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Kazhdan-Lusztig conjecture for Kac-Moody Lie algebras

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0 Introduction

This note is a survey of the results by Beilinson-Bernstein [1], Brylinski-Kashiwara [3], Kashiwara [12], Kashiwara-Tanisaki [13], [15], [16], Casian [4], [5] concerning highest weight modules over symmetrizable Kac-Moody Lie algebras.

The theory of highest weight modules over finite dimensional semisimple Lie algebras with general (not necesarily dominant integral) highest weights was initiated by Verma [25], where he defined the so called Verma modules and proposed the problem of determining their composition factors with multiplicities. This problem is equivalent to the one of determining the characters of the infinite dimensional irreducible highest weight modules with general highest weights. The composition factors were determined by the works of Verma [25] and Bernstein-Gelfand-Gelfand [2] (the corresponding result for symmetrizable Kac-Moody Lie algebras is due to Kac-Kazhdan [10]), and Jantzen [8] developped the algebraic theory of highest weight modules by which he determined the multiplicities in many cases; however, the general multiplicity formula was not known until the end of 70's.

A remarkable breakthrough was made around 1980. Kazhdan-Lusztig [17] proposed a conjectural multiplicity formula involving the so called Kazhdan-Lusztig polynimials, and soon after it was settled independently by Beilinson-Bernstein [1] and Brylinski-Kashiwara [3] using D-modules on the flag manifolds (see the expositions Hotta-Tanisaki [7], Sekiguchi [21], Tanisaki [22], [23], [24]).

Finally, this result was extended to Kac-Moody Lie algebras by Kashiwara [12], Kashiwara-Tanisaki [13], [15], [16], Casian [4], [5].

The contents of this note is as follows. In §1 we recall fundamental results on highest weight modules, and formulate the multiplicity formulas. A sketch of the proofs for the formulas are given in §2. In §3 we shall explain how Therem 3.5 below, which is related to Lusztig's conjectures concerning quantum groups at roots of unity and semisimple groups in positive characteristics, is deduced from the result in Kashiwara-Tanisaki [16].

1 The character formula

1.1 Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra over $\mathbf{C}$, $\mathfrak{h}$ its Cartan subalgebra, $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ the set of simple roots, $\{h_i\}_{i\in I} \subset \mathfrak{h}$ the set of simple coroots, Δ the set of roots, and Δ^+ the set of positive roots. For each $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}, \quad \mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}.$$

Let W be the Weyl group. It is a Coxeter group with canonical generator system $\{s_i \mid i \in I\}$, where s_i is the simple reflection coresponding to $i \in I$. We denote its length function by $\ell : W \to \mathbf{Z}_{\geq 0}$, and its standard partial order by $\geqq$.

1.2 Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$ define a $\mathfrak{g}$-module $M(\lambda)$ by

$$M(\lambda) = U(\mathfrak{g})/(\sum_{h\in\mathfrak{h}} U(\mathfrak{g})(h - \lambda(h)1) + U(\mathfrak{g})\mathfrak{n}^+).$$

Then $M(\lambda)$ contains a unique maximal proper submodule $K(\lambda)$, and hence the quotient module $L(\lambda) = M(\lambda)/K(\lambda)$ is an irreducible $U(\mathfrak{g})$-module. We call $M(\lambda)$ the Verma module with highest weight λ, and $L(\lambda)$ the irreducible highest weight module with highest weight λ.

For a $\mathfrak{g}$-module M and $\mu \in \mathfrak{h}^*$ set

$$\begin{aligned} M_\mu &= \{m \in M \mid \text{ for any } h \in \mathfrak{h} \text{ there exists some } N \in \mathbf{Z}_{>0} \\ &\qquad\qquad \text{such that } (h - \mu(h))^N m = 0\}. \end{aligned}$$

If $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ and $\dim M_\mu < \infty$ for any $\mu \in \mathfrak{h}^*$, we can define the character of M as a formal infinite sum

$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu e^\mu.$$

Especially we can consider the characters of $M(\lambda)$ and $L(\lambda)$. Since $M(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank 1, we see easily the following.

Proposition 1.1 *For any* $\lambda \in \mathfrak{h}^*$ *we have*

$$\mathrm{ch}(M(\lambda)) = e^\lambda / (\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\dim \mathfrak{g}_\alpha}).$$

Here, we understand that the symbols e^μ satisfy $e^{\mu_1} e^{\mu_2} = e^{\mu_1 + \mu_2}$, and $1/(1 - e^\mu) = \sum_{j=0}^\infty e^{j\mu}$.

Problem 1.2 (Verma) Determine $\mathrm{ch}(L(\lambda))$.

We fix a $\mathbf{Z}$-lattice P such that $\langle P, h_i \rangle \subset \mathbf{Z}$ and $\alpha_i \in P$ for any $i \in I$. In §1 and §2 we shall treat the above problem in the case $\lambda \in P$.

Fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, h_i \rangle = 1$ for any $i \in I$, and set

$$
\begin{aligned}
P^{+} &= \{\lambda \in P \mid \langle \lambda, h_i \rangle \in \mathbf{Z}_{\geq 0} \text{ for any } i \in I\} \\
&= \{\lambda \in P \mid \langle \lambda + \rho, h_i \rangle \in \mathbf{Z}_{>0} \text{ for any } i \in I\} \\
P^{-} &= \{\lambda \in P \mid \langle \lambda + \rho, h_i \rangle \in \mathbf{Z}_{<0} \text{ for any } i \in I\} \\
P_{\text{sing}} &= \{\lambda \in P \mid \langle \lambda + \rho, h_i \rangle = 0 \text{ for some } i \in I\} \\
P_{\text{reg}} &= P \setminus P_{\text{sing}}
\end{aligned}
$$

The following is well known.

Theorem 1.3 (Weyl-Kac) *For any $\lambda \in P^+$ we have*

$$
\operatorname{ch}(L(\lambda)) = \sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch}(M(w(\lambda + \rho) - \rho))
$$

1.3 As is seen from the formula in Theorem 1.3 it is convenient to intoroduce a new shifted action of W on $\mathfrak{h}^*$ given by

$$
w \circ \mu = w(\mu + \rho) - \rho \quad (w \in W, \mu \in \mathfrak{h}^*).
$$

Note that P is preserved under this new action of W.

If $\mathfrak{g}$ is of type A_2, the weights are as in Figure 1. Here, the reflections with respect to the three lines (walls) generate the group corresponding to the shifted action of W. The dots $\bullet$ denote the points in P, and those on the three walls (resp. , resp.) represent the points in P_{sing} (resp. P^+, resp P^-). For type A_2 (or more generally if $\mathfrak{g}$ is of finite type), both of P^+ and P^- are complete set of representatives with respect to the shifted action of W on P_{reg}.

1.4 We first consider the case where $\mathfrak{g}$ is of finite type. It is known that $\operatorname{ch}(L(\lambda))$ for $\lambda \in P_{\text{sing}}$ is a certain limit of the one for $\lambda \in P_{\text{reg}}$ (translation principle, see Jantzen [8]). Hence we can restrict ourselves to the case $\lambda \in P_{\text{reg}}$.

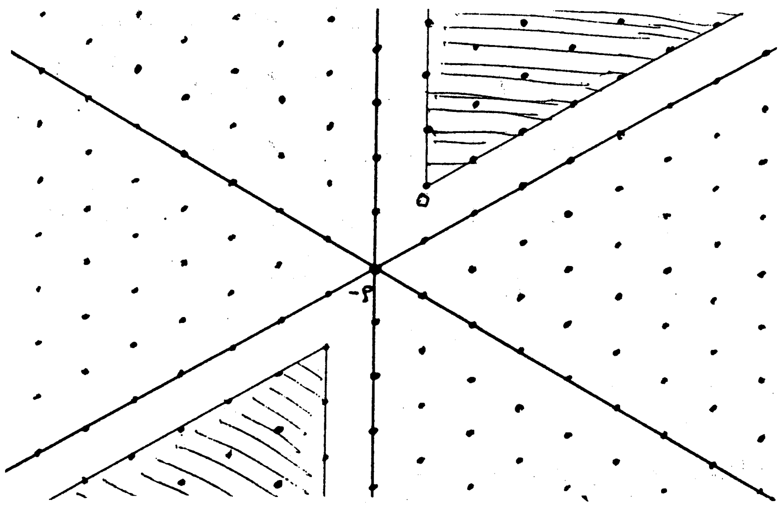

Figure 1: weights for A_2

Proposition 1.4 *Assume that* $\mathfrak{g}$ *is of finite type and that* $\mathrm{rank}\,\mathfrak{g} \leqq 2$.

(i) *For* $\lambda \in P^-$ *and* $w \in W$ *we have*

$$\text{(1)} \qquad \mathrm{ch}(L(w \circ \lambda)) = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} \clubsuit\, \mathrm{ch}(M(y \circ \lambda)),$$

$$\text{(2)} \qquad \mathrm{ch}(M(w \circ \lambda)) = \sum_{y \leqq w} \clubsuit\, \mathrm{ch}(L(y \circ \lambda)).$$

(ii) *For* $\lambda \in P^+$ *and* $w \in W$ *we have*

$$\text{(3)} \qquad \mathrm{ch}(L(w \circ \lambda)) = \sum_{y \geqq w} (-1)^{\ell(w)-\ell(y)} \clubsuit\, \mathrm{ch}(M(y \circ \lambda)),$$

$$\text{(4)} \qquad \mathrm{ch}(M(w \circ \lambda)) = \sum_{y \geqq w} \clubsuit\, \mathrm{ch}(L(y \circ \lambda)).$$

Here $\clubsuit = 1$.

Note that all of (1),...,(4) are equivalent formulas. In fact, (1) and (2) are equivalent by the formula

$$\sum_{y \leqq x \leqq w} (-1)^{\ell(x)-\ell(y)} = \delta_{y,w} \qquad (y \leqq w).$$

Let $w_0 \in W$ be the (unique) element such that $\ell(w_0)$ is maximal. Then we have $w_0 \circ P^+ = P^-$, and $y \leqq w$ if and only if $yw_0 \geqq ww_0$. Hence (3) (resp. (4)) is equivalent to (1) (resp. (2)).

If rank$\mathfrak{g} > 2$, then the situation is not so simple, and ♣'s possibly take integers greater than 1. The Kazhdan-Lusztig conjecture asserts that ♣ can be described using the Kazhdan-Lusztig polynomials.

1.5 We recalll the definition of the Kazhdan-Lusztig polynomials. For a Coxeter system (W, S) let $H(W)$ be the free $\mathbf{Z}[q, q^{-1}]$-module with basis $\{T_w\}_{w \in W}$. We can define a structure of an associative algebra over $\mathbf{Z}[q, q^{-1}]$ on $H(W)$ by

$$T_{w_1}T_{w_2} = T_{w_1 w_2} \qquad (\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)),$$
$$(T_s + 1)(T_s - q) = 0 \qquad (s \in S).$$

Note that $T_e = 1$. This algebra is called the Hecke algebra (or the Iwahori algebra, or the Hecke-Iwahori algebra) of (W, S).

Proposition 1.5 (Kazhdan-Lusztig [17]) *For any $w \in W$ there exists uniquely an element $C_w \in H(W)$ of the form*

$$C_w = \sum_{y \leqq w} P_{y,w}(q) T_y \qquad (P_{y,w}(q) \in \mathbf{Z}[q])$$

satisfying the following conditions.

(a) $P_{w,w} = 1$,

(b) *for $y < w$ we have* $P_{y,w}(q) \in \mathbf{Z}[q^{-1/2}]q^{(\ell(w)-\ell(y)-1)/2} \cap \mathbf{Z}[q]$,

(c) $C_w = q^{\ell(w)} \sum_{y \leqq w} P_{y,w}(q^{-1}) T_{y^{-1}}^{-1}$.

The polynomials $P_{y,w}$ are called the Kazhdan-Lusztig polynomials. We set $P_{y,w} = 0$ unless $y \leqq w$. If $|S| = 2$, we have $P_{y,w} = 1$ for any $y, w \in W$ with $y \leqq w$.

1.6 The answer to the probem 1.2 for finite dimensional semisimple Lie algebras is given by the following.

Theorem 1.6 (Beilinson-Bernstein [1], Brylinski-Kashiwara [3]) *Assume that $\mathfrak{g}$ is of finite type.*

(i) *For* $\lambda \in P^-$ *and* $w \in W$ *we have*

$$\mathrm{ch}(L(w\circ\lambda)) = \sum_{y\leqq w}(-1)^{\ell(w)-\ell(y)}P_{y,w}(1)\mathrm{ch}(M(y\circ\lambda)), \tag{5}$$

$$\mathrm{ch}(M(w\circ\lambda)) = \sum_{y\leqq w}P_{ww_0,yw_0}(1)\mathrm{ch}(L(y\circ\lambda)). \tag{6}$$

(ii) *For* $\lambda \in P^+$ *and* $w \in W$ *we have*

$$\mathrm{ch}(L(w\circ\lambda)) = \sum_{y\geqq w}(-1)^{\ell(w)-\ell(y)}P_{yw_0,ww_0}(1)\mathrm{ch}(M(y\circ\lambda)), \tag{7}$$

$$\mathrm{ch}(M(w\circ\lambda)) = \sum_{y\geqq w}P_{w,y}(1)\mathrm{ch}(L(y\circ\lambda)). \tag{8}$$

This result was conjectured by Kazhdan-Lusztig [17]. Again, all of (5),...,(8) are equivalent formulas. In fact, (5) and (6) are equivalent by the following formula in [17];

$$\sum_{y\leqq x\leqq w}(-1)^{\ell(x)-\ell(y)}P_{y,x}P_{ww_0,xw_0} = \delta_{y,w} \qquad (y \leqq w),$$

and (7) (resp. (8)) is equivalent to (5) (resp. (6)) by the same reason as the one for Proposition 1.4.

1.7 We nextly consider generalizations of Theorem 1.6 to arbitrary symmetrizable Kac-Moody Lie algebras.

In order to illustrate the difference between the finite and the infinite dimensional cases let us draw the figure of weights for $A_1^{(1)}$ (see Figure 2).

The walls corresponding to the reflections generating the shifted action of W are given by

$$y = -\frac{n-1}{n}x, \quad y = -\frac{n+1}{n}x \quad (n = 1, 2, 3, \ldots), \quad x = 0.$$

We have another special wall $y = -x$ which plays a different role. Starting from a point on P^+ (or P^-) and operating the reflections, the point leaps over the walls and move to the next regions successively; however, it is impossible to leap over the special wall $y = -x$. The wall $y = -x$ (of Berlin) divides the city into two parts with different laws.

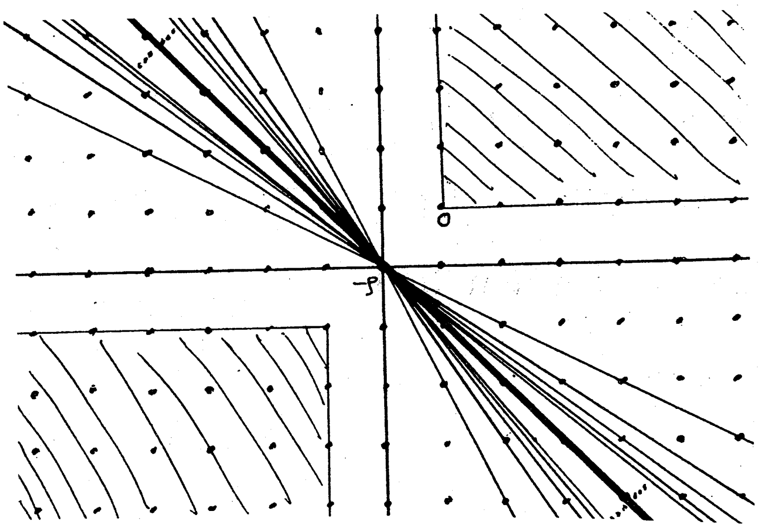

Figure 2: weights for $A_1^{(1)}$

In this note we do not treat the case when the highest weight lies on the wall $y = -x$, where the third law is applied.

Among the formulas (5),..., (8), we can only consider direct generalizations of (5) and (8), since w_0 does not exist. Moreover, the formulas (5) and (8) give different statements for general Kac-Moody Lie algebras.

Theorem 1.7 (Kashiwara(-Tanisaki) [12], [13], Casian [4]) *For any symmetrizable Kac-Moody Lie algebra* $\mathfrak{g}$, *we have*

$$\mathrm{ch}(M(w \circ \lambda)) = \sum_{y \geqq w} P_{w,y}(1)\mathrm{ch}(L(y \circ \lambda))$$

for any $\lambda \in P^+$ *and* $w \in W$.

This result was conjectured by Deodhar-Gabber-Kac [6].

Theorem 1.8 (Kashiwara-Tanisaki [15], Casian [5]) *For any affine Lie algebra* $\mathfrak{g}$, *we have*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\mathrm{ch}(M(y \circ \lambda))$$

for any $\lambda \in P^-$ *and* $w \in W$.

This result was conjectured by Lusztig [19].

Remark The formula (5) does not hold unless $\mathfrak{g}$ is afiine or of finite type.

2 D-modules on the flag manifold

2.1 The scheme of the proofs of Theorem 1.6, Theorem 1.7, Theorem 1.8 are similar. Via the correspondence:

$$\boxed{\mathfrak{g}\text{-modules}} \longleftrightarrow \boxed{D\text{-modules}} \longleftrightarrow \boxed{\text{perverse sheaves}}$$

the problem is translated into the one for perverse sheaves, where the calculation of the intersection cohomology groups for the Schubert varieties gives the answer.

2.2 We first explain the strategy of the proof for Theorem 1.6 which is the prototype of those for Theorem 1.7 and Theorem 1.8.

Let $\mathfrak{g}$ be of finite type. Let G be the connected algebraic group with Lie algebra $\mathfrak{g}$, and let B^+ and B^- be the subgroup of G corresponding to $\mathfrak{b}^+$ and $\mathfrak{b}^-$ respectively. We call the homogeneous space $X = G/B^+$ the flag manifold of G. In general, for a smooth variety Y over $\mathbf{C}$ we denote its structure sheaf, the canonical sheaf and the sheaf of differential operators on Y by $\mathcal{O}_Y$, Ω_Y and D_Y respectively. The action of G on X induces an algebra homomorphism

$$U(\mathfrak{g}) \to \Gamma(X, D_X) \qquad (a \mapsto \partial_a)$$

$$(\partial_a f)(x) = \tfrac{d}{dt} f(\exp(-ta)x)|t = 0 \qquad (a \in \mathfrak{g}, f \in \mathcal{O}_X, x \in X).$$

Let $\mathfrak{z}$ be the center of $U(\mathfrak{g})$, and let $\chi : \mathfrak{z} \to \mathbf{C}$ be the restriction of the algebra homomorphism $U(\mathfrak{g}) \to \mathbf{C}$ given by $a \mapsto 0$ for any $a \in \mathfrak{g}$. Let $\mathbf{M}_0(\mathfrak{g})$ be the category of (left)

$\mathfrak{g}$-modules M such that $zm = \chi(z)m$ for any $z \in \mathfrak{z}$ and $m \in M$, and let $\mathbf{M}(D_X)$ be the category of (left) D_X-modules which are quasi-coherent over $\mathcal{O}_X$. We have two functors

$$\Gamma(X, \bullet): \quad \mathbf{M}(D_X) \to \mathbf{M}_0(\mathfrak{g})$$
$$D_X \otimes_{U(\mathfrak{g})} (\bullet): \quad \mathbf{M}_0(\mathfrak{g}) \to \mathbf{M}(D_X).$$

Theorem 2.1 (Beilinson-Bernstein [1]) *The functors $\Gamma(X, \bullet)$ and $D_X \otimes_{U(\mathfrak{g})} (\bullet)$ give equivalences of abelian categories which are inverses to each other.*

Note that an equivalence of smaller categories is given in Brylinski-Kashiwara [3].

The $\mathfrak{g}$-modules $M(w \circ (-2\rho)) = M(-w\rho - \rho)$ and $L(w \circ (-2\rho)) = L(-w\rho - \rho)$ for $w \in W$ are objects of $\mathbf{M}_0(\mathfrak{g})$, and we have the corresponding objects $\mathcal{M}_w = D_X \otimes_{U(\mathfrak{g})} M(w \circ (-2\rho))$ and $\mathcal{L}_w = D_X \otimes_{U(\mathfrak{g})} L(w \circ (-2\rho))$ of $\mathbf{M}(D_X)$. Then, (5) in Theorem 1.6 for $\lambda = -2\rho$ is equivalent to the formula

$$[\mathcal{L}_w] = \sum_{y \leqq w} (-1)^{\ell(w) - \ell(y)} P_{y,w}(1) [\mathcal{M}_y]$$

in the Grothendieck group $K(\mathbf{M}(D_X))$. Therefore, in order to show Theorem 1.6, we need descriptions of the D_X-modules $\mathcal{M}_w$ and $\mathcal{L}_w$. Set $X_w = B^+ w B^+ / B^+ \subset X$ for $w \in W$. The following is well-known.

Proposition 2.2 (i) *X_w is a locally closed subvariety.*

(ii) $X = \amalg_{w \in W} X_w$.

(iii) $X_w \simeq \mathbf{C}^{\ell(w)}$.

(iv) $\overline{X}_w = \amalg_{y \leqq w} X_y$.

Then we have the following.

Theorem 2.3 *For any $w \in W$ we have*

$$\mathcal{M}_w = \mathcal{H}_{X_w}^{\mathrm{codim} X_w} (\mathcal{O}_X)^*, \mathcal{L}_w = \mathrm{Image}(\mathcal{M}_w \to \mathcal{M}_w^*).$$

Here, $\mathcal{H}^{\mathrm{codim}X_w}_{X_w}$ denotes the functor taking the local cohomology sheaf with support X_w and degree codimX_w. Since $\mathcal{O}_X$ is a D_X-module, $\mathcal{H}^{\mathrm{codim}}_{X_w}(\mathcal{O}_X)$ is also a D_X-module. Moreover it is a regular holonomic D_X-module by the general theory of D-modules. In the category of regular holonomic D_X-modules we have the duality functor $\mathcal{M} \rightsquigarrow \mathcal{M}^*$ given by

$$\mathcal{M}^* = \mathcal{E}xt^{\dim X}_{D_X}(\mathcal{M}, D_X) \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$$

Therefore, $\mathcal{M}_w, \mathcal{M}^*_w, \mathcal{L}_w$ are regular holonomic D_X-modules. Setting $\partial X_w = \overline{X}_w \setminus X_w$, we have $\mathcal{M}_w|X\setminus\partial X_w \simeq \mathcal{M}^*_w|X\setminus\partial X_w \simeq \mathcal{L}_w|X\setminus\partial X_w$, and they correspond to the differential equation satisfied by the delta-function supported on X_w.

For a smooth algebraic variety over $\mathbf{C}$ we denote the the category of perverse sheaves on Y (resp. regular holonomic D_Y-modules) by $\mathbf{P}(Y)$ (resp. $\mathbf{M}_r(D_Y)$). Then an eqivalence of $\mathbf{M}_r(D_Y)$ and $\mathbf{P}(Y)$ is given by

$$DR : \mathbf{M}_r(D_Y) \to \mathbf{P}(Y) \qquad (\mathcal{M} \rightsquigarrow \mathbf{R}\mathcal{H}om_{D_Y}(\mathcal{O}_Y, \mathcal{M})[\dim Y])$$

(the Riemann-Hilbert correspondence). By the general theory we have

$$DR(\mathcal{M}_w) = \mathbf{C}_{X_w}[\ell(w)], \quad DR(\mathcal{L}_w) = {}^{\pi}\mathbf{C}_{X_w}[\ell(w)],$$

where $\mathbf{C}_{X_w}$ (resp. ${}^{\pi}\mathbf{C}_{X_w}$) denotes the zero extention (minimal extension) of the constant sheaf $\mathbf{C}_{X_w}$ on X_w to X. Therefore, Theorem 1.6 for $\lambda = -2\rho$ follows from the following theorem which had been already known before Theorem 1.6 was proved.

Theorem 2.4 (Kazhdan-Lusztig [18]) *In the Grothendieck group $K(\mathbf{P}(X))$ we have*

$$[{}^{\pi}\mathbf{C}_{X_w}[\ell(w)]] = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)[\mathbf{C}_{X_y}[\ell(y)]].$$

Theorem 1.6 for general λ can be proved either by reducing it to the special case $\lambda = -2\rho$ using the translation principle, or by applying the arguments above to twisted differential operators instead of D_X.

2.3 In order to apply the arguments used in the proof of Theorem 1.6 to general Kac-Moody Lie algebras we need their flag manifolds. In this subsection we explain the scheme theoretical construction of the flag manifolds for general Kac-Moody Lie algebras given by Kashiwara [11].

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. For $k \in \mathbf{Z}_{>0}$ set

$$\mathfrak{n}_k^{\pm} = \bigoplus_{\alpha\in\Delta^+,\mathrm{ht}(\alpha)\geqq k} \mathfrak{g}_{\pm\alpha},$$

where for $\alpha = \sum_{i\in I} m_i\alpha_i \in \Delta^+$ we define $\mathrm{ht}(\alpha)$ by $\mathrm{ht}(\alpha) = \sum_{i\in I} m_i$. Define group schemes H, N^+, N^-, B^+, B^- as follows:

$$\begin{aligned} H &= \mathrm{Spec}\mathbf{C}[P], \\ N^{\pm} &= (\text{the projective limit of } \exp(\mathfrak{n}^{\pm}/\mathfrak{n}_k^{\pm})), \\ B^{\pm} &= (\text{the semidirect product of } H \text{ and } N^{\pm}). \end{aligned}$$

Here, for a finite dimensional nilpotent Lie algebra $\mathfrak{a}$ we denote the corresponding unipotent algebraic group by $\exp\mathfrak{a}$. Setting $\hat{\mathfrak{n}}^{\pm} = \prod_{\alpha\in\Delta^+} \mathfrak{g}_{\pm\alpha}$, we have an isomorphism $\exp : \hat{\mathfrak{n}}^{\pm} \to N^{\pm}$ of schemes. Note that H is finite dimensional, while $N^{\pm}$ and $B^{\pm}$ are infinite dimensional (unless $\mathfrak{g}$ is finite dimensional). If $\mathfrak{g}$ is finite dimensional, then the coordinate algebra of the corresponding algebraic group G is a certain dual Hopf algebra of the enveloping algebra $U(\mathfrak{g})$. Kashiwara [11] constructed a scheme G for a general Kac-Moody Lie algebra $\mathfrak{g}$ using a similar method (We omit the details). This G is not a group scheme but a scheme equipped with a locally free left action of B^- and a locally free right action of B^+, and the flag manifold X is constructed as the quotient scheme

$$X = G/B^+.$$

One can also define Schubert varieties X^w and X_w for $w \in W$, which are analogues of B^-wB^+/B^+ and B^+wB^+/B^+ respectively in the finite dimensional case, as locally closed subvarieties of X.

Proposition 2.5 (Kashiwara [11]) (i) $X = \sqcup_{w \in W} X^w$.

(ii) X^w *is isomorphic to* $\mathbf{C}^\infty$ *(unless* $\mathfrak{g}$ *is finite dimensional) and* $\mathrm{codim} X^w = \ell(w)$.

(iii) $\overline{X^w} = \sqcup_{y \geqq w} X^y$.

Proposition 2.6 (Kashiwara-Tanisaki [15]) (i) $\cup_{w \in W} X_w = \sqcup_{w \in W} X_w \subset X$, *and the equality holds only if* $\mathfrak{g}$ *is finite dimensional.*

(ii) X_w *is isomorphic to* $\mathbf{C}^{\ell(w)}$.

(iii) $\overline{X_w} = \sqcup_{y \leqq w} X_y$.

Here, $\mathbf{C}^\infty$ denotes the affine scheme corresponding to the polynomial ring $\mathbf{C}[x_i | i \in \mathbf{N}] =$ (the inductive limit of $\mathbf{C}[x_1, \dots, x_n]$). Hence we have $\mathbf{C}^\infty \simeq$ (the projective limit of $\mathbf{C}^n$). Although X is infinite dimensional, it is a good scheme in the sense that it is locally isomorphic to $\mathbf{C}^\infty$.

2.4 We use left D-modules supported on infinite dimensional Schubert varieties $\overline{X^w}$ in the proof of Theorem 1.7, while in proving Theorem 1.8 right D-modules supported on finite dimensional Schubert varieties $\overline{X_w}$ are used. In the following we shall explain how Theorem 1.8 is proved.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. We first define a category $\mathbf{H}$ of "right holonomic D_X-modules", which plays a fundamental role in the proof. We call a finite dimensional closed subset Z (resp. an open subset Y) of X an admissible closed subset (resp. an admissible open subset) of X if $Z = \cup_{w \in F} X_w$ (resp. $Y = \cup_{w \in F} X^w$) for some finte subset F of W satisfying

$$w \in F,\ y \leqq w \Longrightarrow y \in F.$$

For an admissible closed subset Z of X let $\mathbf{H}(Z)$ be the category of "right holonomic D_X-modules supported in Z". Then the abelian category $\mathbf{H}$ is defined to be the inductive limit of $\mathbf{H}(Z)$ with respect to Z. The definition of $\mathbf{H}(Z)$ is given as follows. For an admissible open subset Y of X containing Z let $\mathbf{H}(Z,Y)$ be the category of "right holonomic D_Y-modules supported in Z". Then $\mathbf{H}(Z)$ is the projective limit of $\mathbf{H}(Z,Y)$ with respect to Y. Finally, the category $\mathbf{H}(Z,Y)$ is defined as follows. For any sufficiently large $k \in \mathbf{Z}_{>0}$ the subgroup $N_k^- = \exp(\prod_{\alpha\in\Delta^+,\mathrm{ht}(\alpha)\geqq k} \mathfrak{g}_{-\alpha})$ of N^- acts on Y locally freely. Then the quotient scheme $Y_k = N_k^-\backslash Y$ is finite dimensional, and the natural morphism $i_k : Z \to Y_k$ is injective. Let $\mathbf{H}(D_{Y_k}, i_k(Z))$ be the category of right holonomic D_{Y_k}-modules supported in $i_k(Z)$, and let $\mathbf{H}(Z,Y,k)$ be the category consisting of $(\mathcal{M}_l)_{l\geqq k} \in \prod_{l\geqq k} \mathrm{Ob}(\mathbf{H}(D_{Y_l}, i_l(Z)))$ such that $\int_{p_{l_2}^{l_1}} \mathcal{M}_{l_1} = \mathcal{M}_{l_2}$ for $l_1 \geqq l_2 \geqq k$, where $p_{l_2}^{l_1} : Y_{l_1} \to Y_{l_2}$ is the natural morphism. Then the category $\mathbf{H}(Z,Y)$ is defined to be the projective limit of $\mathbf{H}(Z,Y,k)$ with respect to k.

Note that what we really treat is not D-modules on infinite dimensional spaces, but certain limits of D-modules on finite dimensional spaces. In this framework we can directly apply the fruitful theory of D-modules on finite dimensional manifolds.

For λ in P let $\mathcal{O}_X(\lambda)$ be the invertible $\mathcal{O}_X$-module corresponding to the $\mathfrak{g}$-equivariant line bundle L_λ on X such that the action of $\mathfrak{b}^+$ on the fiber $(L_\lambda)_{eB^+}$ is given by λ. Define a sheaf $D_X(\lambda)$ of rings of twisted differential operators by

$$D_X(\lambda) = \mathcal{O}_X(-\lambda) \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lambda) \subset \mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X(-\lambda)).$$

We can also define an invertible $\mathcal{O}_{Y_k}$-module $\mathcal{O}_{Y_k}(\lambda)$ and a sheaf $D_{Y_k}(\lambda)$ of twisted rings of differential operators. Then a category $\mathbf{H}(\lambda)$ of "right holonomic $D_X(\lambda)$-modules" and categories $\mathbf{H}(\lambda, Z), \mathbf{H}(\lambda, Z, Y), \mathbf{H}(\lambda, Z, Y, k)$ are defined similarly to $\mathbf{H}$, $\mathbf{H}(Z)$, $\mathbf{H}(Z,Y)$, $\mathbf{H}(Z,Y,k)$ using $D_{Y_k}(\lambda)$ instead of D_{Y_k}.

For $\mathcal{M} \in \mathrm{Ob}(\mathbf{H}(\lambda))$ take a representative $(\mathcal{M}_l)_{l\geqq k} \in \mathbf{H}(\lambda, Z, Y, k)$ and set

$$H^n(X, \mathcal{M}) = (\text{the projevtive limit of } H^n(Y_l, \mathcal{M}_l))$$

for $n \in \mathbf{Z}_{\geqq 0}$. It carries a natural left $\mathfrak{g}$-module structure induced from the action of $\mathfrak{g}$ on

$\mathcal{O}_X(\lambda)$, and we obtain additive functors

$$H^n(X, \bullet) : \mathbf{H}(\lambda) \to \mathbf{M}(\mathfrak{g}) \quad (n \in \mathbf{Z}_{\geqq 0}),$$

where $\mathbf{M}(\mathfrak{g})$ denotes the category of $\mathfrak{g}$-modules.

Remark that the category $\mathbf{H}(\lambda, Z, Y, k)$ is isomorphic to $\mathbf{H}(D_{Y_l}(\lambda), i_l(Z))$ for any single $l \geqq k$. However, in order to define the functor $H^n(X, \bullet)$ we need all $l \geqq k$.

For $w \in W$ we can define objects $\mathcal{M}_w(\lambda)$ (resp. $\mathcal{L}_w(\lambda)$) of $\mathbf{H}(\lambda)$ as the dual meromorphic extension (resp. the minimal extension) of the right D_{X_w}-module Ω_{X_w} to a "right $D_X(\lambda)$-module". Then $\mathcal{L}_w(\lambda)$ is an irreducible object of $\mathbf{H}(\lambda)$, and $\mathcal{M}_w(\lambda)$ has a finite composition series whose composition factors are isomorphic to $\mathcal{L}_y(\lambda)$ for some $y \leqq w$. Let $\mathbf{H}_0(\lambda)$ be the full subcategory of $\mathbf{H}(\lambda)$ consisting of objects of $\mathbf{H}(\lambda)$ which have finite composition series whose composition factors are isomorphic to some $\mathcal{L}_w(\lambda)$.

For $\mathcal{M} \in \mathrm{Ob}(\mathbf{H}_0(\lambda))$ define a $\mathfrak{g}$-submodule $\bar{H}^n(X, \mathcal{M})$ of $H^n(X, \mathcal{M})$ by

$$\bar{H}^n(X, \mathcal{M}) = \bigoplus_{\mu \in P} H^n(X, \mathcal{M})_\mu \subset H^n(X, \mathcal{M}).$$

Then $H^n(X, \mathcal{M})$ is a certain completion of $\bar{H}^n(X, \mathcal{M})$. For a short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{M} \to 0$ in $\mathbf{H}_0(\lambda)$ we have a long exact sequence

$$\cdots \to \bar{H}^n(X, \mathcal{K}) \to \bar{H}^n(X, \mathcal{L}) \to \bar{H}^n(X, \mathcal{M}) \to \bar{H}^{n+1}(X, \mathcal{K}) \to \cdots$$

in $\mathbf{M}(\mathfrak{g})$.

Then Theorem 1.8 follows from the following.

Theorem 2.7 (Kashiwara-Tanisaki [15]) *Let $\mathfrak{g}$ be an affine Lie algebra. For $\lambda \in P^-$ we have the following.*

(i) $\bar{H}^n(X, \mathcal{M}) = 0$ *for any* $n > 0$ *and any* $\mathcal{M} \in \mathrm{Ob}(\mathbf{H}_0(\lambda))$.

(ii) $\bar{H}^0(X, \mathcal{M}_w(\lambda)) = M(w \circ \lambda)$, $\bar{H}^0(X, \mathcal{L}_w(\lambda)) = L(w \circ \lambda)$ *for any* $w \in W$.

(iii) $[\mathcal{L}_w(\lambda)] = \sum_{y \leqq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)[\mathcal{M}_y(\lambda)]$ *in the Grothendieck group of* $\mathbf{H}_0(\lambda)$.

The statements (i) and (ii) are the main results of [15]. The statement (iii) is proved by reducing it to the calculation of the intersection cohomology sheaf of $\overline{X_w}$ via the Riemann-Hilbrt correspondence.

3 The character formula for untwisted affine Lie algebras

3.1 Recently we have generalized Theorem 1.8 for integral highest weights to the one for rational highest weights (Kashiwara-Tanisaki [16]). In this section we shall formulate this result and give its consequence in the case of untwisted affine Lie algebras.

3.2 Let $\mathfrak{g}$ be a (not necesarily untwisted) affine Lie algebra with Cartan subalgebra $\mathfrak{h}$, and let $\mathfrak{g}^\vee$ be the dual affine Lie algebra whose Dinkin diagram is obtained by reversing the arrows in the Dynkin diagram of $\mathfrak{g}$. We identify the Cartan subalgebra of $\mathfrak{g}^\vee$ with the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. Let $\Delta^\vee, \Delta^{\vee+}, \Delta^\vee_{\mathrm{re}} \subset \mathfrak{h}$ be the set of roots, positive roots and real roots for $\mathfrak{g}^\vee$. For $\mu \in \mathfrak{h}^*$ set

$$\begin{aligned}
\Delta^\vee(\mu) &= \{h \in \Delta^\vee \mid (\mu+\rho)(h) \in \mathbf{Z}\}, \\
\Delta^{\vee+}(\mu) &= \Delta^\vee(\mu) \cap \Delta^{\vee+}, \\
\Delta^{\vee+}_{\mathrm{re}}(\mu) &= \Delta^{\vee+}(\mu) \cap \Delta^\vee_{\mathrm{re}}, \\
\Pi^\vee(\mu) &= \Delta^{\vee+}(\mu) \setminus (\Delta^{\vee+}(\mu) + \Delta^{\vee+}(\mu)), \\
\Pi^\vee_{\mathrm{re}}(\mu) &= \Pi^\vee(\mu) \cap \Delta^\vee_{\mathrm{re}},
\end{aligned}$$

and let $W(\mu)$ be the subgroup of W generated by the reflections s_h corresponding to $h \in \Delta^{\vee+}_{\mathrm{re}}(\mu)$. Then $W(\mu)$ is a Coxeter group with canonical generator system $S(\mu) = \langle s_h \mid h \in \Pi^\vee_{\mathrm{re}}(\mu)\rangle$. Let $\ell^\mu : W(\mu) \to \mathbf{Z}_{\geq 0}$ be its length function.

The main result of [16] is the following.

Theorem 3.1 (Kashiwara-Tanisaki [16]) *Let $\lambda \in \mathfrak{h}^*$ be such that $(\lambda+\rho)(h) \in \mathbf{Q}\setminus\mathbf{Z}_{>0}$ for any $h \in \Delta^{\vee+}_{\mathrm{re}}$ and $(\lambda+\rho)(c) \neq 0$, where $c \in \mathfrak{h}$ is a generator of the center of $\mathfrak{g}$. Let*

$w \in W(\lambda)$ be such that $\ell^\lambda(w) = \min\{\ell^\lambda(w') \mid w' \in W(\lambda), w' \circ \lambda = w \circ \lambda\}$. Then we have

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \leqq w} (-1)^{\ell^\lambda(w) - \ell^\lambda(y)} P_{y,w}(1) \mathrm{ch}(M(y \circ \lambda)).$$

Here $\leqq$ and $P_{y,w}$ are the standard partial order and the Kazhdan-Lusztig polynomial for the Coxeter group $W(\lambda)$.

This result was conjectured by Lusztig [20].

3.3 In connection with other Kazhdan-Lusztig type conjectures due to Lusztig concerning quantum groups at roots of unity and semisimple groups in positive characteristics, some special cases of Theorem 3.1 are important. We shall formulate it in the following.

Let $\mathfrak{g}_0$ be a finite dimensional simple Lie algebra with Cartan subalgebra $\mathfrak{h}_0$. We fix a nondegenerate invariant symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}_0$ such that relative to the induced symmetric bilinear form on $\mathfrak{h}_0^*$ we have $(\alpha, \alpha) = 2$ for any long root α. We identify $\mathfrak{h}_0$ with $\mathfrak{h}_0^*$ via this symmetric bilinear form. For a root α of $\mathfrak{g}_0$ we denote the corresponding coroot by $\alpha^\vee$. Let

$$\begin{aligned}
\Delta_0 &= (\text{the set of roots}),\\
\Delta_0^+ &= (\text{the set of positive roots}),\\
\Pi_0 &= \{\alpha_i\}_{i \in I_0} = (\text{the set of simple roots}),\\
\Delta_0^\vee &= (\text{the set of coroots}),\\
W_0 &= \langle s_i \mid i \in I_0 \rangle = (\text{the Weyl group}),\\
\theta &= (\text{the highest root}),\\
\tilde{\theta} &= (\text{the root such that the corresponding coroot is the highest coroot}),\\
Q_0^\vee &= \sum_{\alpha \in \Delta_0^\vee} \mathbf{Z}\alpha,\\
Q_0 &= \sum_{\alpha \in \Delta_0} \mathbf{Z}\alpha,\\
P_0 &= \{\lambda \in \mathfrak{h}_0^* \mid (\lambda, \alpha_i^\vee) \in \mathbf{Z} \quad (i \in I_0)\},\\
P_0^+ &= \{\lambda \in \mathfrak{h}_0^* \mid (\lambda, \alpha_i^\vee) \in \mathbf{Z}_{\geqq 0} \quad (i \in I_0)\}.
\end{aligned}$$

Let $\rho_0 \in \mathfrak{h}_0^*$ be such that $(\rho_0, \alpha_i^\vee) = 1$ for any $i \in I_0$, and set $g = (\rho_0, \theta^\vee) + 1$. Let $r = 1$, or 2, or 3 according as $\mathfrak{g}_0$ is of type $\mathrm{A}, \mathrm{D}, \mathrm{E}$, or $\mathrm{B}, \mathrm{C}, \mathrm{F}$, or G_2.

The untwisted affine Lie algebra $\mathfrak{g}$ corresponding to $\mathfrak{g}_0$ is given by

$$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c,$$

where the bracket product of $\mathfrak{g}$ is given by $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \delta_{n+m,0}(x, y)c$ for any $x, y \in \mathfrak{g}_0$, and $[c, \mathfrak{g}] = 0$. The Cartan subalgebra of $\mathfrak{g}$ is given by

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbf{C}c.$$

Define $\chi \in \mathfrak{h}^*$ by $\chi(\mathfrak{h}_0) = 0$ and $\chi(c) = 1$. Identifying $\mathfrak{h}_0^*$ with a subspace of $\mathfrak{h}^*$ by $\mathfrak{h}_0^*(c) = 0$, we have

$$\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbf{C}\chi.$$

Set $I = I_0 \sqcup \{0\}$. The set of simple coroots for $\mathfrak{g}$ is given by $\{h_i\}_{i \in I}$, where $h_i = \alpha_i^\vee$ for $i \in I_0$ and $h_0 = -\theta^\vee + c = -\theta + c$. Setting $\rho = \rho_0 + g\chi$ we have $\rho(h_i) = 1$ for any $i \in I$. The Weyl group $W \subset GL(\mathfrak{h}^*)$ is identified with the semidirect product of W_0 and $Q_0^\vee$ via

$$\begin{aligned} t_\gamma(\lambda + a\chi) &= (\lambda - a\gamma) + a\chi \qquad (\gamma \in Q_0^\vee, \lambda \in \mathfrak{h}_0^*, a \in \mathbf{C}), \\ x(\lambda + a\chi) &= x(\lambda) + a\chi \qquad (x \in W_0, \lambda \in \mathfrak{h}_0^*, a \in \mathbf{C}), \end{aligned}$$

where t_γ denotes the element of W corresponding to $\gamma \in Q_0^\vee$. Then the canonical generator system $\{s_i\}_{i \in I}$ of W is given by $s_i = s_{\alpha_i}$ for $i \in I_0$ and $s_0 = s_\theta t_\theta$. Here, $s_\alpha \in W_0$ for $\alpha \in \Delta_0$ denotes the reflection corresponding to α.

The sets $\Delta^\vee$, $\Delta^{\vee+}$, $\Delta_{\mathrm{re}}^{\vee+}$ defined in §3.2 are given by

$$\begin{aligned} \Delta^\vee &= \{\alpha^\vee + 2nc/(\alpha, \alpha) \mid \alpha \in \Delta_0, n \in \mathbf{Z}\} \cup \{nc \mid n \in \mathbf{Z} \setminus \{0\}\}, \\ \Delta^{\vee+} &= \{\alpha^\vee + 2nc/(\alpha, \alpha) \mid \alpha \in \Delta_0, n \in \mathbf{Z}_{>0}\} \cup \Delta_0^+ \cup \{nc \mid n \in \mathbf{Z}_{>0}\}, \\ \Delta_{\mathrm{re}}^{\vee+} &= \{\alpha^\vee + 2nc/(\alpha, \alpha) \mid \alpha \in \Delta_0, n \in \mathbf{Z}_{>0}\} \cup \Delta_0^+. \end{aligned}$$

3.4 In the rest of this section we fix $k \in \mathbf{Q}$ such that $-(k+g) > 0$. Write $-(k+g) = l/q$, where l and q are relatively prime positive integers.

For $\lambda \in P_0$ set $\widehat{\lambda} = \lambda + k\chi \in \mathfrak{h}^*$. Note that we have $(\widehat{\lambda} + \rho)(c) = k + g < 0$ for any $\lambda \in P_0$.

Lemma 3.2 *Let* $\lambda \in P_0$.

(i) *If* r *does not divide* q*, then we have*

$$\begin{aligned} \Delta^\vee(\widehat{\lambda}) &= \{\alpha^\vee + 2mqc/(\alpha,\alpha) \mid \alpha \in \Delta_0, m \in \mathbf{Z}\} \cup \{mqc \mid m \in \mathbf{Z} \setminus \{0\}\}, \\ \Pi^\vee(\widehat{\lambda}) &= \{\alpha_i^\vee \mid i \in I_0\} \cup \{-\theta^\vee + qc\}, \\ W(\widehat{\lambda}) &= (\textit{the semidirect product of } W_0 \textit{ and } qQ_0^\vee). \end{aligned}$$

(ii) *If* r *divides* q*, then we have*

$$\begin{aligned} \Delta^\vee(\widehat{\lambda}) &= \{\alpha^\vee + mqc \mid \alpha \in \Delta_0, m \in \mathbf{Z}\} \cup \{mqc \mid m \in \mathbf{Z} \setminus \{0\}\}, \\ \Pi^\vee(\widehat{\lambda}) &= \{\alpha_i^\vee \mid i \in I_0\} \cup \{-\tilde{\theta}^\vee + qc\}, \\ W(\widehat{\lambda}) &= (\textit{the semidirect product of } W_0 \textit{ and } qQ_0). \end{aligned}$$

Let $W^\sharp$ denote the semidirect product of W_0 and Q_0. It is a Coxeter group with canonical generator system $S^\sharp = \{s_{\alpha_i} \mid i \in I_0\} \cup \{s_{\tilde{\theta}} t_{\tilde{\theta}}\}$

Definition

(i) Assume that r does not divide q.

(a) Define a group homomorphism $\Phi_q : W \to W$ by

$$\Phi_q(x) = x \quad (x \in W_0), \qquad \Phi_q(t_\gamma) = t_{q\gamma} \quad (\gamma \in Q_0^\vee).$$

(b) Define an action of W on P_0 by

$$\begin{aligned} x \circ_l \lambda &= x(\lambda + \rho_0) - \rho_0 \quad (x \in W_0, \lambda \in P_0), \\ t_\gamma \circ_l \lambda &= \lambda + l\gamma \qquad (\gamma \in Q_0^\vee, \lambda \in P_0). \end{aligned}$$

(c) Set

$$A_l = \{\lambda \in P_0 \mid (\lambda+\rho_0, \alpha_i) \leqq 0 \quad (i \in I_0), \quad (\lambda+\rho_0, \theta) \geqq -l\,\}.$$

(ii) Assume that r divides q.

(a) Define a group homomorphism $\Phi_q^\sharp : W^\sharp \to W$ by

$$\Phi_q^\sharp(x) = x \quad (x \in W_0), \qquad \Phi_q^\sharp(t_\gamma) = t_{q\gamma} \quad (\gamma \in Q_0^\vee).$$

(b) Define an action of $W^\sharp$ on P_0 by

$$\begin{aligned} x \circ_l^\sharp \lambda &= x(\lambda+\rho_0) - \rho_0 \quad (x \in W_0, \lambda \in P_0), \\ t_\gamma \circ_l^\sharp \lambda &= \lambda + l\gamma \qquad\qquad (\gamma \in Q_0, \lambda \in P_0). \end{aligned}$$

(c) Set

$$A_l^\sharp = \{\lambda \in P_0 \mid (\lambda+\rho_0, \alpha_i) \leqq 0 \quad (i \in I_0), \quad (\lambda+\rho_0, \tilde{\theta}^\vee) \geqq -l\,\}.$$

Lemma 3.3 *Let $\lambda \in P_0$.*

(i) *Asuume that r does not divide q.*

(a) *We have $\mathrm{Im}(\Phi_q) = W(\widehat{\lambda})$, and $\Phi_q : W \to W(\widehat{\lambda})$ is an isomorphism of the Coxeter groups.*

(b) *$\Phi_q(w) \circ \widehat{\lambda} = (w \circ_l \lambda)\widehat{\ }$ for any $w \in W$.*

(c) *A_l is a fundamental domain with respect to the acton $\circ_l$ of W on P_0.*

(d) *We have $\lambda \in A_l$ if and only if $(\widehat{\lambda}+\rho)(h) \notin \mathbf{Z}_{>0}$ for any $h \in \Delta_{\mathrm{re}}^{\vee +}$.*

(ii) *Asuume that r divides q.*

(a) *We have $\mathrm{Im}(\Phi_q^\sharp) = W(\widehat{\lambda})$, and $\Phi_q^\sharp : W^\sharp \to W(\widehat{\lambda})$ is an isomorphism of the Coxeter groups.*

(b) $\Phi_q^\sharp(w) \circ \widehat{\lambda} = (w \circ_l^\sharp \lambda)\widehat{}$ *for any* $w \in W^\sharp$.

(c) $A_l^\sharp$ *is a fundamental domain with respect to the acton* $\circ_l^\sharp$ *of* $W^\sharp$ *on* P_0.

(d) *We have* $\lambda \in A_l^\sharp$ *if and only if* $(\widehat{\lambda} + \rho)(h) \notin \mathbf{Z}_{>0}$ *for any* $h \in \Delta_{\mathrm{re}}^{\vee +}$.

Lemma 3.4 (i) *Asuume that* r *does not divide* q. *Let* $w \in W$ *and* $\lambda \in A_l$.

(a) *If* $w \circ_l \lambda \in P_0^+$, *then* $s_i w < w$ *for any* $i \in I_0$.

(b) *If* $s_i w < w$ *for any* $i \in I_0$, *then* $w \circ_l \lambda \in P_0^+ - \rho_0$.

(ii) *Asuume that* r *divides* q. *Let* $w \in W$ *and* $\lambda \in A_l^\sharp$.

(a) *If* $w \circ_l^\sharp \lambda \in P_0^+$, *then* $s_i w < w$ *for any* $i \in I_0$.

(b) *If* $s_i w < w$ *for any* $i \in I_0$, *then* $w \circ_l^\sharp \lambda \in P_0^+ - \rho_0$.

3.5 For $\mu \in P_0^+$ we define $\mathfrak{g}$-modules $\tilde{M}_k(\mu)$ and $\tilde{L}_k(\mu)$ as follows. Set $\mathfrak{p} = \mathfrak{g}_0 \otimes \mathbf{C}[t] \oplus \mathbf{C}c$. It is a maximal parabolic subalgebra of $\mathfrak{g}$ corresponding to the subset I_0 of I. Let $L_0(\mu)$ be the finite dimensional irreducible $\mathfrak{g}_0$-module with highest weight μ. We regard it as a $U(\mathfrak{p})$-module via the algebra homomorphism $\epsilon_k : U(\mathfrak{p}) \to U(\mathfrak{g}_0)$ given by $\epsilon_k(\mathfrak{g}_0 \otimes t\mathbf{C}[t]) = \{0\}$, $\epsilon_k(c) = k$ and $\epsilon_k(x) = x$ for any $x \in \mathfrak{g}_0$. Set $\tilde{M}_k(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_0(\mu)$. It is obviously a highest weight module with highest weight $\widehat{\mu}$. Let $\tilde{L}_k(\mu)$ be its unique irreducible quotient. Note that $\tilde{M}_k(\mu)$ (resp. $\tilde{L}_k(\mu)$) is a quotient of $M(\widehat{\mu})$ (resp. is isomorphic to $L(\widehat{\mu})$).

Theorem 3.5 (i) *Assume that* r *does not divide* q. *Let* $\lambda \in A_l$ *and* $w \in W$ *be such that* $w \circ_l \lambda \in P_0^+$ *and* $\ell(w) = \min\{\ell(w') \mid w' \circ_l \lambda = w \circ_l \lambda\}$. *Then we have*

$$\mathrm{ch}(\tilde{L}_k(w \circ_l \lambda)) = \sum_{y \in W, y \leqq w, y \circ_l \lambda \in P_0^+} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \mathrm{ch}(\tilde{M}_k(y \circ_l \lambda)).$$

Here, the length function ℓ, *the standard partial order* $\leqq$, *and the Kazhdan-Lusztig polynomial* $P_{y,w}$ *are those for* W.

(ii) *Assume that r divides q. Let $\lambda \in A_l^\sharp$ and $w \in W^\sharp$ be such that $w \circ_l^\sharp \lambda \in P_0^+$ and $\ell(w) = \min\{\ell(w') \mid w' \circ_l^\sharp \lambda = w \circ_l^\sharp \lambda\}$. Then we have*

$$\mathrm{ch}(\tilde{L}_k(w \circ_l^\sharp \lambda)) = \sum_{y \in W^\sharp, y \leqq w, y \circ_l^\sharp \lambda \in P_0^+} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \mathrm{ch}(\tilde{M}_k(y \circ_l^\sharp \lambda)).$$

Here, the length function ℓ, the standard partial order $\leqq$, and the Kazhdan-Lusztig polynomial $P_{y,w}$ are those for $W^\sharp$.

We need the following in order to deduce Theorem 3.5 from Theorem 3.1.

Lemma 3.6 (i) *For $\mu \in P_0^+$ we have*

$$\sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(M(x \circ \hat{\mu})) = \mathrm{ch}(\tilde{M}_k(\mu)).$$

(ii) *For $\mu \in P_0$ such that $(\mu + \rho_0, \alpha_i^\vee) = 0$ for some $i \in I_o$ we have*

$$\sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(M(x \circ \hat{\mu})) = 0.$$

Proof. (i) For $\nu \in P_0$ let $M_0(\nu)$ be the Verma module of $\mathfrak{g}_0$ with highest weight ν, and regard it as a $U(\mathfrak{p})$-module via ϵ_k. Then we have $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_0(\nu) = M(\hat{\nu})$ for any $\nu \in P_0$, and $\mathrm{ch}(L_0(\mu)) = \sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(M_0(x(\mu + \rho_0) - \rho_0))$ for any $\mu \in P_0^+$. The last equality is Weyl's character formula. Since $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\bullet)$ is an exact functor, we have

$$\begin{aligned}
& \sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(M(x \circ \hat{\mu})) \\
= \; & \sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(M((x(\mu + \rho_0) - \rho_0)^\wedge)) \\
= \; & \sum_{x \in W_0} (-1)^{\ell(x)} \mathrm{ch}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_0(x(\mu + \rho_0) - \rho_0)) \\
= \; & \mathrm{ch}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_0(\mu)) \\
= \; & \mathrm{ch}(\tilde{M}_k(\mu)).
\end{aligned}$$

(ii)

$$
\begin{aligned}
& \sum_{x\in W_0} (-1)^{\ell(x)}\mathrm{ch}(M(x\circ\hat{\mu})) \\
= & \sum_{x\in W_0, xs_i>x} (-1)^{\ell(x)}(\mathrm{ch}(M(x\circ\hat{\mu}))-\mathrm{ch}(M(xs_i\circ\hat{\mu}))) \\
= & \sum_{x\in W_0, xs_i>x} (-1)^{\ell(x)}(\mathrm{ch}(M(x\circ\hat{\mu}))-\mathrm{ch}(M(x\circ\hat{\mu}))) \\
= & \ 0.
\end{aligned}
$$

■

We also note the following general result.

Lemma 3.7 (Kazhdan-Lusztig [17]) *Let (W',S') be a Coxwter system, and let $S''\subset S'$ and $w\in W'$ be such that $sw<w$ for any $s\in S''$. Then $P_{y,w}=P_{sy,w}$ for any $y\in W'$, $s\in S''$.*

Proof of Theorem 3.5. (i) Set $E=\{y\in W \mid y\leqq w\}$, $F=\{y\in E\mid s_iy<y \ (i\in I_0)\}$. By Lemma 3.4 we have $s_iw<w$ for any $i\in I_0$, and hence the product map $W_0\times F\to E$ is bijective. Moreover, for any $y\in F$ we have $y\circ_l\lambda\in P_0^+-\rho_0$. Set $F_1=\{y\in F\mid y\circ_l\lambda\in P_0^+\}$, $F_2=F\setminus F_1$. By Lemma 3.3, Theorem 3.1, Lemma 3.7, we have

$$
\begin{aligned}
& \mathrm{ch}(\tilde{L}_k(w\circ_l\lambda)) \\
= & \ \mathrm{ch}(L((w\circ_l\lambda)^\frown)) \\
= & \ \mathrm{ch}(L(\Phi(w)\circ\hat{\lambda})) \\
= & \sum_{y\in E}(-1)^{\ell(w)-\ell(y)}P_{y,w}(1)\mathrm{ch}(M(\Phi(y)\circ\hat{\lambda})) \\
= & \sum_{y\in F}(-1)^{\ell(w)-\ell(y)}P_{y,w}(1)\sum_{x\in W_0}(-1)^{\ell(x)}\mathrm{ch}(M(\Phi(xy)\circ\hat{\lambda})) \\
= & \sum_{y\in F}(-1)^{\ell(w)-\ell(y)}P_{y,w}(1)\sum_{x\in W_0}(-1)^{\ell(x)}\mathrm{ch}(M(x\circ(y\circ_l\lambda)^\frown)).
\end{aligned}
$$

By Lemma 3.6 wee see that if $y\in F_1$ (resp. F_2), then

$$
\sum_{x\in W_0}(-1)^{\ell(x)}\mathrm{ch}(M(x\circ(y\circ_l\lambda)^\frown))=\mathrm{ch}(\tilde{M}_k(y\circ_l\lambda)^\frown)) \qquad (\text{resp.} \ =0).
$$

Moreover, for $y \in W$ we have $y \in F_1$ if and only if $y \circ_l \lambda \in P_0^+$, by Lemma 3.4. Hence we have obtained the desired formula.

The statement (ii) is proved similarly. ■

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