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The number of compact leaves of a one-dimensional foliation on the $2n-1$ dimensional sphere S^{2n-1} associated with a holomorphic vector field.

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Introduction

Let $Z=\sum_{i=1}^{n} f_i(z)\partial/\partial z_i$ be a holomorphic vector field in some neighborhood of the $2n$-dimensional closed disk $\bar{D}^{2n}(1)=\{z\in \mathbf{C}^n \mid \|z\|\leq 1\}$ in $\mathbf{C}^n$. We denote by $\mathcal{F}(Z)$ the foliation defined by the solutions of Z. In this paper we will prove the following

THEOREM A. *If the $2n-1$ dimensional sphere $S^{2n-1}(1)$, which is the boundary $\partial\bar{D}^{2n}(1)$ of $\bar{D}^{2n}(1)$, is transverse to $\mathcal{F}(Z)$ then the number of the compact leaves of the foliation $\mathcal{F}(Z)|_{S^{2n-1}(1)}$ is $1,2,\ldots,n$ or ∞.*

In [5], A. Douady and the author proved the following Poincaré-Bendixson type theorem for a holomorphic vector field.

THEOREM 0.1 (A. Douady and T. Ito). *If $S^{2n-1}(1)$ is transverse to $\mathcal{F}(Z)$, then each leaf L of $\mathcal{F}(Z)$ which crosses $S^{2n-1}(1)$ tends to the unique singular point P of Z in $\bar{D}^{2n}(1)$. Furthermore, since we can move P to the origin $\mathbf{0}$ of $\mathbf{C}^n$ by the Möbius transformation, we see that the sphere $S^{2n-1}(r)=\{z\in \mathbf{C}^n \mid \|z\|=r\}$ is transverse to $\mathcal{F}(Z)$ for any real number r, $0<r\leq 1$.*

In the case $n=2$ we used Theorem 0.1 as well as the existence theorem of separatrix proved by C. Camacho and P. Sad ([3]) to obtain an affirmative answer to a special case of the Seifert conjecture:

COROLLARY 0.2 ([5]). *Under the hypothesis of Theorem 0.1, the foliation $\mathcal{F}(Z)|_{S^3(1)}$ on $S^3(1)$ has at least one compact leaf.*

We use Theorem 0.1 to prove the following

THEOREM B. *Under the hypothesis of Theorem 0.1, the set of the eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ of the $n\times n$ matrix $\left(\dfrac{\partial f_i}{\partial z_j}(0)\right)$ belongs to the Poincaré domain.*

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The proof of Theorem A follows from Theorem 0.1, Theorem B and the Poincaré-Dulac theorem ([6], [4]. See §3).

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1. Examples

To shed some light on Theorem A, we give some examples in this section.

EXAMPLE 1.1. Let λ_1 and λ_2 be non-zero complex numbers. Assume that λ_1/λ_2 is not a negative real number. Consider $Z = \lambda_1 z_1 \partial/\partial z_1 + \lambda_2 z_2 \partial/\partial z_2$ on $\mathbf{C}^2$. For any positive real number r, the 3-dimensional sphere $S^3(r)$ is transverse to $\mathcal{F}(Z)$. The solution set L_w of Z with the initial condition $w = (w_1, w_2)$ is $\{ (z_1, z_2) = (w_1 e^{\lambda_1 T}, w_2 e^{\lambda_2 T}) \in \mathbf{C}^2 | T \in \mathbf{C} \}$. In particular, if w_1 is different from zero we may write

$$z_2 = w_2 e^{\lambda_2/\lambda_1 \log(z_1/w_1)}. \tag{1.1}$$

Case (i). If $\lambda_2/\lambda_1 = q/p$ is a positive rational number every leaf of $\mathcal{F}(Z)|_{S^3(1)}$ is compact. This is a Seifert fibration over $S^3(1)$. In the case where λ_2/λ_1 is equal to 1, $\mathcal{F}(Z)|_{S^3(1)}$ is a Hopf fibration. In this case we have infinitely many compact leaves.
Case (ii). If λ_2/λ_1 is either positive irrational or non-real, then $\{ (z_1, 0) \in \mathbf{C}^2 | |z_1| = 1 \}$ and $\{ (0, z_2) \in \mathbf{C}^2 | |z_2| = 1 \}$ are compact leaves of $\mathcal{F}(Z)|_{S^3(1)}$. The equation (1.1) implies that the set $L_w \cap S^3(1)$ is not a compact leaf when every w_i is different from zero. In this case $\mathcal{F}(Z)|_{S^3(1)}$ has exactly two compact leaves.

EXAMPLE 1.2. Let λ and ϵ be two non-zero complex numbers. Consider $Z = \lambda z_1 \partial/\partial z_1 + (\lambda z_2 + \epsilon z_1)\partial/\partial z_2$. The solution set L_w is $\{ (z_1, z_2) = (w_1 e^{\lambda T}, (w_2 + \epsilon w_1 T) e^{\lambda T}) | T \in \mathbf{C} \}$. If w_1 is different from zero we may write

$$z_2 = \left(w_2 + \frac{\epsilon w_1}{\lambda} \log\left(\frac{z_1}{w_1}\right)\right)\left(\frac{z_1}{w_1}\right). \tag{1.2}$$

If $r > 0$ is small $S^3(r)$ is transverse to $\mathcal{F}(Z)$. If $r > 0$ is large, on the other hand, $S^3(r)$ is not transverse to $\mathcal{F}(Z)$. In the case where $S^3(r)$ is transverse to $\mathcal{F}(Z)$, the set $\{ (0, z_2) \in \mathbf{C}^2 | |z_2| = r \}$ is a compact leaf of $\mathcal{F}(Z)$. The equation (1.2) implies that the leaf $L_w \cap S^3(r)$ is not compact if w_1 is different from zero. Thus $\mathcal{F}(Z)|_{S^3(r)}$ has exactly one compact leaf.

EXAMPLE 1.3. Let λ and a be two non-zero complex numbers. Let k be an integer bigger than two. Consider $Z = \lambda z_1 \partial/\partial z_1 + (k\lambda z_2 + a z_1^k)\, \partial/\partial z_2$. The solution set L_w of Z is

$$\begin{aligned} z_1 &= w_1 e^{\lambda T} \qquad \text{and} \\ z_2 &= \left(w_2 + \int_0^T a w_1^k e^{k\lambda T} \cdot e^{-k\lambda T} dT \right) e^{k\lambda T} \\ &= (w_2 + a w_1^k T) e^{k\lambda T}. \end{aligned}$$

If w_1 is different from zero we may write

$$z_2 = \left(w_2 + \frac{a w_1^k}{\lambda} \log(z_1/w_1) \right) \left(\frac{z_1}{w_1}\right)^k. \tag{1.3}$$

For a small $r > o$, $S^3(r)$ is transverse to $\mathcal{F}(Z)$ and the set $\{ (0, z_2) \in \mathbf{C}^2 | \, |z_2| = r \}$ is a compact leaf of $\mathcal{F}(Z)|_{S^3(r)}$. We see from the equation (1.3) that $L_w \cap S^3(r)$ fails to be compact if $w_1 \neq 0$. Thus $\mathcal{F}(Z)|_{S^3(r)}$ has one and only one compact leaf.

We mention that we investigated in ([5]) a global property of contact sets between spheres and $\mathcal{F}(Z)$.

2. The non-existence of transversal maps

Let μ_i $(1 \leq i \leq n)$ be non-zero complex numbers. Assume that the set $\{\mu_1, \dots, \mu_n\}$ belongs to the Siegel domain. Consider a linear vector field $Z = \sum_{i=1}^{n} \mu_i z_i \partial/\partial z_i$ on $\mathbf{C}^n$. To prove Theorem B we need a non-existence theorem of a transversal map f of a manifold to the foliation $\mathcal{F}(Z)$.

THEOREM 2.1. *Let μ_1 and μ_2 be non-zero complex numbers. Consider $Z = \mu_1 z_1 \partial/\partial z_1 + \mu_2 z_2 \partial/\partial z_2$ on $\mathbf{C}^2$. Let M be a closed connected C^∞-manifold of dimension either two or three. If μ_1/μ_2 is a negative real number, then there exists no C^∞-map φ of M to $\mathbf{C}^2$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$.*

PROOF. Suppose that there exists a C^∞-map φ of M to $\mathbf{C}^2$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$. We may select a negative rational number $-p/q$ sufficiently close to μ_1/μ_2 such that $\varphi(M)$ is transverse to $\mathcal{F}(Z')$, where Z' is the linear vector field defined by $Z' = pz_1\partial/\partial z_1 - qz_2\partial/\partial z_2$. The solution L_w of Z' with the initial point $w = (w_1, w_2)$ is $z_1^q z_2^p = w_1^q w_2^p$. Set $F(z_1, z_2) = z_1^q z_2^p$. Then the map $\Phi = |F \circ \varphi| : M \xrightarrow{\varphi} \mathbf{C}^2 \xrightarrow{F} \mathbf{C} \xrightarrow{|\cdot|} \mathbf{R}$ attains a maximal value $\Phi(P)$ at some point $P \in M$. At the point $\varphi(P)$, $\varphi(M)$ is not transverse to $\mathcal{F}(Z')$, but this contradicts our transversality assumption. $\square$

THEOREM 2.2. *Consider a linear vector field $Z = \sum_{i=1}^{n} \mu_i z_i \partial/\partial z_i$ on $\mathbf{C}^n$, $n \geq 3$, where the μ_i's are non-zero complex numbers and the μ_i/μ_j's, $i \neq j$, are imaginary. Let M be a $2n-2$ or $2n-1$-dimensional closed connected C^∞-manifold. If the set $\{\mu_1, \dots, \mu_n\}$ belongs to the Siegel domain, then there is no C^∞-map φ of M to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$.*

PROOF. Let $\Sigma = \{ z \in \mathbf{C}^n | \sum_{i=1}^{n} \mu_i z_i \bar{z}_i = 0 \}$ be the contact set between the spheres $S^{2n-1}(r)$ and $\mathcal{F}(Z)$. Then the set Σ is a cone and $\Sigma - \{0\}$ is a submanifold of dimension $2n - 2$. C. Camacho, N. H. Kuiper and J. Palis proved the following

Fact ([2]). If we take a point $w \in \Sigma - \{0\}$, the distance between L_w and the origin 0 of $\mathbf{C}^n$ attains a unique minimum at w and $L_w \cap \Sigma = \{w\}$. Further the set $W = \{ z \in \mathbf{C}^n | 0 \notin \bar{L}_z \}$ of leaves which do not tend to 0 is diffeomorphic to $(\Sigma - \{0\}) \times \mathbf{C}$. The projection map $\pi : W \to \Sigma - \{0\}$ indicates that each leaf $L \subset W$ corresponds to the point $L \cap \Sigma$.

Assume that there exists a C^∞-map φ of M to $\mathbf{C}^n$ such that $\varphi(M)$ is transverse to $\mathcal{F}(Z)$. The transversality condition implies that the restricted map $\pi|_{W \cap \varphi(M)}$: $W \cap \varphi(M) \to \Sigma - \{0\}$ is a submersion. Since $\pi(W \cap \varphi(M))$ is open closed connected in $\Sigma - \{0\}$, $\pi(W \cap \varphi(M))$ is equal to $\Sigma - \{0\}$. This contradicts the fact that $\pi(W \cap \varphi(M))$ is bounded. $\square$

We will conclude this section by proving Theorem B.

PROOF OF THEOREM B. We calculated in [5] that the index of Z at the origin is one. Hence every eigenvalue of the matrix $(\partial f_i/\partial z_j(0))$ is different from zero.

It follows from Theorem 0.1 that for small enough $r_1 > 0$ the linear part $Z^{(1)} = \sum_{i=1}^{n}(\sum_{j=1}^{n} \partial f_i/\partial z_j(0)z_j)\partial/\partial z_i$ of Z is transverse to $S^{2n-1}(r_1)$. Suppose that the set $\{\lambda_1, \ldots, \lambda_n\}$ does not belong to the Poincaré domain. We may choose an $n \times n$ matrix $A = (a_{ij})$ close enough to $(\partial f_i/\partial z_j(0))$ that the set of the eigenvalues of A satisfies the conditions of Theorem 2.1 or Theorem 2.2. The sphere $S^{2n-1}(r_1)$ is transverse to $\mathcal{F}(\tilde{Z}^{(1)})$, where $\tilde{Z}^{(1)}$ is the linear vector field defined by $\tilde{Z}^{(1)} = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}z_j)\partial/\partial z_i$. This is a contradiction to Theorem 2.1 or Theorem 2.2. □

3. Proof of Theorem A

We recall first a theorem due to H. Poincaré ([6]) and H. Dulac ([4]), which we shall call the Poincaré-Dulac linearization and polynomialization at an isolated singular point of a holomorphic vector field.

Let $Z = \sum_{i=1}^{n} f_i(z)\partial/\partial z_i$ be a holomorphic vector field defined on some neighborhood of the origin 0 of $\mathbf{C}^n$. Assume that the origin is an isolated singular point of Z.

THEOREM 3.1 (H. Poincaré and H. Dulac). *If the set of eigenvalues of the matrix $(\partial f_i/\partial z_j(0))$ belongs to the Poincaré domain, then there exists a biholomorphic map Φ of some neighborhood of* 0 *to another neighborhood of* 0 *in* $\mathbf{C}^n$, $\Phi(z) = w$, $\Phi(0) = 0$, *such that* $\Phi_* Z = W$ *with*

$$W = \lambda_1 w_1 \partial/\partial w_1 + \sum_{i=2}^{n}(\lambda_i w_i + b_i w_{i-1} + P_i(w_1, \ldots, w_{i-1}))\partial/\partial w_i,$$

where the b_i*'s are either* 0 *or* 1 *defined by the Jordan block of* $(\partial f_i/\partial z_j(0))$ *and the* $P_i(w_1, \ldots, w_{i-1})$*'s are polynomials defined as follows:*
Let $m_i = (m_i(1), \ldots, m_i(i-1))$ *be an* $(i-1)$*-tuples of non-negative integers such that* $\sum_{k=1}^{i-1} m_i(k) \geq 2$ *and* $\lambda_i = \sum_{k=1}^{i-1} m_i(k)\lambda_k$. *Define* P_i *by* $P_i(w_1, \ldots, w_{i-1}) = \sum_{m_i} a_{m_i} w_1^{m_i(1)} \cdots, w_{i-1}^{m_i(i-1)}$. *Here the* a_{m_i} *are complex numbers.*

We note for example in the case where $n = 2$ the W is one of the following:

1. $W = \lambda_1 w_1 \partial/\partial w_1 + \lambda_2 w_2 \partial/\partial w_2$.
2. $W = \lambda w_1 \partial/\partial w_1 + (\lambda w_2 + w_1)\partial/\partial w_2$.
3. $W = \lambda w_1 \partial/\partial w_1 + (k\lambda w_2 + a w_1^k)\partial/\partial w_2$.

We are now ready to prove Theorem A.

PROOF OF THEOREM A. We may assume, using the Möbius transformation, that the unique singular point is the origin 0 of $\mathbf{C}^n$. By the grace of Theorem B and Theorem 3.1 we may select a sufficiently small number $r_0 > 0$ so that $\mathcal{F}(Z)|_{\bar{D}^{2n}(r_0)}$ is biholomorphic to $\mathcal{F}(W)|_{\Phi(\bar{D}^{2n}(r_0))}$. Then $\mathcal{F}(Z)|_{S^{2n-1}(r_0)}$ has 1, 2,..., n or infinitely many compact leaves. By Theorem 0.1 $\mathcal{F}(Z)|_{S^{2n-1}(r_0)}$ is C^ω-diffeomorphic to $\mathcal{F}(Z)|_{S^{2n-1}(1)}$. This completes the proof of Theorem A. □

REMARK. M. Brunella and P. Sad ([1]) proved the following theorem. Define a linear hyperbolic foliation $\mathcal{L}_\lambda$ in $\mathbf{C}^2$ by $xdy + \lambda ydx = 0$, $\lambda \in \mathbf{C} - \mathbf{R}$.

THEOREM (M. Brunella and P. Sad). *Let* $\Omega \subset \mathbf{C}^2$ *be a generalized bidisc and let* $\mathcal{F}$ *be a holomorphic foliation defined in a neighborhood of* $\overline{\Omega}$ *and transverse to* $\partial\Omega$. *Then there exists a locally injective holomorphic map* ϕ *which sends a neighborhood of* $\overline{\Omega}$ *to a neighborhood of* 0 *in* $\mathbf{C}^2$ *and such that* $\mathcal{F} = \phi^*(\mathcal{L}_\lambda)$ *for some* $\lambda \in \mathbf{C}\backslash\mathbf{R}$.

Furthermore ϕ is injective as a map between spaces of leaves, i.e. for every leaf $L \in \mathcal{L}_\lambda$ the preimage $\phi^{-1}(\phi(\overline{\Omega}) \cap L)$ is exactly one leaf of $\mathcal{F}|_{\overline{\Omega}}$.

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