uparrow \lambda > \lambda$. *Then*

$$Ext^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda)) \simeq \hat{A}^\beta / H_\beta \hat{A}^\beta \quad \textit{in } \mathbf{Mod}_{\hat{A}^\beta}.$$

Given a short exact sequence $0 \to Z_\beta(\beta \uparrow \lambda) \to Q \to Z_\beta(\lambda) \to 0$ *in* $\mathcal{C}_\beta$, Q *is projective in* $\mathcal{C}_\beta$ *iff the sequence generates* $Ext^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ *over* $\hat{A}^\beta$.

e° Combinatorial categories

(e1) In order to glue together all the $\mathfrak{sl}_2$-categories $\mathcal{C}_\beta$ to recover $\mathcal{C}_\wedge = \mathcal{C}_{\hat{A}^\emptyset}$, we introduce a combinatorial category $\mathcal{K}(\Omega)$ for each W_a-orbit Ω. An object of $\mathcal{K}(\Omega)$ is a family $(\mathcal{M}(\lambda))_{\lambda \in \Omega}$ of $\hat{A}^\emptyset$-modules of finite type, only finitely many nonzero members, together with $\hat{A}^\beta$-submodules $\mathcal{M}(\lambda, \beta)$ of finite type for each $\lambda \in \Omega$ and $\beta \in R^+$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $\mathcal{M}(\lambda)$ if $\beta \uparrow \lambda = \lambda$). A morphism of $\mathcal{K}(\Omega)$ is $(\psi_\lambda)_{\lambda \in \Omega} \in \prod_{\lambda \in \Omega} \mathcal{C}_\emptyset(\mathcal{M}(\lambda), \mathcal{M}'(\lambda))$ such that for each $\lambda \in \Omega$ and $\beta \in R^+$

$$(\psi_\lambda \oplus \psi_{\beta\uparrow\lambda})\mathcal{M}(\lambda, \beta) \subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda > \lambda$$
$$\psi_\lambda \mathcal{M}(\lambda, \beta) \subseteq \mathcal{M}'(\lambda, \beta) \quad \text{if } \beta \uparrow \lambda = \lambda.$$

(e2) We define a functor $\mathcal{V}_\Omega : \mathcal{C}_\wedge(\Omega) \to \mathcal{K}(\Omega)$, that depends on the choice of $e^\beta(\lambda) \in \mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$ for $\lambda \in \Omega$ and $\beta \in R^+$ with $\beta \uparrow \lambda > \lambda$. If $M \in \mathcal{C}_\wedge(\Omega)$, set $(\mathcal{V}_\Omega M)(\lambda) = \mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset)$, and if $\beta \uparrow \lambda = \lambda$, let $(\mathcal{V}_\Omega M)(\lambda, \beta) = \mathcal{C}_\beta(Z_\beta(\lambda), M^\beta)$. If $\beta \uparrow \lambda > \lambda$, represent $e^\beta(\lambda)$ by a short exact sequence in $\mathcal{C}_\beta$

$$0 \longrightarrow Z_\beta(\beta \uparrow \lambda) \longrightarrow Q^\beta(\lambda) \longrightarrow Z_\beta(\lambda) \longrightarrow 0.$$

Tensoring with $\hat{A}^\emptyset$, the sequence splits uniquely to yield an isomorphism $Q^\beta(\lambda)^\emptyset \simeq Z_\emptyset(\beta \uparrow \lambda) \oplus Z_\emptyset(\lambda)$. We set $(\mathcal{V}_\Omega M)(\lambda, \beta)$ to be the image of the composite of the natural maps

$$\begin{aligned}
&\mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) \longrightarrow \mathcal{C}_\emptyset(Q^\beta(\lambda)^\emptyset, M^\emptyset) \\
&\qquad \xrightarrow{\sim} \mathcal{C}_\emptyset(Z_\emptyset(\lambda) \oplus Z_\emptyset(\beta \uparrow \lambda), M^\emptyset) \xrightarrow{\sim} (\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{C}_\Omega M)(\beta \uparrow \lambda).
\end{aligned}$$

(e3) Let $\mathcal{FC}_\wedge(\Omega)$ be the full subcategory of $\mathcal{C}_\wedge(\Omega)$ consisting of all $\hat{A}$-flat objects.

Theorem (*cf.* **[AJS], Proposition 9.4**) *Choose all $e^\beta(\lambda)$ as generators of $Ext^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$. Then $\mathcal{V}_\Omega : \mathcal{FC}_\wedge(\Omega) \to \mathcal{K}(\Omega)$ is fully faithful.*

Proof. Let $M, N \in \mathcal{FC}_\wedge(\Omega)$. One must show

$$(\mathcal{V}_\Omega)_{M,N} : \mathcal{C}_\wedge(M, N) \longrightarrow \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$$

is an isomorphism. By (c10) and (d14)(2)

$$\begin{aligned}
\mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) &= \mathcal{C}_\emptyset\Big(\coprod_{\lambda \in \Omega} (M^\emptyset)_{\{\lambda\}}, \coprod_{\lambda \in \Omega} (N^\emptyset)_{\{\lambda\}}\Big) \\
&\simeq \prod_\lambda \mathcal{C}_\emptyset\Big((M^\emptyset)_{\{\lambda\}}, \coprod_{\mu \in \Omega} (N^\emptyset)_{\{\mu\}}\Big) \simeq \prod_\lambda \mathcal{C}_\emptyset((M^\emptyset)_{\{\lambda\}}, (N^\emptyset)_{\{\lambda\}}) \\
&\simeq \prod_\lambda \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), (M^\emptyset)_{\{\lambda\}}), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), (N^\emptyset)_{\{\lambda\}})) \quad \text{by (d15)} \\
&\simeq \prod_\lambda \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)).
\end{aligned}$$

Hence $(\mathcal{V}_\Omega)_{M,N}$ is injective:

$$\text{(3)} \quad \begin{array}{ccc}
\mathcal{C}_\wedge(M, N) & \xrightarrow{(\mathcal{V}_\Omega)_{M,N}} & \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N) \\
\wedge| & & \wedge| \\
\mathcal{C}_\wedge(M, N^\emptyset) & \curvearrowright & \prod_{\lambda \in \Omega} \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)) \\
\wr| & & \| \\
\mathcal{C}_\emptyset(M^\emptyset, N^\emptyset) & \xrightarrow{\sim} & \prod_\lambda \mathbf{Mod}_{\hat{A}^\emptyset}(\mathcal{C}_\emptyset(Z_\emptyset(\lambda), M^\emptyset), \mathcal{C}_\emptyset(Z_\emptyset(\lambda), N^\emptyset)) \\
h & \longmapsto & h \circ ?.
\end{array}$$

To see the surjectivity, let $\psi \in \mathcal{K}(\Omega)(\mathcal{V}_\Omega M, \mathcal{V}_\Omega N)$. By (1) there is $h \in \mathcal{C}_\emptyset(M^\emptyset, N^\emptyset)$ such that for each $\lambda \in \Omega$

$$\psi_\lambda = h \circ ? \quad \text{in } \mathbf{Mod}_{\hat{A}^\emptyset}((\mathcal{V}_\Omega M)(\lambda), (\mathcal{V}_\Omega N)(\lambda)).$$

Let $\beta \in R^+$. For $\lambda \in \Omega$ let $Q^\beta(\lambda)$ be the middle term of the short exact sequence representing $e^\beta(\lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $Z^\beta(\lambda)$ if $\beta \uparrow \lambda = \lambda$). If $\beta \uparrow \lambda > \lambda$, one gets a commutative diagram

$$\begin{array}{ccc}
\mathcal{C}_\beta(Q^\beta(\lambda), M^\beta) & \dashrightarrow & \mathcal{C}_\beta(Q^\beta(\lambda), N^\beta) \\
\wedge| & & \wedge| \\
\mathcal{C}_\beta(Q^\beta(\lambda), M^\emptyset) & \xrightarrow[u \longmapsto h \circ u]{} & \mathcal{C}_\beta(Q^\beta(\lambda), N^\emptyset) \\
\wr| & & |\wr \\
\mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, M^\emptyset) & & \mathcal{C}_\emptyset(Q^\beta(\lambda) \otimes_{\hat{A}^\beta} \hat{A}^\emptyset, N^\emptyset) \\
\wr| & & |\wr \\
(\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{V}_\Omega M)(\beta \uparrow \lambda) & \xrightarrow[\psi_\lambda \oplus \psi_{\beta\uparrow\lambda}]{} & (\mathcal{V}_\Omega N)(\lambda) \oplus (\mathcal{V}_\Omega N)(\beta \uparrow \lambda),
\end{array}$$

hence each $h \circ u$, $u \in \mathcal{C}_\beta(Q^\beta(\lambda), M^\beta)$ factors through N^β. Likewise if $\beta \uparrow \lambda = \lambda$.

On the other hand, one can write by definition $\coprod Z_\beta(\lambda_i) \twoheadrightarrow M^\beta$ with the coproduct running over some $\lambda_i \in \Omega$, hence $\coprod Q^\beta(\lambda_i) \twoheadrightarrow M^\beta$, i.e., there are $u_i \in \mathcal{C}_\beta(Q^\beta(\lambda_i), M^\beta)$ such that $M^\beta = \sum \operatorname{im}(u_i)$. Then

$$h(M^\beta) = h \sum \operatorname{im}(u_i) = \sum \operatorname{im}(h \circ u_i) \subseteq N^\beta.$$

Hence $h(M) = h(\bigcap_{\beta \in R^+} M^\beta) \leq \bigcap_{\beta \in R^+} h(M^\beta) \leq \cap_\beta N^\beta = N$, the last equality following from (d5). Consequently, h arises from $\mathcal{C}_\wedge(M, N)$ with $(\mathcal{V}_\Omega)_{M,N}(h) = \psi$, as desired.

(e4) To get a characteristic free description of $\mathcal{C}_\wedge(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A}))$, it is now enough to find a characteristic free description of

$$\mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A})))$$

with $\Omega_0 = W_a \cdot_k 0$. Define $\mathcal{Z}_{\nu_i}(\hat{A}) \in \mathcal{K}(\Delta_i)$ by setting for each $\mu \in \Delta_i =$

$W_a \cdot_k \nu_i$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu) = \begin{cases} \hat{A}^{\emptyset} & \text{if } \mu = \nu_i \\ 0 & \text{otherwise,} \end{cases}$$

and for each $\beta \in R^+$

$$\mathcal{Z}_{\nu_i}(\hat{A})(\mu, \beta) = \begin{cases} \hat{A}^{\beta} & \text{if } \mu = \nu_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) $$\mathcal{V}_{\Delta_i}(Z_\wedge(\nu_i)) \simeq \mathcal{Z}_{\nu_i}(\hat{A}).$$

(e5) We want next to construct a translation functor $\mathcal{T} : \mathcal{K}(\Omega) \to \mathcal{K}(\Gamma)$ for W_a-orbits Ω and Γ such that $\mathcal{V}_\Gamma \circ T_\Omega^\Gamma \simeq \mathcal{T} \circ \mathcal{V}_\Omega$. We will consider only the case that for each $\lambda \in \Omega$ there is a unique $\mu \in \Gamma$ that lies in the closure of the facet of λ, which we will denote by λ_Γ.

Put $T = T_\Omega^\Gamma$ and $T' = T_\Gamma^\Omega$. For each $\lambda \in \Omega$ choose an isomorphism $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$. Let $\beta \in R^+$ with $\beta \uparrow \lambda > \lambda$. Define

$$t[f_\lambda, f_{\beta\uparrow\lambda}] : \mathrm{Ext}^1_{\mathcal{C}_\wedge}(Z_\wedge(\lambda), Z_\wedge(\beta \uparrow \lambda)) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}_\wedge}(Z_\wedge(\lambda_\Gamma), Z_\wedge((\beta \uparrow \lambda)_\Gamma))$$

by sending each short exact sequence $0 \to Z_\wedge(\beta \uparrow \lambda) \to Q \to Z_\wedge(\lambda) \to 0$ to the bottom horizontal exact sequence of the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TZ_\wedge(\beta \uparrow \lambda) & \longrightarrow & TQ & \longrightarrow & TZ_\wedge(\lambda) & \longrightarrow & 0 \\ & & {\scriptstyle f_{\beta\uparrow\lambda}}\uparrow & & \| & & \uparrow{\scriptstyle f_\lambda} & & \\ 0 & \longrightarrow & Z_\wedge((\beta \uparrow \lambda)_\Gamma) & \longrightarrow & TQ & \longrightarrow & Z_\wedge(\lambda_\Gamma) & \longrightarrow & 0, \end{array}$$

where the top horizontal sequence is the one obtained by hitting T on the first exact sequence.

Assume first $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$, so that $(\beta \uparrow \lambda)_\Gamma = \beta \uparrow \lambda_\Gamma$. Let $W_\beta = \langle s_\beta \rangle \ltimes \mathbb{Z}\beta \leq W_a$ and let $T_0 = T_{W_\beta \cdot_k \lambda}^{W_\beta \cdot_k \lambda_\Gamma}$, $T_0' = T_{W_\beta \cdot_k \lambda_\Gamma}^{W_\beta \cdot_k \lambda}$. As $\beta \uparrow \lambda_\Gamma > \lambda_\Gamma$,

$$T_0' Z_\beta(\lambda_\Gamma) \simeq Z_\beta(\lambda) \quad \text{and} \quad T_0' Z_\beta(\beta \uparrow \lambda_\Gamma) \simeq Z_\beta(\beta \uparrow \lambda).$$

Let $f_\lambda' = \mathrm{adj}_1^{-1}(f_\lambda^{-1}) \in \mathcal{C}_\beta(Z_\beta(\lambda), T_0' Z_\beta(\lambda_\Gamma))^\times$ using the adjunction adj_1 with respect to T_0 and T_0', and likewise $f_{\beta\uparrow\lambda}'$. Define

$$t[f_\lambda', f_{\beta\uparrow\lambda}'] : \mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda_\Gamma), Z_\beta((\beta \uparrow \lambda)_\Gamma)) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta \uparrow \lambda))$$

just like $t[f_\lambda, f_{\beta\uparrow\lambda}]$ replacing T by T_0'. Then (cf. [AJS], (10.6)(1))

$$(1) \qquad t[f_\lambda, f_{\beta\uparrow\lambda}] \circ t[f'_\lambda, f'_{\beta\uparrow\lambda}] = \mathrm{id}_{\mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda_\Gamma), Z_\beta((\beta\uparrow\lambda)_\Gamma))}.$$

Suppose we have chosen an $\hat{A}^\beta$-generator $e^\beta(\lambda)$ (resp. $e^\beta(\lambda_\Gamma)$) of

$$\mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta\uparrow\lambda)) \quad (\text{resp. } \mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda_\Gamma), Z_\beta((\beta\uparrow\lambda)_\Gamma))).$$

Then one can write with some a^β_λ and $b^\beta_\lambda \in \hat{A}^\beta$

$$t[f_\lambda, f_{\beta\uparrow\lambda}]e^\beta(\lambda) = a^\beta_\lambda e^\beta(\lambda_\Gamma) \quad \text{and} \quad t[f'_\lambda, f'_{\beta\uparrow\lambda}]e^\beta(\lambda_\Gamma) = b^\beta_\lambda e^\beta(\lambda).$$

By (1) and (d16)

$$(2) \qquad a^\beta_\lambda b^\beta_\lambda \in 1 + H_\beta \hat{A}^\beta \quad \text{in } \hat{A}^\beta/H_\beta\hat{A}^\beta.$$

Assume next $\beta\uparrow\lambda_\Gamma = \lambda_\Gamma = (\beta\uparrow\lambda)_\Gamma$. Define an isomorphism

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}] \in \mathbf{Mod}_{\hat{A}^\beta}(\mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta\uparrow\lambda)), \hat{A}^\beta H_\beta^{-1}/\hat{A}^\beta)^\times$$

as follows (cf. [AJS], Proposition 8.14). Let $e \in \mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta\uparrow\lambda))$ represented by a short exact sequence

$$0 \longrightarrow Z_\beta(\beta\uparrow\lambda) \xrightarrow{i} Q \xrightarrow{j} Z_\beta(\lambda) \longrightarrow 0.$$

As $eH_\beta = 0$ in $\mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\lambda), Z_\beta(\beta\uparrow\lambda))$ by (d16), there is a unique $j' \in \mathcal{C}_\beta(Z_\beta(\lambda), Q)$ such that $j\circ j' = H_\beta \mathrm{id}_{Z_\beta(\lambda)}$ (cf. [B2], (X.119) Proposition 4/ (X.120) Corollary 3(ii)): one has a commutative diagram of short exact sequences with the top sequence representing eH_β

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_\beta(\beta\uparrow\lambda) & \longrightarrow & Z_\beta(\beta\uparrow\lambda)\oplus Z_\beta(\lambda) & \longrightarrow & Z_\beta(\lambda) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow H_\beta & & \\
0 & \longrightarrow & Z_\beta(\beta\uparrow\lambda) & \xrightarrow[i]{} & Q & \xrightarrow[j]{} & Z_\beta(\lambda) & \longrightarrow & 0.
\end{array}$$

As $TZ_\beta(\lambda) \simeq Z_\beta(\lambda_\Gamma) \simeq TZ_\beta(\beta\uparrow\lambda)$, Te splits by (d16) to yield $i' \in \mathcal{C}_\beta(TQ, TZ_\beta(\beta\uparrow\lambda))$ such that $i'\circ Ti = \mathrm{id}_{TZ_\beta(\beta\uparrow\lambda)}$. Then one can write

$$f^{-1}_{\beta\uparrow\lambda}\circ i'\circ Tj'\circ f_\lambda = a\,\mathrm{id}_{Z_\beta(\lambda_\Gamma)} \quad \text{for some } a \in \hat{A}^\beta.$$

One can check a is independent of the choice of the representative of e (cf. [AJS], (8.13)). Set

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}]e = a.$$

Then one can write with some $\bar{a}^\beta_\lambda \in \hat{A}^\beta H_\beta^{-1}$ and $\bar{b}^\beta_\lambda \in \hat{A}^\beta H_\beta$

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}]e^\beta(\lambda) = \bar{a}^\beta_\lambda + \hat{A}^\beta \quad \text{and} \quad \theta[f_\lambda, f_{\beta\uparrow\lambda}]^{-1}(\frac{1}{H_\beta} + \hat{A}^\beta) = \bar{b}^\beta_\lambda \frac{1}{H_\beta} e^\beta(\lambda),$$

in which case $\bar{a}^\beta_\lambda \bar{b}^\beta_\lambda \in 1 + \hat{A}^\beta H_\beta$ in $\hat{A}^\beta / \hat{A}^\beta H_\beta$.

Define now $\mathcal{T} : \mathcal{K}(\Omega) \to \mathcal{K}(\Gamma)$ and $\mathcal{T}' : \mathcal{K}(\Gamma) \to \mathcal{K}(\Omega)$ as follows. If $\mathcal{M} \in \mathcal{K}(\Omega)$, set

$$(\mathcal{T}\mathcal{M})(\mu) = \coprod_{\substack{\lambda\in\Omega \\ \lambda_\Gamma = \mu}} \mathcal{M}(\lambda) \quad \forall \mu \in \Gamma,$$

and for each $\beta \in R^+$ set $(\mathcal{T}\mathcal{M})(\mu, \beta) =$

$$\begin{cases} \coprod_{\substack{\lambda\in\Omega, \beta\uparrow\lambda=\lambda \\ \lambda_\Gamma=\mu}} \mathcal{M}(\lambda,\beta) \oplus \coprod_{\substack{\lambda\in\Omega,\beta\uparrow\lambda>\lambda \\ \lambda_\Gamma=\mu=(\beta\uparrow\lambda)_\Gamma}} ((\bar{b}^\beta_\lambda, 1)\mathcal{M}(\lambda,\beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta \uparrow \mu = \mu \\ \coprod_{\substack{\lambda\in\Omega \\ \lambda_\Gamma=\mu}} ((b^\beta_\lambda, 1)\mathcal{M}(\lambda,\beta) + H_\beta \mathcal{M}(\lambda)_\beta) & \text{if } \beta\uparrow\mu > \mu, \end{cases}$$

where $\mathcal{M}(\lambda)_\beta = \mathcal{M}(\lambda) \cap \mathcal{M}(\lambda, \beta)$. If $\mathcal{N} \in \mathcal{K}(\Gamma)$, set

$$(\mathcal{T}'\mathcal{N})(\lambda) = \mathcal{N}(\lambda_\Gamma) \quad \forall \lambda \in \Omega,$$

and for each $\beta \in R^+$ set $(\mathcal{T}'\mathcal{N})(\lambda, \beta) =$

$$\begin{cases} \mathcal{N}(\lambda_\Gamma, \beta) & \text{if } \beta\uparrow\lambda = \lambda \\ (a^\beta_\lambda, 1)\mathcal{N}(\lambda_\Gamma,\beta) + \mathcal{N}(\lambda_\Gamma)_\beta & \text{if } \beta\uparrow\lambda_\Gamma > \lambda_\Gamma \\ \mathcal{N}(\lambda_\Gamma,\beta) \oplus \mathcal{N}(\lambda_\Gamma + p\beta, \beta) & \text{if } (\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma + p\beta \\ \{(x + \bar{a}^\beta_\lambda y, y) \mid x, y \in \mathcal{N}(\lambda_\Gamma, \beta)\} & \text{if } (\beta\uparrow\lambda)_\Gamma = \lambda_\Gamma \text{ and } \beta\uparrow\lambda > \lambda. \end{cases}$$

Although $\mathcal{T}$ and $\mathcal{T}'$ depend on the choices of a^β_λ, b^β_λ, and $\bar{a}^\beta_\lambda$ in their classes modulo $\hat{A}^\beta$, and $\bar{b}^\beta_\lambda$ modulo $\hat{A}^\beta H_\beta^2$, the restriction of $\mathcal{T}$ (resp. $\mathcal{T}'$) to the image of $\mathcal{V}_\Omega$ (resp. $\mathcal{V}_\Gamma$) is independent of those choices (cf. [AJS], Remark 10.10).

(e6) **Proposition** (*cf.* **[AJS], Proposition 10.11)** *One has natural isomorpisms*

$$\mathcal{V}_\Gamma \circ T \simeq \mathcal{T} \circ \mathcal{V}_\Omega \quad \textit{and} \quad \mathcal{V}_\Omega \circ T' = \mathcal{T}' \circ \mathcal{V}_\Gamma.$$

(e7) Let $\mathcal{H}$ be the set of reflexion hyperplanes for the $\cdot_k$-action of W_a on

$X \otimes_{\mathbb{Z}} \mathbb{R}$. If $H = \{\nu \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \nu + \rho, \gamma^\vee \rangle = mp\}$, $\gamma \in R^+$ and $m \in \mathbb{Z}$, then we set $\gamma = \alpha(H)$. Also we will write for each $\nu \in X \otimes_{\mathbb{Z}} \mathbb{R}$

$$\nu \gtrless H \quad \text{iff} \quad \langle \nu + \rho, \gamma^\vee \rangle \gtrless mp.$$

If $\beta \in R^+$, let $\mathcal{H}(\beta) = \{H \in \mathcal{H} \mid s_\beta(\alpha(H)) < 0\}$. If $\lambda, \mu \in X$ with μ lying in the closure of the facet of λ, set in the fractional field $\mathrm{Frac}(\hat{A})$ of $\hat{A}$

$$C^\beta(\lambda, \mu) = \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda > H}} h_{-\alpha(H)} \prod_{\substack{H \in \mathcal{H}(\beta) \\ \mu \in H, \lambda < H}} \frac{1}{h_{\alpha(H)}},$$

where $h_\alpha = d_\alpha H_\alpha$ in Case 1 (resp. $\log K_\alpha = \sum_{j \geq 1} \frac{(-1)^{j+1}}{j} (K_\alpha - 1)^j$ in Case 2) for each $\alpha \in R$. If $H^\mu_\beta \in \mathcal{H}$ with $\alpha(H^\mu_\beta) = \beta$ and $\mu \in H^\mu_\beta$, then

$$C^\beta(\lambda, \mu) \in \begin{cases} (\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu > \mu \\ \frac{1}{h_\beta}(\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda < H^\mu_\beta \\ h_\beta(\hat{A}^\beta)^\times & \text{if } \beta \uparrow \mu = \mu \text{ and } \lambda > H^\mu_\beta. \end{cases}$$

(e8) Let $s \in \Sigma_a$, $\mu_s \in X \cap \bar{\mathfrak{A}}_k$ with $C_{W_a}(\mu_s) = \{1, s\}$, and $\Gamma_s = W_a \cdot_k \mu_s$. Let $\mathcal{G} = \{\Gamma_s, \Delta_i \mid s \in \Sigma_a, i \in [1, n_0]\}$. We can now state a highlight of [AJS], difficult

Theorem of good choice (*cf.* **[AJS], Theorem 13.4)** *For $\lambda \in \Omega_0$ let $\lambda_\Gamma \in \Gamma$ in the closure of the alcove of λ, $\Gamma \in \mathcal{G}$. One can simultaneously choose $\hat{A}^\beta$-generators $e^\beta(\mu)$ of $\mathrm{Ext}^1_{\mathcal{C}_\beta}(Z_\beta(\mu), Z_\beta(\beta \uparrow \mu))$ for each $\mu \in \Omega_0 \cup (\cup_{\Gamma \in \mathcal{G}} \Gamma)$ and $\beta \in R^+$ with $\beta \uparrow \mu > \mu$, and $f_\lambda \in \mathcal{C}_\wedge(Z_\wedge(\lambda_\Gamma), TZ_\wedge(\lambda))^\times$ for each $\lambda \in \Omega_0$ and $\Gamma \in \mathcal{G}$ such that for each $\lambda \in \Omega_0$, $\beta \in R^+$ and $\Gamma \in \mathcal{G}$*

$$t[f_\lambda, f_{\beta\uparrow\lambda}] e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma) e^\beta(\lambda_\Gamma) \quad \text{if } \beta \uparrow \lambda_\Gamma > \lambda_\Gamma,$$

and

$$\theta[f_\lambda, f_{\beta\uparrow\lambda}] e^\beta(\lambda) = C^\beta(\lambda, \lambda_\Gamma) + \hat{A}^\beta \quad \text{if } \beta \uparrow \lambda_\Gamma = \lambda_\Gamma = (\beta \uparrow \lambda)_\Gamma.$$

(e9) With the good choices of the $e^\beta(\lambda)$'s redefine functors $\mathcal{V}_{\Omega_0}$, $\mathcal{V}_{\Delta_i}$, $i \in [1, n_0]$, $\mathcal{V}_{\Gamma_s}$, $s \in \Sigma_a$, so that the combinatorial functors $\mathcal{T}_s$, $\mathcal{T}'_s$, and $\mathcal{T}'_i$ corresponding to $T^{\Gamma_s}_{\Omega_0}$, $T^{\Omega_0}_{\Gamma_s}$, and $T^{\Omega_0}_{\Delta_i}$, respectively, involve only the constants

$C^\beta(\lambda, \lambda_\Gamma)$, $\lambda \in \Omega_0$, $\Gamma \in \mathcal{G}$. If we set $\mathcal{Q}^{[i]}(\hat{A}) = \mathcal{T}_{i_1}\mathcal{T}'_{i_1} \dots \mathcal{T}_{i_r}\mathcal{T}'_{i_r}\mathcal{T}'_i\mathcal{Z}_{\nu_i}(\hat{A})$ with $\mathcal{T}_{i_j} = \mathcal{T}_{s_{i_j}}$ and $\mathcal{T}'_{i_j} = \mathcal{T}'_{s_{i_j}}$ (cf. (d4)), then

$$\mathcal{Q}^{[i]}(\hat{A}) \simeq \mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})) \quad \text{in } \mathcal{K}(\Omega_0).$$

(e10) Let $S = S(\mathbb{Z}R)$ the symmetric algebra of $\mathbb{Z}R$. Put $S_k = S \otimes_{\mathbb{Z}} k$. Recall the h_α from (e7). If $\hat{S}_k$ is the completion of S_k with respect to the maximal ideal generated by all $\alpha \in R$, one has a k-algebra isomorphism

(1) $$\hat{S}_k \longrightarrow \hat{A} \quad \text{via} \quad \alpha \longmapsto h_\alpha \quad \forall \alpha \in R.$$

Through the isomorphism one can regard $C^\beta(\lambda, \lambda_\Gamma)$ living in $\mathrm{Frac}(S)$ for each $\lambda \in \Omega_0$ and $\Gamma \in \mathcal{G}$. Hence one can define combinatorial categories $\mathcal{K}(\Omega_0, S)$, $\mathcal{K}(\Gamma_s, S)$, $\mathcal{K}(\Delta_i, S)$, combinatorial translation functors $\mathcal{T}_s$, $\mathcal{T}'_s$, $\mathcal{T}'_i$ between them, and $\mathcal{Z}_{\nu_i}(S)$, $\mathcal{Q}^{[i]}(S)$ by copying the definitions of $\mathcal{K}(\Omega_0)$, etc., with $\hat{A}$ replaced by S and with $S^\emptyset = S[\frac{1}{\alpha} \mid \alpha \in R^+]$ and $S^\beta = S[\frac{1}{\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$. Note that $h_\alpha \in H_\alpha \hat{A}^\times$.

More generally, let $A \in \mathbf{Alg}_S$ with $\alpha \neq 0$ in A for any $\alpha \in R$. For a W_a-orbit Ω define $\mathcal{K}(\Omega, A)$ likewise. If $A' \in \mathbf{Alg}_A$ with $\alpha \neq 0$ in A' for all $\alpha \in R$, define a functor of extension of scalars $\mathcal{K}(\Omega, A) \to \mathcal{K}(\Omega, A')$, written $\mathcal{M} \mapsto \mathcal{M}_{A'}$, by

$$\mathcal{M}_{A'}(\lambda) = \mathcal{M}(\lambda) \otimes_{A^\emptyset} A'^\emptyset \simeq \mathcal{M}(\lambda) \otimes_A A' \quad \forall \lambda \in \Omega$$

and for each $\beta \in R^+$ by setting $\mathcal{M}_{A'}(\lambda, \beta)$ equal to the image of

$$\mathcal{M}(\lambda, \beta) \otimes_{A^\beta} A'^\beta \simeq \mathcal{M}(\lambda, \beta) \otimes_A A'$$

in $\mathcal{M}_{A'}(\lambda) \oplus \mathcal{M}_{A'}(\beta \uparrow \lambda)$ if $\beta \uparrow \lambda > \lambda$ (resp. $\mathcal{M}_{A'}(\lambda)$ if $\beta \uparrow \lambda = \lambda$). The translation functors $\mathcal{T}_s$, $\mathcal{T}'_s$, $\mathcal{T}'_i$ commute with functors of extension of scalars. In particular,

(2) $$\mathcal{Q}^{[i]}(S)_{S_k} \simeq \mathcal{Q}^{[i]}(S_k) \quad \text{and} \quad \mathcal{Q}^{[i]}(S)_{\hat{A}} \simeq \mathcal{Q}^{[i]}(\hat{A}).$$

Note that

(3) $\mathcal{K}(\Omega, A)$ *is independent of* k.

For let $\mu \in \Omega \cap \bar{\mathfrak{A}}_k$ and $W_\Omega = C_{W_a}(\mu)$. Define a category $\mathcal{K}(W_a/W_\Omega, A)$ just like $\mathcal{K}(\Omega, A)$. An object of $\mathcal{K}(W_a/W_\Omega, A)$ is a family of $A^\emptyset$-modules $(\mathcal{M}(wW_\Omega))_{wW_\Omega \in W_a/W_\Omega}$, almost all members 0, together with A^β-submodules

$\mathcal{M}(wW_\Omega, \beta)$, $wW_\Omega \in W_a/W_\Omega$ and $\beta \in R^+$, of $\mathcal{M}(wW_\Omega) \oplus \mathcal{M}((\beta \uparrow w)W_\Omega)$ if $(\beta \uparrow w)W_\Omega \neq wW_\Omega$ (resp. $\mathcal{M}(wW_\Omega)$ if $(\beta \uparrow w)W_\Omega = wW_\Omega$), where $\beta \uparrow w \in W_a$ such that $(\beta \uparrow w) \cdot_k 0 = \beta \uparrow (w \cdot_k 0)$. Then one has an isomorphism

$$\mathcal{K}(\Omega, A) \longrightarrow \mathcal{K}(W_a/W_\Omega, A) \quad \text{via} \quad \mathcal{M} \longmapsto \mathcal{M}'$$

with $\mathcal{M}'(wW_\omega) = \mathcal{M}(w \cdot_k \mu)$ and $\mathcal{M}'(wW_\Omega, \beta) = \mathcal{M}(w \cdot_k \mu, \beta)$ for each $w \in W_a$ and $\beta \in R^+$.

(e11) **Lemma** (*cf.* **[AJS], Lemma 14.8)** *If A' is flat over A, then for each $\mathcal{M}, \mathcal{N} \in \mathcal{K}(\Omega, A)$*

$$\mathcal{K}(\Omega, A)(\mathcal{M}, \mathcal{N}) \otimes_A A' \simeq \mathcal{K}(\Omega, A')(\mathcal{M}_{A'}, \mathcal{N}_{A'}).$$

(e12) **Theorem** (*cf.* **[AJS], Lemma 14.9)** *Assume $p >> 0$ in Case 1. Then for each $i, j \in [1, n_0]$,*

$$\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)).$$

Proof. We first rewrite the left hand side as $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_{\mathbb{Z}} k$. For each $\lambda \in \Omega_0$ and $\beta \in R^+$ let

$$\mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 = \begin{cases} (\mathcal{Q}^{[i]}(S)(\lambda) \oplus \mathcal{Q}^{[i]}(S)(\beta \uparrow \lambda))/\mathcal{Q}^{[i]}(\lambda, \beta) & \text{if } \beta \uparrow \lambda > \lambda \\ \mathcal{Q}^{[i]}(S)(\lambda)/\mathcal{Q}^{[j]}(S)(\lambda, \beta) & \text{if } \beta \uparrow \lambda = \lambda. \end{cases}$$

One has (cf. [AJS], Lemma 14.15(b)/(14.16)) for each $\lambda \in \Omega_0$ and $\beta \in R^+$

(1) $$\mathcal{Q}^{[i]}(S)(\lambda) \textit{ is } S^\emptyset\textit{-free of finite rank,}$$

(2) $$\mathcal{Q}^{[i]}(S)(\lambda, \beta) \textit{ is } S^\beta\textit{-free of finite rank,}$$

and

(3) $$\mathcal{Q}^{[i]}(S)(\lambda, \beta)^0 \textit{ has no } p\textit{-torsion.}$$

Consider a natural map

$$\psi : \coprod_{\lambda \in \Omega_0} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \longrightarrow$$
$$\coprod_{\lambda \in \Omega_0} \coprod_{\beta \in R^+} \mathbf{Mod}_{S^\beta}(\mathcal{Q}^{[i]}(S)(\lambda, \beta), \mathcal{Q}^{[j]}(S)(\lambda, \beta)^0).$$

Then $\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) = \ker\psi$. By (3) the codomain of ψ has no p-torsion, hence $\operatorname{im}\psi$ has no p-torsion. Then $\operatorname{Tor}_1^{\mathbb{Z}}(\operatorname{im}\psi, k) = 0$. On the other hand, for each $\lambda \in \Omega_0$

$$
\begin{aligned}
&\mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda))\\
&\qquad \simeq \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset, \mathcal{Q}^{[j]}(S)(\lambda) \otimes_{S^\emptyset} S_k^\emptyset) \quad \text{by (e10)(2)}\\
&\qquad \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{S^\emptyset} S_k^\emptyset \quad \text{by (1)}\\
&\qquad \simeq \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k.
\end{aligned}
$$

Hence if ψ_k is the analogue of ψ over S_k, one gets a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\ker\psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda\in\Omega_0} \mathbf{Mod}_{S^\emptyset}(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k & \longrightarrow & (\operatorname{im}\psi) \otimes_{\mathbb{Z}} k & \longrightarrow & 0\\
(4) & & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & \ker(\psi_k) & \longrightarrow & \coprod_{\lambda\in\Omega_0} \mathbf{Mod}_{S_k^\emptyset}(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda)) & \longrightarrow & \operatorname{im}(\psi_k) & \longrightarrow & 0.
\end{array}
$$

Consequently,

(5) $\mathcal{K}(\Omega, S)(\mathcal{Q}^{[i]}(S)(\lambda), \mathcal{Q}^{[j]}(S)(\lambda)) \otimes_{\mathbb{Z}} k$ *injects into* $\mathcal{K}(\Omega, S_k)(\mathcal{Q}^{[i]}(S_k)(\lambda), \mathcal{Q}^{[j]}(S_k)(\lambda))$.

Suppose $\operatorname{Tor}_1^{\mathbb{Z}}(\operatorname{coker}\psi, k) = 0$. Then from (2) one gets as in (4) a commutative diagram of exact sequences

$$
\begin{array}{ccccc}
0 & \longrightarrow & (\operatorname{im}\psi) \otimes_{\mathbb{Z}} k & \longrightarrow & \coprod_{\lambda,\beta} \mathbf{Mod}_{S^\beta}(\mathcal{Q}^{[i]}(S)(\lambda,\beta), \mathcal{Q}^{[j]}(S)(\lambda,\beta)^0) \otimes_{\mathbb{Z}} k\\
(6) & & \downarrow & & \downarrow \wr\\
0 & \longrightarrow & \operatorname{im}(\psi_k) & \longrightarrow & \coprod_{\lambda,\beta} \mathbf{Mod}_{S_k^\beta}(\mathcal{Q}^{[i]}(S_k)(\lambda,\beta), \mathcal{Q}^{[j]}(S_k)(\lambda,\beta)^0).
\end{array}
$$

As the left vertical arrow is surjective by (4), $(\operatorname{im}\psi) \otimes_{\mathbb{Z}} k \simeq \operatorname{im}(\psi_k)$. Then the 5-lemma applied to (4) yields $(\ker\psi) \otimes_{\mathbb{Z}} k \simeq \ker(\psi_k)$, i.e., the bijectivity in (5).

Finally, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{coker}\,\psi, k) = 0$ automatically in Case 2. In Case 1 $\mathrm{coker}\,\psi$ is of finite type in $\mathbf{Mod}_S$, hence (cf. [M], Theorem 6.5) $|\,\mathrm{Ass}(\mathrm{coker}\,\psi)\,| < \infty$. Also (cf. [M], Theorem 6.1) $\mathrm{coker}\,\psi$ has a p-torsion iff $p \in \bigcup_{\mathfrak{p}\in \mathrm{Ass}(\mathrm{coker}\,\psi)} \mathfrak{p}$. As each $\mathfrak{p}$ contains a unique prime of $\mathbb{Z}$, $\mathrm{coker}\,\psi$ has no p-torsion for $p >> 0$, in which case $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{coker}\,\psi, k) = 0$.

(e13) **Remark** Theorem e.12 holds, in fact, for $p \geq h$ (cf. [AJS], Theorem 16.7). Its proof, however, requires introduction of $\mathbb{Z}$-graded combinatorial categories $\tilde{\mathcal{K}}(\Omega_0, S)$ and $\tilde{\mathcal{K}}(\Omega_0, S_k)$ (cf. [AJS], Lemma 16.6), that are also relevant to the question of the Koszulity of $U^{[p]}(\mathfrak{g})$ and $\mathfrak{u}(k)$ in [AJS], §§17/18.

(e14) Regarding $\mathbb{Z}$ as S-algebra via $\alpha \mapsto 0$ for each $\alpha \in \Sigma$, let $\mathcal{E}_{[i],[j]}(\mathbb{Z}) = \mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S \mathbb{Z}$.

Corollary *Let $i, j \in [1, n_0]$.*

(i) *$\mathcal{E}_{[i],[j]}(\mathbb{Z})$ is independent of k.*

(ii) *One has a k-linear isomorphism $\mathcal{E}_{[i],[j]}(\mathbb{Z}) \otimes_{\mathbb{Z}} k \simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k))$.*

Proof. (i) follows from (e10)(3). The left hand side of (ii) is isomorphic to

$$\begin{aligned}
&\mathcal{K}(\Omega_0, S)(\mathcal{Q}^{[i]}(S), \mathcal{Q}^{[j]}(S)) \otimes_S S_k \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \\
&\simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k), \mathcal{Q}^{[j]}(S_k)) \otimes_{S_k} \hat{A} \otimes_{\hat{A}} k \quad \text{by (e12/13)} \\
&\simeq \mathcal{K}(\Omega_0, S_k)(\mathcal{Q}^{[i]}(S_k)_{\hat{A}}, \mathcal{Q}^{[j]}(S_k)_{\hat{A}}) \otimes_{\hat{A}} k \quad \text{by (e11) as } \hat{A} \simeq \hat{S}_k \text{ is flat over } S_k \\
&\simeq \mathcal{K}(\Omega_0, \hat{A})(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e10)(2)} \\
&= \mathcal{K}(\Omega_0)(\mathcal{Q}^{[i]}(\hat{A}), \mathcal{Q}^{[j]}(\hat{A})) \otimes_{\hat{A}} k \\
&\simeq \mathcal{K}(\Omega_0)(\mathcal{V}_{\Omega_0}(Q^{[i]}(\hat{A})), \mathcal{V}_{\Omega_0}(Q^{[j]}(\hat{A}))) \otimes_{\hat{A}} k \\
&\simeq \mathcal{C}_\wedge(Q^{[i]}(\hat{A}), Q^{[j]}(\hat{A})) \otimes_{\hat{A}} k \quad \text{by (e3)} \\
&\simeq \mathcal{C}_k(Q^{[i]}(\hat{A}) \otimes_{\hat{A}} k, Q^{[j]}(\hat{A}) \otimes_{\hat{A}} k) \quad \text{by (d3) as } Q^{[i]}(\hat{A}) \text{ is projective} \\
&\simeq \mathcal{C}_k(Q^{[i]}(k), Q^{[j]}(k)).
\end{aligned}$$

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