数理解析研究所講究録
955巻 1996年 80-92

Non-vanishing Wronskian determinants and Riemann Problem for Hypergeometric Function F_D

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0. Introduction. According to M. Yoshida at Kyushu University, hypergeometric functions (see [KF] for example) are very famous but not so familiar. Therefore we begin with some elementary facts. It is defined by Gauss's hypergeometric series

$$F(\alpha,\beta,\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha,n)}{(\gamma,n)}\frac{(\beta,n)}{(1,n)}x^{n},$$

where $(\alpha,n)=\alpha(\alpha+1)\cdots(\alpha+n-1)=\dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$. It has an integral representation and satisfies Euler's differential equation

$$x(1-x)F''+[\gamma-(\alpha+\beta+1)x]F'-\alpha\beta F=0. \tag{0.1}$$

The solutions of (0.1) are locally holomorphic on the Riemann sphere punctured at three regular singularities $0,1,\infty$, and those of the determining equation at each point are

$$0,1-\gamma\ (x=0);\ 0,\gamma-\alpha-\beta\ (x=1);\ \alpha,\beta\ (x=\infty).$$

In 1880, Appel [A] generalized it to two variables and defined four series

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)}{(\gamma,m+n)}\frac{(\beta,m)(\beta',n)}{(1,m)(1,n)}x^{m}y^{n},$$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)}{(\gamma,m)(\gamma',n)}\frac{(\beta,m)(\beta',n)}{(1,m)(1,n)}x^{m}y^{n},$$

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m)(\alpha',n)}{(\gamma,m+n)}\frac{(\beta,m)(\beta',n)}{(1,m)(1,n)}x^m y^n,$$

$$F_4(\alpha,\beta,\gamma,\gamma';x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha,m+n)}{(\gamma,m)(\gamma',n)}\frac{(\beta,m+n)}{(1,m)(1,n)}x^m y^n,$$

Each of them satisfies a Pfaffian equation and has also integral representations.

In 1893, G. Lauricella [L], generalizing them to n variables, defined also four series F_A, F_B, F_C, F_D. The most interesting one among them is

$$\begin{aligned}&F_D(\alpha,\beta_1,\cdots,\beta_n,\gamma;x_1,x_2,\cdots,x_n)\\&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\alpha,m_1+\cdots+m_n)}{(\gamma,m_1+\cdots+m_n)}\frac{(\beta_1,m_1)\cdots(\beta_n,m_n)}{(1,m_1)\cdots(1,m_n)}x^{m_2}\cdots x^{m_n}.\end{aligned}$$

It coincides with $F_1(\alpha,\beta_1,\beta_2,\gamma;x_1,x_2)$ if $n=2$ and with $F(\alpha,\beta,\gamma;x)$ if $n=1$, and satisfies the Pfaffian differential equation,

(0.2)

$$\begin{cases} x_i(x_i-1)\partial_i^2F+\left[x_i(x_i-1)\displaystyle\sum_{1\le\alpha\le n,\alpha\ne i}\frac{1-\lambda_\alpha}{x_i-x_a}+\lambda_{0i}-1+(2-\lambda_{0i}-\lambda_{in+1})x_i\right]\partial_iF\\ \qquad\qquad +(\lambda_i-1)\displaystyle\sum_{1\le\alpha\le n,\alpha\ne i}\frac{x_\alpha(x_\alpha-1)}{x_i-x_\alpha}\partial_\alpha F+\lambda_\infty(1-\lambda_i)F=0\\ (x_i-x_j)\partial_i\partial_jF+(\lambda_j-1)\partial_iF-(\lambda_i-1)\partial_jF=0 \qquad (i\ne j)\end{cases},$$

where $\lambda_0=1-\gamma+\sum_{i=1}^{n}\beta_i$, $\lambda_i=1-\beta_i\ (1\le i\le n)$, $\lambda_{n+1}=\gamma-\alpha$, $\lambda_\infty=\alpha$, $\lambda_{ij}=\lambda_i+\lambda_j-1$ and $\partial_i=\dfrac{\partial}{\partial x_i}$.

In 1857, Riemann [R] gave $F(\alpha,\beta,\gamma;x)$ a new point of view. He studied the problem called after him and proved that the function which has the same singularities as the hypergeometric function reduces to $F(\alpha,\beta,\gamma;x)$. His method was to construct the differential equation (0.1) which F satisfies.

Generalizing his theory to n variable case, there arises the

Conjecture. If a function F has the same singularities as F_D, then F is equal to F_D, that is, F satisfies the Pfaffian equation (0.2).

This is true. As for exact statements, see paragraph 3. The main idea to prove is the following:

Let F be any branch and $f_0, f_1, \cdots, f_n$ be some linearly independent branches. Then we have differential equations

$$W(f_0, f_1, \cdots, f_n, F; 1, \partial_1, \cdots, \partial_n, \partial_i\partial_j) = 0$$

for any i, j ; as to the notation $W(\cdots)$, see (1.1). By the cofactor development with respect to the last column, we have

$$A_{ij,0}F + A_{ij,1}\partial_1 F + \cdots + A_{ij,n}\partial_n F + A\partial_i\partial_j F = 0, \tag{0.3}$$

where the coefficients are determined explicitly by

$$A_{ij,k} = W(f_0, f_1, \cdots, f_n; 1, \partial_1, \cdots, \hat{\partial}_k, \cdots, \partial_n, \partial_i\partial_j)$$

and

$$A = W(f_0, f_1, \cdots, f_n; 1, \partial_1, \cdots, \partial_n).$$

This conjecture was examined by Picard [P] for Appell's F_1 in 1881 and Terada [T] for Lauricella's F_D in 1973. In 1993, Kato [Ka] proved it for F_4.

But Deligne, in a letter to the author, pointed out the imcompliteness of the proof: that is, in order to show the equation (0.3) is not trivial, $A \neq 0$ must be demonstrated. Kato evitated it by stronger assumptions. So we give here a proof of the non-trivialness of (0.3). The essential tool is:

For a given finite dimensional vector space generated by germs of holomorphic functions, there exists a regular Wronskian matrix which is as simple as possible, and, by using it, one can construct a convenient non-trivial system of partial differential equations.

1. Preliminaries. At first basic definitions and notations are collected.

Let $\mathcal{M}$ be the set of germs of meromorphic functions at the origine of $C^n(x_1, x_2, \cdots, x_n)$ and $\mathcal{D}$ be the set of differential opertors of the form

$$T := \partial_1^{d_1} \partial_2^{d_2} \cdots \partial_n^{d_n}.$$

And $d(T) := d_1 + d_2 + \cdots + d_n$ will be called the degree of T.

$\mathcal{V}_m$ means the set of $(m+1)$-dimensional vector space over C which is generated by $m+1$ elements of $\mathcal{M}$ and $\mathcal{L}(f_0, f_1, \cdots, f_m)$ means the element of $\mathcal{V}_m$ having $\{f_0, f_1, \cdots, f_m\}$ as a base.

(1.1) **Definition.** For $f_0, f_1, \cdots, f_m \in \mathcal{M}$ and differential operators $T_0, T_1, \cdots, T_m$, we will call

$$W(f_0, f_1, \cdots, f_m; T_0, T_1, \cdots, T_m) := \begin{vmatrix} T_0 f_0 & T_0 f_1 & \cdots & T_0 f_m \\ T_1 f_0 & T_1 f_1 & \cdots & T_1 f_m \\ & \cdots & \cdots & \\ T_m f_0 & T_m f_1 & \cdots & T_m f_m \end{vmatrix}$$

the Wronskian of $f_0, f_1, \cdots, f_m$ with respect to $T_0, T_1, \cdots, T_m$.

And a sequence $T_0, T_1, \cdots, T_l$ of differential operators will be called regular with respect to $L \in \mathcal{V}_m$ or L -regular if there exist $f_0, f_1, \cdots, f_l \in L$ such that

$$W(f_0, f_1, \cdots, f_l; T_0, T_1, \cdots, T_l) \neq 0.$$

(1.2) **Definition.** A base of $L \in \mathcal{V}_m$ defines an analytic mapping from a Zarisky open set of a neighborhood of the origine to the m -dimensional projective space. The rank of this mapping is well-defined, which will be called the rank of L and expressed by $r = r(L)$.

(1.3) **Definition.** For two elements $T = \partial^{d_1} \partial^{d_2} \cdots \partial^{d_n}, T' = \partial^{d'_1} \partial^{d'_2} \cdots \partial^{d'_n} \in \mathcal{D}$, we will say $T < T'$ if one of the following conditions (1),(2) holds.

(1) $d(T) < d(T')$

(2) $d(T) = d(T')$ and, by the lexicographical order for the sequences of exponents, $(d_1, d_2, \cdots d_n) < (d'_1, d'_2, \cdots d'_n)$ holds.

Thus $\mathcal{D}$ can be regarded as a totally ordered set.

(1.4) **Definition.** A finite set $\mathcal{O} = \{T_0, T_1, \cdots, T_m\}$ of elements of $\mathcal{D}$ is said an order ideal, if $T \in \mathcal{O}$ and $T > T' \in \mathcal{D}$ imply $T' \in \mathcal{O}$.

2. Regular order ideal and differential equations. Here it will be shown that every element of $L \in \mathcal{V}_m$ satisfies some differential equations which will be used to solve the Riemann problem for hypergeometric function of Lauricella and will be also useful for similar problems for other functions and for general theories of holonomic systems of partial differential equations.

(2.1) **Lemma.** Let a sequence $T_0, T_1, \cdots, T_{l-1} \in \mathcal{D}$ be regular with respect to $L \in \mathcal{V}_m$ $(l \leq m)$, then there exists $T_l := \partial_i T_j \in \mathcal{D}$ $(1 \leq i \leq n,\ 0 \leq j \leq l-1)$ such that the sequence $T_0, T_1, \cdots, T_l$ is also L-regular.

Proof. By the assumption there exist linearly independent elements $f_0, f_1, \cdots, f_l \in \mathcal{M}$ such that $W(f_0, f_1, \cdots, f_{l-1}; T_0, T_1, \cdots, T_{l-1}) \neq 0$. And let

$$W(f_0, f_1, \cdots, f_l; T_0, T_1, \cdots, T_{l-1}, \partial_i T_j) = 0 \tag{1}$$

holds for any i and j $(1 \leq i \leq n,\ 0 \leq j \leq l-1)$.

Let $C_0, C_1, \cdots, C_{l-1} \in \mathcal{M}$ be the solution of the simultaneous equations

$$\begin{cases} T_0 f_l = C_0 T_0 f_0 + C_1 T_0 f_1 + \cdots + C_{l-1} T_0 f_{l-1} \\ T_1 f_l = C_0 T_1 f_0 + C_1 T_1 f_1 + \cdots + C_{l-1} T_1 f_{l-1} \\ \cdots\cdots\cdots\cdots\cdots \\ T_{l-1} f_l = C_0 T_{l-1} f_0 + C_1 T_{l-1} f_1 + \cdots + C_{l-1} T_{l-1} f_{l-1} \end{cases} \tag{2}$$

As $W(f_0, f_1, \cdots, f_{l-1}; T_0, T_1, \cdots, T_{l-1}) \neq 0$, C_k are uniquely determined. In fact, by the formula of Cramer, we have

$$C_k = \frac{W(f_0, \cdots, f_{k-1}, f_l, f_{k+1}, \cdots, f_{l-1}; T_0, \cdots, T_{l-1})}{W(f_0, \cdots, f_{l-1}; T_0, \cdots, T_{l-1})}$$

By (1) and (2), the following equations are also true for any i ($1 \leq i \leq n$).

$$\begin{cases} \partial_i T_0 f_l = C_0 \partial_i T_0 f_0 + C_1 \partial_i T_0 f_1 + \cdots + C_{l-1} \partial_i T_0 f_{l-1} \\ \partial_i T_1 f_l = C_0 \partial_i T_1 f_0 + C_1 \partial_i T_1 f_1 + \cdots + C_{l-1} \partial_i T_1 f_{l-1} \\ \cdots\cdots\cdots\cdots\cdots \\ \partial_i T_{l-1} f_l = C_0 \partial_i T_{l-1} f_0 + C_1 \partial_i T_{l-1} f_1 + \cdots + C_{l-1} \partial_i T_{l-1} f_{l-1} \end{cases} \tag{3}$$

Operate ∂_i on each equation of (2) and compare with (3). Then we have

$$\begin{cases} (\partial_i C_0) T_0 f_0 + (\partial_i C_1) T_0 f_1 + \cdots + (\partial_i C_{l-1}) T_0 f_{l-1} = 0 \\ (\partial_i C_0) T_1 f_0 + (\partial_i C_1) T_1 f_1 + \cdots + (\partial_i C_{l-1}) T_1 f_{l-1} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ (\partial_i C_0) T_{l-1} f_0 + (\partial_i C_1) T_{l-1} f_1 + \cdots + (\partial_i C_{l-1}) T_{l-1} f_{l-1} = 0 \end{cases}.$$

As $W(f_0, f_1, \cdots, f_{l-1}; T_0, T_1, \cdots, T_{l-1}) \neq 0$, we have $\partial_i C_k = 0$ for any i and k . Therefore all C_k are constant, which contradicts to the linear independence of $f_0, f_1, \cdots f_l$.

The following theorem is due to Noumi ([N],Theorem 1.1). But, as his proof is not so easy to understand, we will prove it again elementarily.

(2.4) **Theorem.** For $L \in \mathcal{V}_m$, there exist an order ideal $\{T_0, T_1, \cdots, T_m\}$ which is regular with respect to L.

Proof. Put T_0 = identity, and, after the L-regular order ideal $\mathcal{O}' = \{T_0, T_1, \cdots, T_{l-1}\}$ is determined, we can, by lemma (2.1), choose as T_l the minimal element of the form $\partial_i T_j \in \mathcal{D}$ such that $\mathcal{O} = \{T_1, \cdots, T_l\}$ is L-regular. Put $T_l = \partial_i T_j$. If $\mathcal{O}$ is not an order ideal, there exist b and

k such that $i \neq b$, $T_j = \partial_b T_k$ and $\partial_i T_k \notin \mathcal{O}'$. Because $T_l = \partial_i \partial_b T_k > \partial_i T_k \notin \mathcal{O}'$, there exist $f_0, f_1, \cdots, f_l$ such that $W(f_0, f_1, \cdots, f_{l-1}; T_1, \cdots, T_{l-1}) \neq 0$ and $W(f_0, f_1, \cdots, f_{l-1}; T_1, \cdots, T_{l-1}, \partial_i T_k) = 0$. Therefore, for some $A_a \in \mathcal{M}$, $\partial_i T_k \sim \sum_{a=0}^{l-1} A_a T_a$ holds, which means $\partial_i T_k f = \sum_{a=0}^{l-1} A_a T_a f$ for any $f \in L$.

Now as $T_l = \partial_b \partial_i T_k \sim \sum_{a=0}^{l-1} (\partial_b A_a) T_a + \sum_{a=0}^{l-1} A_a (\partial_b T_a)$, there exists $p (= 0, 1, \cdots, l-1)$ such that $T_0, T_1, \cdots, T_{l-1}, \partial_b T_p$ is regular. By the minimality of T_l, $\partial_b T_p > \partial_b \partial_i T_k = T_l$ and therefore $T_p > \partial_i T_k$ holds. But, as $\partial_i T_k \notin \mathcal{O}'$ implies the regularity of $T_0, T_1, \cdots, T_{p-1}, \partial_i T_k$, it contradicts to the minimality of T_p, which completes the proof.

(2.5) **Corollary.** Let $L = \mathcal{L}(f_0, f_1, \cdots, f_m) \in \mathcal{V}_m$, $\{T_k\} \subset \mathcal{D}$ $(0 \leq k \leq m)$ be the L-regular order ideal just constructed above and T be any element of $\mathcal{D}$. Then any $F \in L$ satisfies the following differential equation with $A \neq 0$

$$A_0 F + A_1 T_1 F + \cdots + A_m T_m F + ATF = 0, \tag{4}$$

where $A_k = W(f_0, f_1, \cdots, f_m; T_0, \cdots, T_{k-1}, T_m, T_{k+1}, \cdots, T_m)$ $(0 \leq k \leq m)$

$$A = W(f_0, f_1, \cdots, f_m; T_0, T_1, \cdots, T_m).$$

and

Moreover, if r is the rank of $\mathcal{L}$ and T is of degree 1, we have

$$A_{r+1} = A_{r+2} = \cdots = A_m = 0.$$

Therefore (4) is a partial differential equation of first order, and the coefficients can be calculated explicitly by means of T_k and $f_k (0 \leq k \leq m)$.

Proof. For any $F \in L$, $m+2$ elements of L being linearly dependent, we have

$$W(f_0, f_1, \cdots, f_m, F; 1, T_1, \cdots, T_m, T) = 0. \tag{5}$$

As the regularity of $T_0, T_1, \cdots, T_m$ means $A \neq 0$, the statements follow imediately by the cofactor development of the left side of (5) with respect to the last column. The second part is evident by the definition of the rank and by the fact that $W(f_0, f_1, \cdots, f_{r+1}; 1, T_1, \cdots, T_r, T) = 0$ holds for any $r+2$ elements $f_0, f_1, \cdots, f_{r+1}$ of L.

3. The Wronskian of the hypergeometric function F_D . Let X be the product of n Riemann spheres with inhomogeneous coordinates $x_1,\ x_2,\ \cdots,\ x_n$ and

$$D := X - \bigcup_{0\le i\le j\le\infty} S_{ij}$$

be a domain on X, where $x_0 = 0$, $x_{n+1} = 1$, $x_\infty = \infty$ and

$$S_{ij} = \{x_i = x_j\}\ (i, j = 0, 1, \cdots, n+1, \infty, i \neq j).$$

And, in the sequal, let $\mathcal{F}$ be a locally constant sheaf on D, whose stalk at $x \in D$ is a vector space $\mathcal{F}_x$ over $\boldsymbol{C}$ generated by $n+1$ linearly independent germs of holomorphic functions, and $f_0, f_1, \cdots, f_n$ be linearly independent sections of $\mathcal{F}$ on a simply connected domain D_0 made of D by means of slitting.

(3.1) **Definition.** We will say that a sheaf $\mathcal{F}$ on D is of exponent $(1, n; \lambda,\ \mu)$ on S_{ij}, if, for every generic point $\xi \in S_{ij}$, there exist a neighbourhood U of ξ, a defining function x_{ij} of S_{ij} on U and holomorphic functions $g_0, g_1, \cdots, g_n$ such that, at every point x of $U - S$, $\mathcal{F}_x$ is generated by following functions

$$\begin{cases} x_{ij}^{\lambda} g_0,\ x_{ij}^{\mu} g_1,\ x_{ij}^{\mu} g_2, \cdots, x_{ij}^{\mu} g_n & (\lambda - \mu \notin \boldsymbol{Z}), \\ x_{ij}^{\lambda} g_0 \log x_{ij} + g_1,\ x_{ij}^{\lambda} g_0,\ x_{ij}^{\mu} g_2, \cdots, x_{ij}^{\mu} g_n & (0 \geq \lambda - \mu \in \boldsymbol{Z}), \\ x_{ij}^{\mu} g_0 \log x_{ij} + x_{ij}^{\lambda} g_1,\ x_{ij}^{\mu} g_0,\ x_{ij}^{\mu} g_2, \cdots, x_{ij}^{\mu} g_n & (0 < \lambda - \mu \in \boldsymbol{Z}). \end{cases}$$

(3.2) **Definition.** We will say that $\mathcal{F}$ is of rank r if the rank of the stalk $\mathcal{F}_x$ at a point x is r, and that a sequence of differential operators is regular with respect to $\mathcal{F}$ if it is so with respect to $\mathcal{F}_x$. Not depending on x, they are well defined.

(3.3) **Theorem**. Assume that a sheaf $\mathcal{F}$ is of exponent $(1,n;\lambda_{ij},0)$ $(0 \leq i < j \leq n+1)$ on S_{ij} and $(1,n;\lambda_\infty, 1-\lambda_i)$ on $S_{i\infty}(1 \leq i \leq n)$ where $\lambda_{ij} = \lambda_i + \lambda_j - 1$ and $\lambda_0, \lambda_1, \cdots, \lambda_{n+1}, \lambda_\infty$ are complex constants with $\sum_{i=0}^{\infty} \lambda_i = n+1$ and $\lambda_i \notin \boldsymbol{Z}$. Then the sequence $1, \partial_1, \cdots, \partial_n$ is regular with respect to $\mathcal{F}$.

Before demonstrating, we prepare some propositions and definitions.

(3.4) **Definition.** A rational function will be said of order d on S_{ij} if $\frac{f}{x_{ij}^d}$ is holomorphic at generic points, and of strict order d if it is of order d but not of order $d+1$.

(3.5) **Proposition**. Let $f_0, f_1, \cdots, f_n$ be linearly independent sections of $\mathcal{F}$ on D_0 and $T_k := \partial_1^{d_{k1}} \partial_2^{d_{k2}} \cdots \partial_n^{d_{kn}}$ $(0 \leq k \leq n)$ be differential operators, then

$$W(f_0, f_1, \cdots, f_n; T_0, T_1, \cdots, T_n) \cdot \prod_{0 \leq i < j \leq n+1} (x_i - x_j)^{-\lambda_{ij}}$$

is a rational function and is of order $-\max\{d_{ai} + d_{aj} \mid 1 \leq a \leq n\}$ on S_{ij} $(1 \leq i \leq n, 0 \leq j \leq n+1, i \neq j)$ and of order $n + \sum_{a=1}^{n} d_{ai}$ on $S_{i\infty}$ $(1 \leq i \leq n)$.

Proof. Generally, if $\mathcal{F}$ is of exponent $(1,n;\lambda,0)$ on S_{ij} $(i,j \neq \infty)$, then $T_k\mathcal{F}$ is also a locally constant sheaf of exponent $(1,n;\lambda - d_{ki} - d_{kj}, 0)$ on S_{ij} and if $\mathcal{F}$ is of exponent $(1,n;\lambda,\mu)$ on $S_{i\infty}$ $(i \neq \infty)$, then $T_k\mathcal{F}$ is of exponent $(1,n;\lambda + d_{ki}, \mu + d_{ki})$ on $S_{i\infty}$.

(3.6) **Proposition.** Suppose that the equation (3.7) below has, at a generic point of S_{nj}, a solution of the form

$$\begin{cases} (x_n - x_j)^\lambda f(x) & (\lambda \notin \boldsymbol{Z}) \\ (x_n - x_j)^\lambda f(x) \log(x_n - x_j) + f_1(x) & (\lambda \in \boldsymbol{Z},\ \lambda \geq 0) \\ f_1(x) \log(x_n - x_j) + (x_n - x_j)^\lambda f(x) & (\lambda \in \boldsymbol{Z},\ \lambda < 0) \end{cases}$$

where $f_1(x)$ and $f(x)$ are holomorphic at this point and $f(x)$ does not vanish identically on S_{nj}. And assume that the order of every $A_i (0 \leq i \leq r)$ on S_{nj} is not less than the strict order of A_n on S_{nj}, then A_j has the same strict order as A_n. Therefore $A_n \neq 0$ implies $A_j \neq 0$.

Proof. In (3.7), replace F with one of these solutions, and a contradicton comes out if this proposition is not true.

Proof of (3.3) . It is sufficient to show the rank r is equal to n. So suppose $r < n$.

Choose the order ideal $\{1, T_1, T_2, \cdots, T_n\}$ regular with respect to $\mathcal{F}$ which was constructed in (2.4). If the rank of $\mathcal{F}$ is r, then we can suppose $T_k = \partial_k$ $(1 \leq k \leq r)$ without reducing the generality. If $r \neq n$, then, for any section $F \in \mathcal{F}$ on D_0, we have a differential equation.

$$A_0 F + A_1 \partial_1 F + \cdots + A_r \partial_r F + A_n \partial_n F = 0, \tag{3.7}$$

where

$$A = A_n = W(f_0, f_1, \cdots, f_n; T_0, T_1, \cdots, T_n) \cdot \prod_{0 \leq \alpha < \beta \leq n+1} (x_\alpha - x_\beta)^{-\lambda_{\alpha\beta}} \neq 0,$$

and, for other i,

$$A_i = W(f_0, f_1, \cdots, f_n; T_0, T_1, \cdots, T_{i-1}, \partial_n, T_{i+1}, \cdots, T_n) \cdot \prod_{0 \leq \alpha < \beta \leq n+1} (x_\alpha - x_\beta)^{-\lambda_{\alpha\beta}}.$$

Since $T_i = \partial^{d_{i1}} \partial^{d_{i2}} \cdots \partial^{d_{ir}}$, the order of A is

$$\begin{cases} n & \text{on } S_{n\infty} \\ -\max\{d_{ai} \mid 0 \leq a \leq n\} & \text{on } S_{ni} \ (i \neq n, \infty) \end{cases} .$$

Put $\sum_{i=1}^{r} \sum_{a=0}^{k} \max\{d_{ai} \mid 0 \leq a \leq k\} = -m_k$ and let $T_k = \partial_{i_k} T_{j_k}$. Then it is evident that $m_k = m_{k-1} + 1$ or $m_k = m_{k-1}$ and the former case never occurs if ∂_{i_k} is not a component of T_{j_k}. As $m_0 = 0$, we have $m_n \leq n$. Because the order of A is n on $S_{n\infty}$ and A has, regarded as function of x_n, poles only on S_{ni} $(1 \le i \le r)$, the equality $m_n = n$ must hold. This means that $m_k = m_{k-1} + 1$ is true for every k, which takes place if and only if T_k are, up to the suffices,

$$1, \partial_1, \partial_1^2, \cdots, \partial_1^{n_1}, \partial_2, \cdots, \partial_2^{n_2}, \cdots, \partial_r, \cdots, \partial_r^{n_r} \tag{3.8}$$

and

$$n_1 + n_2 + \cdots n_r = n$$

holds .

Therefore we have the following table of orders

	S_{in}	S_{jn}	$S_{n\infty}$
A_k	$-n_i$	-1	$n+1$.
A_n	$-n_i$	0	n

where $k \ne n,\ 1 \le i \le r,\ j = 0, r+1, \cdots n-1, n+1.$ So $A_k \prod_{a=1}^{r}(x_n - x_a)^{n_a}$ are of order 0 on S_{ni} $(0 \le i \le r)$ and $A_n \prod_{a=1}^{r}(x_n - x_a)^{n_a}$ is of strict order 0 on S_{in}. By (3.6), none of A_i $(i = 0, 1, \cdots r)$ vanish, for A_n is of strict order $-n_i$ on S_{ni}.

If $n_1 > 1$, operate $\partial_1^{n_1-1}$ on the equation (3.7), then we can easily see that there exist $i(= 2, \cdots, r, n)$ and $m \in \boldsymbol{Z}(0 \le m \le n_1 - 1)$ such that the sequence

$$1, \partial_1, \partial_1^2, \cdots, \partial_1^{n_1-1}, \partial_1^{n_1-1-m}\partial_i, \partial_2, \cdots, \partial_2^{n_2}, \cdots, \partial_r, \cdots, \partial_r^{n_r}$$

is regular. By (3.8), i must be equal to n.

With this sequence, we can construct a new differential equation

$$B_0 F + B_1 \partial_1 F + \cdots + B_r \partial_r F + B_n \partial_n F = 0, \tag{6}$$

which is essentially the same as (3.7), otherwise, by eliminating the term of $\partial_n F$, we see $r(\mathcal{F}) < r$.

If $r < n-1$, let $p = n-1$. Then the orders $B_k (k \neq n)$ are as below, where $2 \leq i \leq r$ and $j = 0, r+1, \cdots, n-1, n+1$.

$$\begin{cases} S_{p1} & S_{pi} & S_{pj} & S_{pn} & S_{p\infty} \\ 1-n_1 & -n_i & 0 & -1 & n \end{cases},$$

so $B_k(x_n - x_p)(x_p - x_1)^{n_1-1} \prod_{i=2}^{r} (x_n - x_i)^{n_i}$ are constant in x_p, which contradicts to (3.6) and that $\mathcal{F}$ is of exponent $(1, n; \lambda_{pn}, 0)$ on S_{pn}.

Therefore $r = n-1$, and we have again the equation (6) by means of $1, \partial_1, \partial_2, \cdots, \partial_{n-1}, \partial_1\partial_n$, for example. Each B_k $(k \neq n, 1, 0)$ is of order -1 on S_{kj} $(j \neq k, 0, n+1)$ and n on $S_{k\infty}$, so $B_k = 0$ $(k \neq n, 1, 0)$ and therefore, by (3.6), $n = 2$ must hold.

Consequently by means of $1, \partial_1, \partial_2, \partial_1\partial_2$, we have

$$B_0 F + B_1 \partial_1 F + B_2 \partial_2 F = 0.$$

By the following table of orders on each singularities,

	S_{01}	S_{12}	S_{13}	S_{20}	S_{23}	$S_{1\infty}$	$S_{2\infty}$
B_0	-1	-2	-1	-1	-1	4	4
B_1	-1	-2	-1	-1	-1	3	4
B_2	-1	-2	-1	-1	-1	4	3

we can consider, by multiplying some factor, $B_0 = e, B_1 = ax_1 + b, B_2 = cx_2 + d$, where a, b, c, d, e are constant. By the situation of $\mathcal{F}$ on S_{01} and S_{13}, we see $a = b = 0$; similarly $c = d = 0$, which completes the proof.

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