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Kyoto University
CURVATURE OF CURVILINEAR 4-WEBS
AND PENCILS OF ONE FORMS

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ABSTRACT. A curvilinear \( n \)-web \( W = (F_1, \ldots, F_n) \) is a configuration of \( n \) curvilinear foliations \( F_i \) on a surface. When \( n = 3 \), Bott connections of \( F_i \) extend naturally to a unique affine connection, which is called Chern connection. For \( 3 < n \), this is the case if and only if the modulus of tangents to the leaves of \( F_i \) at a point is constant. An \( n \)-web is associative if the modulus is constant and weakly associative if Chern connections of all 3-subwebs have equal curvature form. We give a geometric interpretation of the curvature form in terms of fake billiard in §2, and prove that a weakly associative \( n \)-web is associative if the foliations are members of a pencil (linear family of dim 2) of 1-forms. This result completes the classification of weakly associative 4-webs initiated by Poincaré, Mayrhofer and Reidemeister for the flat case.

A curvilinear \( n \)-web on a surface \( S \) is an \( n \)-tuple of foliations of codimension 1, \( W = (F_1, \ldots, F_n) \). In this paper we assume that \( S \) is real analytic and connected and \( F_i \) is defined by a real meromorphic 1-form \( \omega_i \): of which coefficients are locally fractions of real analytic functions. \( W \) is non singular at a \( p \in S \) if \( \omega_i \) and \( \omega_i \wedge \omega_j \) are analytic and non zero at \( p \) for \( i \neq j \). \( \Sigma(W) \) denotes the set of those \( p \) where \( W \) is singular. \( W \) is diffeomorphic to an \( n \)-web \( W' = (F'_1, \ldots, F'_n) \) on \( S' \) if there exists an analytic diffeomorphism of \( S \) to \( S' \) sending \( F_i \) to \( F'_i \) for \( i = 1, \ldots, n \). An \( m \)-subweb of \( W \) is an \( m \)-tuple of members of \( W \).

First let \( n = 3 \) and assume \( W \) is non singular at \( p \). Since the defining 1-forms \( \omega_i, i = 1, 2, 3 \) on a surface are linearly dependent, we may assume \( \omega_1 - 2\omega_2 + \omega_3 \) vanishes identitically on a neighbourhood of \( p \). Then there
exists a unique 1-form \( \theta \) on the complement of \( \Sigma(W) \) such that \( d\omega_i = \theta \wedge \omega_i \) for \( i = 1, 2, 3 \) [1]. The exterior derivative \( d\theta \) is independent of the forms \( \omega_i \) defining \( F_i \) as well as the permutation of the suffix \( i \). \( d\theta \) is called the web curvature form of \( W \) and denoted \( K(W) \). Bott connections of \( F_1, F_2, F_3 \) defined by the transverse dynamics extend to unique affine connection without torsion the so-called Chern connection on the complement of \( \Sigma(W) \) (see §1 for the definition). And the leaves of \( F_i \) are geodesics of the connection.

Chern connection has the connection form \( \Theta = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \) with respect to the coframe \( \omega_1, \omega_2 \) and the curvature form \( d\Theta = \begin{pmatrix} K(W) & 0 \\ 0 & K(W) \end{pmatrix} \) [1,5,10].

A 3-web is hexagonal (or flat) if the web curvature vanishes identically. It is classically known that a hexagonal 3-web is locally diffeomorphic to the 3-web by parallel lines on the plane (see Fig 0. and §1).

A non singular 4-web \( W = (F_1, \ldots, F_4) \) possesses the relative and absolute invariants: web curvature forms of 3-subwebs and the cross ratio of tangents to the leaves of \( W \) passing to a point which is a special case of the basic affinor in higher dimensional webs (see [5] for the definition). The higher covariant derivatives of the cross ratio generate all other absolute invariants [3,4].
We call $n$ curvilinear foliations as well as an $n$-web are associative if their Bott connections extend to equal affine connection, in other words, all 3-subwebs have equal Chern connection on the complement of the singular locus. It is easy to see that $n$ foliations ($3 < n$) are associative if and only if the modulus of tangents to the leaves passing through a point is constant. Clearly if an $n$-web is associative, it is weakly associative, i.e. Chern connections of 3-subwebs have equal curvature form. But the converse is not true in general. Poincaré [3], Mayrhofer [8,9] and Reidemeister [11] proved

**Theorem 0.** Let $W$ be a germ of non singular 4-web on a surface. Assume that all 3-subwebs are hexagonal. Then $W$ is diffeomorphic to a germ of the 4-web formed by 4 pencils of lines on the plane (Fig.1).

3-subwebs of the 4-web in the theorem are hexagonal (curvature vanishes), but their Chern connections are not equal. Henaut[7] gives a simple proof of Theorem 0. Goldberg [4] proved a similar result by a different approach. This paper is devoted to finding all weakly associative $n$-webs.

Before stating our result we prepare some notions. A pencil of meromorphic one forms $P = \{\omega_t\}, \omega_t = (1 - t)\omega_0 + t\omega_1, t \in \mathbb{R}$ defined on $S$ is non singular at $p$ if $\omega_s$ and $\omega_s \wedge \omega_t$ are analytic and non zero at $p$ for distinct $s, t$. We denote the set of those $p$ where $P$ is singular by $\Sigma(P)$. The web
curvature form $K(P)$ for $P$ is the 2-form $d\theta$ defined on the complement of $\Sigma(P)$, where $\theta$ is the unique 1-form called the connection form of $P$ such that $d\omega_t = \theta \wedge \omega_t$. Clearly all members of $P$ are associative and all triples of the members form 3webs which have equal web curvature form $K(P)$.


Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on $S$. In this paper geodesics mean the leaves of the foliations. A geodesic triangle is a smooth triangle $\Delta = \Delta(E_1, E_2, E_3)$ with the edges $E_i$ in a leaf $L_{\sigma(i)} \in F_{\sigma(i)}$ for $i = 1, 2, 3$ transversal at the vertices $V_{jk} = E_j \cap E_k, j \neq k$. Here $\sigma$ is a permutation of $\{1, 2, 3\}$ and the convention $E_{i+3} = E_i$ is used. The orientation of the $\Delta$ and the edge $E_i$ are given by $\partial \Delta = E_1 + E_2 + E_3$ and $\partial E_i = V_{[i, (i+1)]} - V_{[(i-1), i]}$ (see Fig. 2). Define $\sigma(\Delta) = 1$ or $-1$ alternatively if the orientation is clockwise or anti-clockwise.

From later on we assume the permutation $\sigma$ is trivial unless otherwise stated.

Let $W = (F_1, F_2, F_3, F_4)$ be a 4-web. A Schläfli configuration is a quadruple of geodesic triangles $\Delta_1 = \Delta(E_2, E_3, E_4), \Delta_2 = \Delta(E_1', E_3', E_4'), \Delta_3 = \Delta(E_1'', E_2'', E_4'') \subset \Delta_2$ and $\Delta_4 = \Delta(E_1''', E_2''', E_3''') \subset \Delta_1$ with the following properties (see Fig. 3).
(1) The edges with suffix $i$ are contained in a common leaf $L_i \in F_i$ for $i = 1, \ldots, 4$.

(2) $\Delta_j, \Delta_k$ has the common vertex $V_{m,n} = E_m \cap E_n$, where \{j, k, m, n\} = \{1, 2, 3, 4\},

(3) $\Delta_2 + \Delta_4 = \Delta_1 + \Delta_3$, where $\Delta_i$ denotes also the underlying set of $\Delta_i$.

(4) The 3-subweb $W_i$ is non singular on a neighbourhood of $\Delta_i$ for $i = 1, \ldots, 4$.

In other words a Schl"afli configuration is formed by leaves of $F_1, \ldots, F_4$ in general position. The goal of this paper is to prove the following generalization of Theorem 0.
**Theorem 1.** Assume 3-subwebs of a 4-web $W = (F_1, F_2, F_3, F_4)$ are non hexagonal. Then the following conditions are equivalent.

1. $F_i$ is defined by a 1-form $\omega_i$ in a pencil of meromorphic 1-forms $P = \{\omega_t\}$.
2. The cross ratio $C(F_4, F_3, F_2, F_1)$ of tangents to the leaves of $F_i, i = 1, ..., 4$ passing through a point is constant on the complement of $\Sigma(W)$.
3. $F_1, ..., F_4$ are weakly associative: The web curvature form $K(F_1, ..., \hat{F}_i, ..., F_4)$ of the 3-subweb $W_i = (F_1, ..., \hat{F}_i, ..., F_4)$ is independent of $i$.
4. For any Schl"{a}fli configuration $(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$,

$$
\sum_{i=1}^{4} (-1)^i \int_{\Delta_i} K(F_1, ..., \hat{F}_i, ..., F_4) = 0.
$$

5. $F_1, ..., F_4$ are associative: Bott connections of $F_1, ..., F_4$ extend to equal affine connection on the complement of $\Sigma(W)$.

In the last section we prove a generalization of the theorem for $m$-webs of $\mathbb{R}^n$, $n < m$, of codimension one.

All results in this paper remain valid replacing real analyticity with $C^3$-smoothness. The argument is local, so from now on we assume $S$ is a connected domain of $\mathbb{R}^2$.

1. **Bott connection and Chern connection.** Bott connection of a non singular foliation is defined by the differential of the transverse dynamics. To state more precisely in our setting, recall the integrability condition

$$
d\omega_i = \theta \wedge \omega_i,
$$

where $\omega_i$ is the defining one form of $F_i$ and $\omega_1 - 2\omega_2 + \omega_3 = 0$. The 1-form $\theta$ defines the (partial) connection of the normal bundle of the foliation $F_i$ along the leaves as follows. Let $L$ be a leaf of $F_i$, $p, q \in L$, and $C \subset L$ a smooth curve joining $p$ to $q$. The parallel transport $T(X)$ of a vector $X$ normal to $L$ at $p$ along $C$ is defined by the relation

$$
\omega_i(T(X)) = \exp(\int_C \theta) \cdot \omega_i(X).
$$
To extend Bott connection to an affine connection on $S$, consider an (infinitesimally) small geodesic triangle $\Delta$ with vertex $p$. By the transverse dynamics along $C$, $\Delta$ is transported to a unique (infinitesimally small) geodesic triangle $\Delta'$ with vertex $q$ (Fig. 4).

This transportation determines a linear map of the tangent spaces $T_pS$ to $T_qS$. The linear map is defined also for all piecewise geodesics by composition of those linear maps along the geodesic pieces. It is easy to see this transportation determines an affine connection, and the connection form with respect to the coframes $\omega_1, \omega_2$ is $\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$. This connection is called Chern connection of the 3-web $W$. Chern connection is in other words the unique common extension of Bott connections of $F_1, F_2, F_3$. The structure group of the connection is $\mathbb{R}^*$: the group of similar transformations, and the holonomy map along a closed cycle $C$ is

$$\exp(\int_C \theta) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume that $\theta$ is closed, i.e. the web curvature form vanishes identically. Then $\tilde{\omega}_i = \exp(-\int \theta) \cdot \omega_i$ is closed and $\tilde{\omega}_1 - 2\tilde{\omega}_2 + \tilde{\omega}_3 = 0$. By integrating the equation, we obtain the developping map $(\int \tilde{\omega}_1, \int \tilde{\omega}_2, \int \tilde{\omega}_3)$ of $S$ to the hyperplane $H = \{(u_1, u_2, u_3) \in \mathbb{R}^3 | u_1 - 2u_2 + u_3 = 0\}$, which sends the leaves of the web to the lines defined by $u_i =$ const. in $H$. Thomsen (c.f. [3]) proved that the Hexagonality of 3-webs is equivalent to the closure condition of the piecewise geodesic hexagon as in Fig. 5.
In general this hexagon is not closed (Fig. 6).

Now assume that the foliations $F_1, F_2, F_3$ are defined by the level functions $x, y$ and an $f(x, y)$ such that $f(t, 0) = f(0, t) = t$ and $f(t, t) = 2t$: $f(x, y) = x + y + k(x - y)xy + \cdots$. Then the web curvature form for the 3-web of this form is presented as

$$K(W) = \frac{\partial^2}{\partial x \partial y} \left( \log \frac{f_x}{f_y} \right) dx \wedge dy$$

and the return map $R(x)$ as in Fig. 6 is written in a coordinate $x$ on the leaf $L$ centered at $p$ as follows.

$$R(x) = x + kx^3 + \cdots.$$
To see the for of $R$ it suffices to notice $R'(x) = 1 + 3kx^2 + \cdots$ is the linear term of the holonomy at $x$ anti-clockwisely along the "non-closed hexagon" and the area of the "hexagon" is proportional to $3x^2$. In the next section we interprete the curvature in terms of the fake billiard.

2. Transverse dynamics and fake billiard. Let $W = (F_1, F_2, F_3)$ be a non singular 3-web on a surface and assume all leaves are simply connected. Let $L_i$ be a leaf of $F_i$, $p, q \in L_i$ and $j, k \neq i$. The transverse dynamics

$$T_{pq}^{jk} : L_j(p), p \to L_k(q), q$$

is a germ of diffeomorphism of the leaf $L_j(p)$ of $F_j$ passing through $p$ to the leaf $L_k(q)$ of $F_k$ passing through $q$, which assigns to $x \in L_j(p)$ sufficiently close to $p$ the unique intersection point $y \in L_i(x) \cap L_k(q)$. Fake billiard along a boundary of an oriented geodesic triangle $\Delta = \Delta(E_1, E_2, E_3)$ is the return map

$$T_{\partial \Delta}^i = T_{i+2} \circ T_{i+1} \circ T_i : L_{i+1}(v_{i-1,i}), v_{i-1,i} \to L_{i+1}(v_{i-1,i}), v_{i-1,i}$$

where $T_j$ denotes the transverse dynamics along the edge $E_j$

$$T_{v_{j-1,j}v_{j,j+1}}^{j+1,j+2} : L_{j+1}(v_{j-1,j}), v_{j-1,j} \to L_{j+2}(v_{j,j+1}), v_{j,j+1},$$

![Diagram of transverse dynamics and fake billiard](https://example.com/diagram.png)
Clearly $T_{i}^{j\Delta}, i = 1, 2, 3$ are conjugate with each other: $T_{i+1} \circ T_{i+1}^{i} \circ T_{i} = T_{i}^{j\Delta}$. We denote the derivative of $T_{i}^{j\Delta}$ at $v_{i-1,i}$ by $dT_{i}^{j\Delta}$.

**Lemma 1.** Fake billiard along an oriented geodesic triangle $\Delta = \Delta(F_{1}, F_{2}, F_{3})$ has the following derivative at the origin.

$$\partial T_{\Delta} = -\exp\left(-\sigma(\Delta) \int_{\Delta} K(F_{1}, F_{2}, F_{3})\right),$$

where $\sigma(\Delta) = 1$ or $-1$ respectively $\Delta$ is the clockwise orientation or anticlockwise.

**Proof.** Assume $\Delta$ and $F_{1}, F_{2}, F_{3}$ are defined by the level functions $f, x$ and $y$ as in Fig.7 bis.

Then $\sigma(\Delta) = -1$. Let $v_{31} = (a, 0), v_{12} = (0, a)$ and $T_{v_{31}v_{12}}^{22}(a, y) = (0, y')$. 

\[\text{Fig. 7. bis}\]
Then we obtain

\[ \log \left( \frac{dy'}{dy}(0) \right) = \int_{v_{v1} v_{12}} \left( \frac{f_x}{f_y} \right)_y dx \]

\[ = \int_{v_{v1} v_{12}} \left( \frac{f_x}{f_y} \right)_y \frac{dx}{dy} dy \]

\[ = \int_{v_{v1} v_{12}} \left( \frac{f_x}{f_y} \right)_y dy \]

\[ = \int_{v_{v1} v_{12}} \left( \log \frac{f_x}{f_y} \right)_y dy. \]

Let \( T_{v_{12} v_{23}}(0, y') = (x, a). \) Then

(2) \[ \log \frac{dx}{dy'}(y' = a) = \log \frac{f_x}{f_y}(0, a). \]

Let \( T_{v_{12} v_{23}}(x, a) = (x, y''). \) Then

(3) \[ \log \frac{d''}{dx}(x = 0) = \log \frac{f_x}{f_y}(0, 0). \]

From (2) and (3), we obtain

(4) \[ \log \frac{dy''}{dy'} = \log \frac{dy''}{dx} - \log \frac{dy'}{dx} \]

\[ = \log \frac{f_x}{f_y}(0, 0) - \log \frac{f_x}{f_y}(0, a) \]

\[ = \int_{v_{12} v_{23}} \left( \log \frac{f_x}{f_y} \right)_y dy. \]

From (1) and (4)

\[ \log \left( \frac{-dy''}{dy} \right) = \int_{v_{31} v_{12} v_{23}} \left( \log \frac{f_x}{f_y} \right)_y dy \]

\[ = \int_{\Delta} \left( \log \frac{f_x}{f_y} \right)_{xy} dx \wedge dy \]

\[ = \int_{\Delta} K(W). \]
This completes the proof.

Let $W = (F_1, F_2, F_3, F_4)$ be a non singular 4-web on a surface. Define the cross ratio by

$$C(k_1, k_2, l_1, l_2) = \frac{(a_1 - b_1)(a_2 - b_2)}{(a_1 - b_2)(a_2 - b_1)}$$

for the lines $k_i = \{y = a_i x\}, l_i = \{y = b_i x\}, i = 1, 2$, and define the cross ratio of tangents to the leaves $L_i(p) \in F_i$ passing through $p$ by

$$C(p) = C(T_p L_4(p), T_p L_3(p), T_p L_2(p), T_p L_1(p)).$$

From now on we assume $1 < C(k_1, k_2, l_1, l_2) < \infty$ (this holds uniformly on the connected component of the surface).

**Lemma 2.** Let $\Delta_3 = \Delta(E_1, E_2, E_4)$ be a geodesic triangle of the 3-subweb $(F_1, F_2, F_4)$ (Fig.8). Then the following fake billiard along $\partial \Delta_3$

$$\tilde{T}_{\Delta_3} = T_{v_{41} v_{12}}^{34} \circ T_{v_{24} v_{41}}^{33} \circ T_{v_{12} v_{24}}^{43} : L_4(v_{12}), v_{12} \rightarrow L_4(v_{12}), v_{12}$$

has the following derivative at $v_{12}$

$$d\tilde{T}_{\Delta_3} = \frac{C(v_{41})}{C(v_{41}) - 1} (1 - C(v_{24})) \, dT_{\Delta_3}.$$
Proof. Let $f_i$ be a local level function of $F_i$ defined on a neighbourhood of $\Delta_3$. Let

$$T_{v_{41}}^{32} : L_3(v_{41}), v_{41} \to L_2(v_{41}), v_{41}$$
$$T_{v_{24}}^{13} : L_1(v_{24}), v_{24} \to L_3(v_{24}), v_{24}$$

denote the transverse dynamics respectively along the leaves of $F_1, F_2$ such that

$$f_1 \circ T_{v_{41}}^{32} = f_1, \quad f_2 \circ T_{v_{24}}^{13} = f_2.$$ 

It is easy to see

$$d(f_4 \circ T_{v_{41}}^{32}) = (1 - C(v_{v_{41}})) df_4,$$
$$d(f_4 \circ T_{v_{24}}^{13}) = \frac{(C(v_{v_{41}}))}{(C(v_{v_{41}}) - 1)} df_4,$$

from which

$$d(f_4 \circ T_{v_{41}}^{32} \circ T_{v_{24}, v_{41}}^{33} \circ T_{v_{24}}^{13}) = \frac{(C(v_{v_{41}}))}{(C(v_{v_{41}}) - 1)} (1 - C(v_{v_{24}})) df_4.$$

By definition we obtain

$$\tilde{T}_{\Delta_3} = T_{v_{41}}^{24} \circ T_{v_{41}}^{32} \circ T_{v_{24}, v_{41}}^{33} \circ T_{v_{24}}^{13} \circ T_{v_{12}, v_{24}}^{41},$$

from which we obtain the statement.

Lemma 3. Let $\Delta_1, \Delta_2, \Delta_3$ be as in Fig.9. Then

$$d\tilde{T}_{\Delta_3} = dT_{\Delta_2} \cdot dT_{\Delta_1} \cdot C(v_{34}),$$

where $-\Delta_1$ denotes the triangle $\Delta_1$ with reverted orientation.
Proof. Let $T_{\Delta_{3}} v_{13}, T_{-\Delta_{1}} v_{13}, T_{\Delta_{3}} v_{24}$ denote fake billiard along $\partial \Delta_{2}, -\partial \Delta_{1}, \partial \delta_{3}$ starting at $v_{13}, v_{24}$. It is easy to see fake billiard $T_{\Delta_{3}} v_{13} \circ T_{-\Delta_{1}} v_{13}$ is conjugate with

$$\hat{T}_{\Delta_{3}} v_{24} \circ T_{v_{34} v_{24}}^{13} \circ T_{v_{34} v_{13}}^{24} \circ T_{v_{24} v_{34}}^{32} = \hat{T}_{\Delta_{3}} v_{24} \circ T_{v_{34} v_{24}}^{13} \circ T_{v_{34} v_{24}}^{21} \circ T_{v_{24} v_{34}}^{32},$$

where $T_{v_{34}}^{21}$ is defined by $f_{3} \circ T_{v_{34}}^{21} = f_{3}, f_{3}$ being the defining function of $F_{3}$. Differentiating the equality we obtain the statement with the equality.

$$C(v_{34}) \cdot df_{4} \circ T_{v_{34}}^{21} = df_{4}.$$

Lemma 4. Let $\Delta_{1}, \Delta_{2}, \Delta_{3}$ be as in Fig.9. Then

$$dT_{\Delta_{1}} \cdot dT_{-\Delta_{2}} \cdot dT_{\Delta_{3}} = -\frac{C(v_{41}) - 1}{C(v_{41})} \cdot \frac{C(v_{34})}{C(v_{24}) - 1}.$$

Proof. The statement follows from Lemmas 2 and 3.

Proposition 5. Let $(\Delta_{1}, \Delta_{2}, \Delta_{3}, \Delta_{4})$ be a Schl"afli configuration. Then

$$\sum (-1)^{i} \int_{\Delta_{i}} K(F_{1}, \ldots, \hat{F}_{i}, \ldots, F_{4})$$

$$= \log \frac{C(v_{41}) - 1}{C(v_{41})} \cdot \frac{C(v_{23}) - 1}{C(v_{23})} \cdot \frac{1}{C(v_{24}) - 1} \cdot \frac{1}{C(v_{13}) - 1} \cdot \frac{1}{C(v_{34}) \cdot C(v_{12})}.$$

Proof. We may assume $v_{34} = (0,0), F_{3}, F_{4}$ are defined by the coordinate functions $y, x$ and $F_{1}, F_{2}$ by functions $f, g$ respectively. Let $v_{12} = (a, b)$ and $P_{1} = (0, b), P_{2} = (a, 0)$. Let $\square$ denote the geodesic rectangle with the vertices $v_{12}, P_{1}, v_{34}, P_{2}$, and let $\Delta_{1}'$ (resp. $\Delta_{2}'$) denote the geodesic triangle with the vertices $v_{12}, v_{24}, P_{1}$ (resp. $v_{12}, v_{13}, P_{2}$) and let $\Delta_{1}'' = \Delta_{4} + \Delta_{2}', \Delta_{2}'' = \Delta_{3} + \Delta_{1}'$ (see Fig.10).
Then
\[ \int_{\square} K(F_{1,3}F, F_{4}) - \int_{\square} K(F_{2}, F_{3}, F_{4}) = \log \frac{kC(v_{12})}{C(P_{1})C(P_{2})} \]

The alternative sum in the equality of the proposition is

\[ \{-\int_{\triangle_{1}'} K(F_{2}, F_{3}, F_{4}) + \int_{\Delta_{2}''} K(F_{1}, F_{3}, F_{4}) - \int_{\triangle_{4}} K(F_{1}, F_{2}, F_{3})\} \]

By Lemma 1 and Lemma 4

\[ = \log \left\{ -\frac{C(v_{41})(C(v_{41}) - 1)}{(C(v_{24}) - 1)C(v_{41})} \left\{ + \frac{C(v_{12})C(v_{34})}{C(P_{1})C(P_{2})} \right\} \{-\frac{C(v_{41})(C(v_{23}) - 1)}{(C(v_{13}) - 1)C(v_{23})} \right\} \]

\[ = \log \left\{ \frac{C(v_{41}) - 1}{C(v_{41})} \cdot \frac{C(v_{23}) - 1}{C(v_{23})} \cdot \frac{1}{C(v_{24}) - 1} \cdot \frac{1}{C(v_{13}) - 1} \cdot C(v_{34}) \cdot C(v_{12}) \right\} \]
3. **Proof of Theorem 1.** The implications (1) $\rightarrow$ (2), (3) $\rightarrow$ (4), (1) $\rightarrow$ (5) are clear. The implication (1) $\rightarrow$ (3) follows from the uniqueness of the 1-form $\theta(P)$.

**Proof of (5) $\rightarrow$ (3).** For each 3-subweb $W' = (F_i, F_j, F_k)$ if Bott connections of $F_i, F_j, F_k$ extend to equal affine connection, it is Chern connection of $W'$. Therefore common extension of all Bott connections is Chern connection of 3-subwebs.

**Proof of (2) $\rightarrow$ (1).** Let $\omega_1, \omega_2$ be meromorphic 1-forms defining $F_1, F_2$ respectively. Then $\omega_3$ is presented as $\omega_3 = \lambda \omega_1 + \mu \omega_2$ with meromorphic functions $\lambda, \mu$ on the surface $S$. Now we may assume $\lambda = \mu = 1/2$ replacing $\omega_1, \omega_2$. Similarly assume $F_4$ is defined by $\omega_4 = \lambda' \omega_1 + \mu' \omega_2$. Then the cross ratio $C(F_4, F_3, F_2, F_1) = -\lambda'/\mu'$ of the leaves of $F_4, F_3, F_2, F_1$ is constant $c \neq 0$ by assumption. Therefore we may assume $\omega_4 = c \omega_1 + \omega_2$.

**Proof of (4) $\rightarrow$ (3).** Consider the Schlaffi configuration such that $v_{12} = v_{24} = v_{41}$ as in Fig.11.

By the hypothesis (4)

$$\int_{\Delta_1} K(W_1) = \int_{\Delta_2} K(W_2) + \int_{\Delta_2} K(W_2),$$

where $K(W_i)$ denotes the web curvature form of the 3-subweb of $W$ forgetting the $i$-th foliation. This tells that the integral of the curvature form over a geodesic triangle can be calculated by decomposing into small geodesic
triangles. Now decompose the $\Delta_1$ into infinitely many geodesic triangles of the same type as $\Delta_2$ as in Fig.12.

\[\text{Fig.12}\]

Then it follows that

\[
\int_{\Delta_1} K(W_1) = \int_{\Delta_1} K(W_2),
\]

from which $K(W_1) = K(W_2)$.

Proof of (3), (4) $\rightarrow$ (2). We may assume $W$ is non singular and $F_1, F_2, F_3, F_4$ are defined by level functions $f, g$ and the coordinate functions $y, x$ respectively. By a suitable coordinate transformation of the $y$ we may assume $f(t, 0) = f(0, t) = g(t, 0) = t$, and applying Poincaré linearization theorem to the dynamics $t \rightarrow g(0, t)$ we may assume $g(0, t) = kt$. Here $k$ is the cross ratio $C(F_4, F_3, F_2, F_1)$ at the origin, and $f, g$ are of the form

\[
f(x, y) = x + y + mxy + O,
g(x, y) = x + kx + nxy + O',
\]

$O, O'$ being the remainder terms of $x, y$ of order 3 which vanishes identically on the $x$ and $y$ axes. It is easy to see that only similar transformations $(x, y) \rightarrow (cx, cy)$ respect this normal form. Therefore the ratio $(m : n)$ as well as $k$ gives rise to an absolute invariant of 4-webs. By definition, the cross ratio at $(x, y)$ is

\[
C(x, y) = C(F_4, F_3, F_2, F_1)(x, y) = \frac{(k + nx)(1 + my)}{(1 + ny)(1 + mx)} + O''.
\]
Denote \( C(x, 0) = C(x) \) and \( C(0, y) = D(y) \) for simplicity. On the \( x \)-axis this restricts to

\[
C(x) = \frac{k + nx}{1 + mx} + \cdots = k + (n - km)x + \cdots.
\]

By Proposition 5 applied to the Schläfli configuration as in Fig. 11 we obtain

\[
\frac{C(kx)}{C(x)} = \frac{C(kx) - 1}{C(x) - 1}.
\]

With the initial condition \( C(0) = k, C'(0) = n - km \) this equation admits a unique solution

\[
C(x, 0) = C(x) = k + \frac{(n - km)x}{1 - \frac{(n-km)x}{k-1}}.
\]

Similarly we obtain

\[
C(0, y) = D(y) = k + \frac{k(m - n)y}{1 - \frac{k(m-n)y}{k-1}}.
\]

By the hypothesis (3) \( K(W_1) = K(W_2) \). Hence

\[
\frac{\partial^2}{\partial x \partial y} \log \left( \frac{f_x}{f_y} \right) = \frac{\partial^2}{\partial x \partial y} \log \left( \frac{g_x}{g_y} \right),
\]

from which

\[
\frac{\partial^2}{\partial x \partial y} \log C(x, y) = \frac{\partial^2}{\partial x \partial y} (\log (\frac{-f_x}{f_y}) - \log (\frac{-g_x}{g_y})) = 0.
\]

Therefore

\[
C(x, y) = C(x)D(y)/k.
\]

Assume a 4-web \( W' = (F_1', \ldots, F_4') \) (not necessarily of the above normal form) is defined by the level functions \( f, g, y, x \), and assume the cross ratio function \( C'(x, y) \) is a product of two linear fractions \( C'(x), D'(y) \) of \( x \) and \( y \). Let \( \phi \) and \( \psi \) be the diffeomorphisms of the \( x \)-axis and the \( y \)-axis, which
normalizes $W'$ to the above normal form. Since the cross ratio is an absolute invariant, we obtain

$$C'(x, y) = C(\phi(x), \psi(y)).$$

First assume $n \neq m, km$. Then $C(x), D(y)$ are not constant and it follows from the above equality that $\phi, \psi$ are also linear fractions. Since for the normal form the transverse dynamics of $F_1, F_2$ sending the $x$-axis to the $y$-axis respecting the origin are linear maps, those dynamics for $W'$ are linear fractions. This argument applies to germs of the normal form $W$ at all points on a neighbourhood of the origin, and implies that the transverse dynamics sending horizontal lines to vertical lines are all linear fractions. It is easy to see that this implies also the transverse dynamics of $F_1, F_2$ among the horizontal lines as well as the vertical lines are also linear fractions in the coordinates $x$ and $y$ (Fig.13).

Therefore we may assume that the level functions $f, g$ are linear fractions in $x, y$ when it is restricted to the horizontal and the vertical lines defined by the coordinate functions $y, x$ respectively. We may write as

$$f = \frac{a(x)y + b(x)}{1 - c(x)y} = b + (a + bc)y + (ac + bc^2)y^2 + \cdots,$$

$b$ being a linear fraction of $x$. The Schwarzian derivative of $f$ in $x$ is

$$[f : x] = \frac{f_{xxx}}{f_x} - \frac{3}{2} \left( \frac{f_{xx}}{f_x} \right)^2.$$
and write it as a series in $y$ with function coefficients

\[ S_0 + S_1 y + S_2 y^2 + \cdots. \]

The initial term is Schwarzian derivative of $b$, and $S_1, S_2$ are Schwarzian derivatives of the first two and three terms of the expansion of $f$. Since $f$ is a linear fraction in $x$ for all fixed $y$, all these coefficients vanish. It then follows that $a, b$ and $c$ are rational functions of $x$, hence $f$ is a rational function of $x, y$. Again since $f$ is linear fraction in $x$ and $y$, $f$ is of the form

\[ f(x, y) = \frac{c_1 xy + c_2 x + c_3 y + c_4}{c'_1 xy + c'_2 x + c'_3 y + c'_4}. \]

Now we will show that the curvature form $K(F_1, F_3, F_4)$ of the 3-subweb $(F_1, F_3, F_4)$ vanishes at the origin. (This can be also seen by straight forward calculation.) Recall the rotation map $R$ defined as in §1, Fig 6. Since $R$ is defined by composing the various transverse dynamics respecting the origin, all of which are linear fractions by the above form of $f$, $p$ is also a linear fraction of $x$. On the other hand the rotation map has the expansion $R(x) = x + k(0, 0)x^3 + \cdots$ and the second order term is missing. Therefore $R$ is the identity and in particular the web curvature vanishes at the origin. This argument applies at all point in the domain of definition $S$ to imply that the curvature form vanishes identically. This contradicts the hypothesis of the theorem.

Next assume $n = m$ and $n \neq km$. Then by the same argument as the above case $D(y)$ is constant and

\[ C(x, y) = C(x, 0) = C(x) = k + \frac{(n - km)x}{1 - \frac{(n - km)x}{k-1}} \]

is not constant. In this case exchange the roles of $F_4$ and $F_1$ (or $F_2$) in the above argument. Then it reduces to the first case $n \neq m, km$, since $C$ is not constant on the leaves of $F_1, F_2$ and also $F_3$.

Similar argument applies to the case $n = km$ and $n \neq m$. The rest is the case $m = n = 0$. Clearly this implies that the cross ratio function is locally constant. By the analyticity, the cross ratio is constant on the domain of definition. This completes the proof of Theorem 1.
4. Proof of Theorem 0 and associative webs and weakly associative webs of codimension 1 in higher dimension.

First we give an elementary proof of Theorem 0. By the form of $f$, the foliation $F_3$ is a pencil of conics with two base points $p_1, p_2$. By a Möbius transformation $(\phi(x), \psi(y))$ sending $p_1$ to $(0,0)$ and $p_2$ to $(\infty, \infty)$, we may assume $F_3$ is a pencil of lines with base point $(0,0)$. In particular a leaf of $F_3$ meet a leaf of $F_1, F_2$ at a single point. Now consider the 4-th foliation $F_4$. The same argument as in the previous section applies to imply that $F_4$ is a pencil of conics and a leaf meets each member of the pencils of lines $F_1, F_2, F_3$ at a single point. It then follows that the leaves of $F_4$ are lines hence $F_4$ is a pencil of lines. This completes the proof of Theorem 0.

Let $W = (F_1, \ldots, F_{n+1})$ be a non singular (in general position) $(n+1)$-web of codimension 1 on an open subset of $\mathbb{R}^n$. Chern connection $\gamma_i$ of $W$ is an extention of Bott connection of the $i$-th foliation $F_i$ (see [5] for the definition). In the case $n = 3$ twice the average of the curvature forms of $\gamma_1, \ldots, \gamma_{n+1}$ had been already found by Blaschke[1], so we call it Blaschke curvature form. Let $F_i$ be given by the $i$-th coordinate function of $\mathbb{R}^n$ for $i = 1, \ldots, n$ and let $F_{n+1}$ be defined by a function $f$. Define Blaschke curvature form by

$$d\Gamma = d(- \sum_{i=1,\ldots,n} (\log f_{x_i}) x_i dx_i) = \frac{1}{2} \sum_{i,j=1,\ldots,n} (\log \frac{f_{x_i}}{f_{x_j}}) x_i x_j dx_i \wedge dx_j.$$ 

It is easily seen that $d\Gamma$ restricts to the web curvature form of the 3-web on $x_i x_j$-plane cut out by $x_i, x_j$ and $f$. Conversely this property characterizes Blaschke curvature form.

**Proposition 6** [1]. Blaschke curvature form restricts to the web curvature form of the 3-web on the intersections of the leaves of $n-2$ foliations cut out by the remaining 3 foliations.

We call $n+2$ foliations of codimension 1 are associative if the modulus of tangent hyperplanes to the leaves of foliations passing through a point is constant. And we call $n+2$ foliations of codimension 1 are weakly associative if for all $n+1$-subwebs Blaschke curvature forms are equal. It is not difficult to see that if the modulus of the tangent hyperplane is constant, $n+2$ foliations are associative hence weakly associative. In the following we discuss the converse.
Assume that Blaschke curvature forms are equal. Then the curvilinear 4-web on the intersection of $n - 2$ foliations cut out by the remaining 4 foliations is weakly associative by Proposition 2. Assume that those curvilinear 4-webs are not hexagonal i.e. all 3-subwebs are not hexagonal. By Theorem 1 the cross ratio of tangents to the leaves of curvilinear 4-web is constant on each leaves of the $(n - 2)$-intersection. We claim this implies that the cross ratio of the 4-web is constant on the connected domain of definition. It then follows that the modulus of the tangent hyperplanes is constant on the domain of definition. To prove the claim it suffices to prove for the case $n = 3$, by induction on $n$. For simplicity assume that 5 foliations are defined by the coordinate functions $x, y, z$ and $f$ and $g$. By the above argument the cross ratio of the curvilinear 4-web on each level hyperplane of $z$ is constant. Notice that the cross ratio of the curvilinear 4-web on $z$-level hyperplane is determined by those on the level hyperplanes of $y$ and $z$. Those cross ratios are constant on the $z$-axis by the same argument. Therefore the cross ratio of the 4-web on the $z$-level planes is constant on $z$-axis hence the modulus of tangent hyperplanes as well as the cross ratio is locally constant. We can state the result in the following general form.

**Theorem 7.** Assume $m$ foliations $F_1, \ldots, F_m$, $n + 2 \leq m$ of codimension 1 on an $n$-manifold are non singular, in general position and also the curvilinear 4-webs on the intersection of $n - 2$ foliations cut out by the remaining 4 foliations are not hexagonal. Then the following conditions are equivalent.

1. $F_1, \ldots, F_m$ are associative: the modulus of tangent planes to the leaves of $F_1, \ldots, F_m$ passing through a point is constant.

2. $F_1, \ldots, F_m$ are weakly associative: Blaschke curvature forms of $(n+1)$-subwebs are equal.

For the hexagonal case the statement is not true. In fact $m$ pencils of hyperplanes on $\mathbb{R}^n$ satisfies (2) but (1). The author does not know if there exist other such examples. It seems imporant to classify all such webs, generalizing Theorem 0.

Clearly a non singular associative $m$-web of codimension 1 on an $n$-manifold, $n < m$, is defined by an $m$-tuple of members of $n$-dimensional linear family of one forms $L = \{\omega_v\}_{v \in \mathbb{R}^n}$. It is easily seen that if the $m$ points in the projectivization $PL = P^{n-1}$ are non degenerate and not contained in a quadric hypersurface, all members of $L$ are integrable by Frobenius theorem. We say such $m$ foliations are generic. In [10] the author proved
Proposition 8. If all members of $L$ are integrable and $3 \leq n$, there exists unique closed one form $\theta$ such that

$$d\omega = \theta \wedge \omega.$$ 

By Proposition 8 we obtain

Proposition 9. If linearly independent integrable one forms $F_1, \ldots, F_m$ (possibly singular) on an $n$-manifold are weakly associative and generic, then the $m$-web $(F_1, \ldots, F_m)$ is parallelizable at non singular point.

Here a non singular $m$-web $(F_1, \ldots, F_m)$ is parallelizable if it is locally diffeomorphic to the $m$-web by $m$ foliations by parallel hyper planes in $\mathbb{R}^n$. In the paper [10] a more detailed suruction of the parallelizable webs is investigated.

REFERENCES

12. Thomsen, See [3].

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