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SINGULARITIES OF FINITE FORMAL TYPE FOR FOLIATIONS OF $(\mathbb{C}^2, 0)$

J.F. MATTEI and E. SALEM[1]

1 Introduction

We consider a germ of a (singular) formal foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ i.e. given by a differential 1-form $\omega = a(x,y)\,dx + b(x,y)\,dy$, where a and b are formal power series in two variables : $a, b \in \mathbb{C}[[x,y]]$.

After desingularization of $\mathcal{F}$ by a finite number of blowing ups at points, we get on a neighbourhood $\widetilde{\mathcal{M}}$ of a divisor $\widetilde{\mathcal{D}}$, a transversally formal foliation (2.1.2) $\widetilde{\mathcal{F}}$ along $\widetilde{\mathcal{D}}$. This foliation is singular only at a finite number of points, at the neighbourhood of which it is locally given by a strictly reduced form (2.2).

We shall compute in this paper the first cohomology group of a distinguished covering (2.3.5) $\mathcal{U}$ of the divisor $\widetilde{\mathcal{D}}$, with values in the sheaf $\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}$ over $\widetilde{\mathcal{D}}$ of transversally formal basic vector fields. By basic vector field, we mean a vector field leaving $\widetilde{\mathcal{F}}$ invariant and which is tangent to $\widetilde{\mathcal{D}}$. The sheaf over $\widetilde{\mathcal{D}}$ of basic vector fields contains as a subsheaf the sheaf $\widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}}$ of transversally formal vector fields which are tangent to $\widetilde{\mathcal{F}}$ and to $\widetilde{\mathcal{D}}$. We denote the quotient sheaf by $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$; it is the sheaf of (transversally formal) transverse vector fields.

The computation of $H^1(\mathcal{U};\widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}})$ is mainly a geometrical problem. It follows from the theorem of Andreotti-Grauert that its dimension is finite, and it has been computed in [7]. The computation of $H^1(\mathcal{U};\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$ is of a different

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nature. In this paper, we show that under some nondegeneracy conditions the dimensions of the spaces $H^1(\mathcal{U};\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$ and $H^1(\mathcal{U};\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}})$ are finite.

More precisely :

Theorem 1.0.1 *Let $\mathcal{F}$ be a formal foliation at the origin of $\mathbb{C}^2$ which is nondegenerate in the following sense :*

1. *$\mathcal{F}$ is nondicritical.*
2. *$\widetilde{\mathcal{F}}$ has no singularity of resonant saddle-node type along $\widetilde{\mathcal{D}}$.*
3. *The holonomy group of each component of $\widetilde{\mathcal{D}}$ of valence ≥ 3 is non abelian.*
4. *Every germ of a transversally formal first integral of $\widetilde{\mathcal{F}}$ at a singular point which is the intersection of a component of $\widetilde{\mathcal{D}}$ of valence ≥ 3 with a chain of valence 2, is constant.*

Then $\dim_{\mathbb{C}} H^1(\mathcal{U};\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$ and $\dim_{\mathbb{C}} H^1(\mathcal{U};\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}})$ are finite.

We also give explicit formula for computing these dimensions (4.0.20) and (4.0.23).

The space $H^1(\mathcal{U};\widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}})$ is the base space of a universal equisingular unfolding of the foliation $\mathcal{F}$ (see [7]). We shall construct in a forthcoming paper, using Theorem (1.0.1), a universal equisingular deformation of $\mathcal{F}$, with fixed local reduced models and fixed holonomy groups. This universal deformation has base space $H^1(\mathcal{U};\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}})$, and is given by a holomorphic family of formal differential 1-forms. We shall also show that the condition of nondegeneracy given above is generic.

This paper is extracted from a paper that will be published elsewhere. The second author gave a talk on Theorem (1.0.1) at the conference "Topology of holomorphic dynamical systems and related topics" at RIMS, Kyoto, in October 1995.

2 Background on transversally formal foliations.

Let M be a holomorphic connected manifold of dimension 2. We shall denote the sheaf of germs of holomorphic functions, holomorphic vector fields, and holomorphic differential forms on M by $\mathcal{O}_M$, $\mathcal{X}_M$ and Λ_M. We refer to [1] and to [2] for the basic notions of ringed spaces and sheaves used in this paragraph.

2.1 Transversally formal foliations.

Definition 2.1.1 *We consider a connected holomorphic manifold M of dimension 2, and an analytic subset $S = (|S|, \mathcal{O}_M/I_S)$ of M, not necessarily reduced. $\widehat{M}^S$ is the ringed space*

$$\widehat{M}^S := \left(|\widehat{M}^S| := |S| \ , \ \mathcal{O}_{\widehat{M}^S} := \widehat{\mathcal{O}}_M^S\right)$$

where $\widehat{\mathcal{O}}_M^S$ is the sheaf of germs of transversally formal functions along S, obtained by completion of $\mathcal{O}_M$ relative to the ideal I_S :

$$\widehat{\mathcal{O}}_M^S := \varprojlim_{k \in \mathbb{N}} \left(\frac{i^{-1}(\mathcal{O}_M)}{i^{-1}({I_S}^{k+1})} \right)$$

and $i : S \hookrightarrow M$ is the inclusion map. We shall say that $\widehat{\mathcal{O}}_M^S$ is a transversally formal space.

We shall consider only analytic subsets S of dimension 0 and monomial divisors, i.e. locally defined by only one equation which is monomial in well chosen coordinates. In that case, the elements of $\widehat{\mathcal{O}}_M^S$ can be written in these coordinates as series :

$$\sum_{k=0}^{\infty} A_k(v)\, u^k \qquad \text{resp.} \qquad \sum_{k=0}^{\infty} \left(A_k^1(u) + A_k^2(v)\right)(uv)^k$$

where S is defined by $u = 0$ resp. $uv = 0$; the coefficients A_k , A_k^1 , A_k^2 being convergent on the same domain.

By extension of the scalars $i^{-1}(\mathcal{O}_M) \hookrightarrow \widehat{\mathcal{O}}_M^S$ we can define the notions of *transversally formal differential 1-form* and of *transversally formal vector field along* S :

$$\widehat{\Lambda}_M^S := i^{-1}(\Lambda_M) \otimes_{i^{-1}(\mathcal{O}_M)} \widehat{\mathcal{O}}_M^S, \qquad \widehat{\mathcal{X}}_M^S := i^{-1}(\mathcal{X}_M) \otimes_{i^{-1}(\mathcal{O}_M)} \widehat{\mathcal{O}}_M^S.$$

When $S = \{m\}$ is a point, we shall denote the modules of germs of formal functions, formal differential 1-forms and formal vector fields on M at the point m by $\widehat{\mathcal{O}}_{M,m}$, $\widehat{\Lambda}_{M,m}$ and $\widehat{\mathcal{X}}_{M,m}$.

Definition 2.1.2 *A transversally formal foliation $\mathcal{F}$ of codimension* 1 *on M along S is a sheaf $\Lambda_{\mathcal{F}}$ of locally free submodules of rank* 1 *of* $\widehat{\Lambda}_M^S$.

Thus, at each point m, the module $\Lambda_{\mathcal{F},m}$ over $\widehat{\mathcal{O}}_{M,m}$ is generated by the germ ω_m of a transversally formal differential 1-form on M along S.

Outside the *singular locus of* $\mathcal{F}$, i.e. the analytic closed subset $Sing(\mathcal{F})$ of S defined by the sheaf of ideals

(1) $$I_{\mathcal{F}} := \Lambda_{\mathcal{F}} \cdot \mathcal{X}_M$$

one has a "regular" foliation of codimension 1. By dividing locally the generators of $\Lambda_{\mathcal{F},m}$ by the *g.c.d.* of their coefficients, one constructs a unique transversally formal foliation, the *saturated foliation sat$\mathcal{F}$ of $\mathcal{F}$* having only isolated singular points.

If $f : \widehat{M'}^{S'} \to \widehat{M}^S$ is a *transversally formal map* (i.e. a morphism of ringed spaces) between two transversally formal spaces we define the *inverse image of $\mathcal{F}$ by f* to be the foliation $f^*\mathcal{F}$ locally given by the inverse image $f^*\omega_m$ of the differential form ω_m which generates $\Lambda_{\mathcal{F},m}$; when the $f^*\omega_m$ are identically zero we say that $(\widehat{M'}^{S'}, f)$ is *an integral manifold of* $\mathcal{F}$.

Definition 2.1.3 *The strict transform of $\mathcal{F}$ by f is the saturated foliation* $f^*\mathcal{F}$.

Definition 2.1.4 *Let $\mathcal{F}$ be a transversally formal foliation of codimension* 1 *on a neighbourhood M of a hypersurface S. The singular locus of $(\mathcal{F}, S)$ is the analytic subset $Sing(\mathcal{F}, S)$ of S defined by the sheaf of ideals*

$$I_{\mathcal{F},S} = sat\ (\Lambda_{\mathcal{F}} \cdot \mathcal{X}_{M,S})$$

where : $\mathcal{X}_{M,S} \subset \mathcal{X}_M$ *is the subsheaf of germs of holomorphic vector fields on* M *tangent to* S *(i.e. to the smooth part of* S*) and, for any ideal* $I := (u_1, \ldots, u_r)$ *of* $\mathcal{O}_{M,m}$*,* $sat\,(I)$ *is the ideal generated by the quotients* $\tilde{u}_j := \dfrac{u_j}{p.g.c.d.(u_1, \ldots, u_r)}$. *A point not in* $Sing(\mathcal{F}, S)$ *is called a regular point of* $(\mathcal{F}, S)$.

One can easily check that :

Proposition 2.1.5 *A point* $m \in S$ *is a regular point for* $(\mathcal{F}, S)$ *if and only if at this point,* $\mathcal{F}$ *is regular,* S *is smooth, and each local irreducible component of* S *is either an integral manifold of* $\mathcal{F}$*, or transverse to* $\mathcal{F}$.

2.2 Strictly reduced forms.

In this paragraph we describe in the context of formal foliations some notions which are classical for holomorphic foliations (see [3], [4], [8]).

The most simple formal invariant associated to a germ of a formal foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$, defined by a differential form

$$\omega = a(x,y)\,dx + b(x,y)\,dy\,, \quad a,\ b\ \in \widehat{\mathcal{O}}_{\mathbb{C}^2,0}$$

is the *algebraic multiplicity* of $\mathcal{F}$ at 0 :

$$(2) \qquad \nu_0(\mathcal{F}) := inf\,\{\nu_0(u)\,;\ u \in I_{\mathcal{F}}\}\ \ (= inf\,\{\nu_0(a)\,;\ \nu_0(b)\}\,)$$

where ν_0 is the valuation at the origin of $\mathbb{C}^2$ relative to the maximal ideal of $\widehat{\mathcal{O}}_{\mathbb{C}^2,0}$.

The *strict tangent cone of* ω*, or of* $\mathcal{F}$, is the subspace C'_ω of $\mathbb{P}^1$ defined by the homogeneous equation $xa_\nu + y\,b_\nu = 0$, where a_ν, b_ν are the homogeneous components of degree $\nu := \nu_0(\mathcal{F})$ of the coefficients a and b. When $C'_\omega = \mathbb{P}^1$, we say that ω or $\mathcal{F}$ is *dicritical at the first order.* In this case the exceptional divisor $D := E^{-1}(0)$ obtained from the origin by the blowing up map $E : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$, is not an integral curve for the saturated foliation $\widetilde{\mathcal{F}} := sat\,(E^*\mathcal{F})$.

Definition 2.2.1 *$\mathcal{F}$ is prereduced if $\mathcal{F}$ is non singular, or if $\mathcal{F}$ is singular and its strict tangent cone consists of two simple points.*

One can see that $\mathcal{F}$ is a singular prereduced foliation if and only if the linear part of ω is diagonalizable : there exists $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq 0$, and coordinates (u, v) at the origin such that the 1-jet of ω is $j^1(\omega) = \lambda_1 udv + \lambda_2 vdu$ and the "eigenvalues" λ_1 , λ_2 satisfy : $\lambda_1 + \lambda_2 \neq 0$.

Definition 2.2.2 *$\mathcal{F}$ is strictly reduced, if $\mathcal{F}$ is singular prereduced and the quotient of the eigenvalues of $j^1(\omega)$ is not a stricly negative rational number.*

Let us now take for $\mathcal{F}$ the germ of a saturated, transversally formal foliation along a divisor with normal crossings S on a holomorphic manifold M of dimension 2. The pair $(\mathcal{F}, S)$ is *prereduced* (resp. strictly reduced) at a point $m \in S$, if one either has :

- m is a regular point for $(\mathcal{F}, S)$ (see (2.1.4)) or
- $m \in Sing(\mathcal{F})$, $\mathcal{F}$ is prereduced (resp. strictly reduced) at m and each irreducible component of S at m is an integral curve of $\mathcal{F}$.

One can check that (see [5], [6]):

Proposition 2.2.3 *Let S be a germ of an analytic curve in $(\mathbb{C}^2, 0)$ with normal crossings, and ω a strictly reduced differential 1-form transversally formal along S. If S is an integral curve of ω, then ω is conjugate (by a transversally formal diffeomorphism along S) to a differential 1-form in the list (we call these models the formal normal forms) :*

1. *Linearizable case :*
   $\omega := \lambda_1 z_1\, dz_2 + \lambda_2 z_2\, dz_1$ *with* $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_2/\lambda_1 \notin \mathbb{Q}_{\leq 0}$
   (a) *Linearizable non resonant case :* $\lambda_2/\lambda_1 \notin \mathbb{Q}_{\geq 0}$
   (b) *Linearizable resonant case :* $\lambda_2/\lambda_1 = p/q$, $p, q \in \mathbb{N}^*$, $(p, q) = 1$
2. *Resonant non linearizable case :*
   $\omega := q\, z_1(1 + \zeta\, (z_1^p z_2^q)^k)\, dz_2 + p\, z_2(1 + (\zeta - 1)\, (z_1^p z_2^q)^k)\, dz_1$ *with* $p, q, k \in \mathbb{N}^*$, $(p, q) = 1, \zeta \in \mathbb{C}$

3. Saddle node case: $\omega := (\zeta z_2^p - p)\, dz_2 + z_2^{p+1}\, dz_1$ *with* $p \in \mathbb{N}^*$, $\zeta \in \mathbb{C}$

In cases 1. and 2., w has two convergent integral manifolds $z_1 = 0$ et $z_2 = 0$. In the third case w has only one convergent integral manifold $z_2 = 0$, the other integral manifold $z_1 = 0$ being only formal.

2.3 Trees of reduction.

We construct a tree with base $\{0\}$, and height h' (which a priori may be infinite) called the *tree of prereduction of* $\mathcal{F}$. It is a commutative diagram

$$\mathsf{A}'(\mathcal{F}) = \left(\mathcal{M}^j, E^j, \Sigma^j, C'^j, \pi_j, \mathcal{D}^j\right)_{j=0,\ldots h'}$$

$$(3)\quad \begin{array}{ccccccccccc}
\mathcal{M}^{h'} & \longrightarrow \cdots \longrightarrow & \mathcal{M}^j & \xrightarrow{E^j} & \mathcal{M}^{j-1} & \longrightarrow \cdots \xrightarrow{E^1} & \mathcal{M}^0 & \xrightarrow{\pi} & \{0\} \\
\cup & & \cup & & \cup & & \cup & & \\
\Sigma^{h'} & \longrightarrow \cdots \longrightarrow & \Sigma^j & \longrightarrow & \Sigma^{j-1} & \longrightarrow \cdots \longrightarrow & \Sigma^0 & & \\
\cup & & \cup & & \cup & & \cup & & \\
C'^{h'} & \longrightarrow \cdots \longrightarrow & C'^j & \longrightarrow & C'^{j-1} & \longrightarrow \cdots \longrightarrow & C'^0 & &
\end{array}$$

defined by :

1. $\mathcal{M}^0 := \mathbb{C}^2$, $\quad \Sigma^0 := \{0\} = C'^0$,
2. $\Sigma^j := Sing(\tilde{\mathcal{F}}^j, \mathcal{D}^j)$ where $\tilde{\mathcal{F}}^j$ is the strict transform of $\mathcal{F}$ by the map E_j, which is the composition of the blowing ups centred at C'^k, $k = 0, \ldots, j-1$, and $\mathcal{D}^j := E_j^{-1}(0)$,
3. $C'^j \subset \Sigma^j$ is the set of points m of $\mathcal{D}^j$ where the pair $\left(\tilde{\mathcal{F}}^j, \mathcal{D}^j\right)$ is not prereduced.

In the same way we can construct a tree of strict reduction denoted by $\mathsf{A}(\mathcal{F})$, with height $h \geq h'$ by replacing in the above definition the sets C'^j by the sets $C^j \subset \Sigma^j$ of points $m \in \mathcal{D}^j$ where the pair $(\tilde{\mathcal{F}}^j, \mathcal{D}^j)$ is not strictly reduced.

Theorem 2.3.1 (of reduction) *[9] [8] The trees of prereduction and strict reduction of a formal foliation at the origin of* $\mathbb{C}^2$ *have finite heights.*

The foliation $\tilde{\mathcal{F}} := \tilde{\mathcal{F}}^h$ is the *strictly reduced foliation associated to* $\mathcal{F}$, $\tilde{\mathcal{D}} := \mathcal{D}^h$ is the *divisor of strict reduction* of $\mathcal{F}$, and $\tilde{E} := \tilde{E}^h : \tilde{\mathcal{M}} \to \mathbb{C}^2$ is the *map of strict reduction.*

Definition 2.3.2 *We say that a formal foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is nondicritical if every irreducible component of the exceptional divisor $\tilde{\mathcal{D}}$ is an integral manifold of $\mathcal{F}$.*

Definition 2.3.3 *The set I of critical elements for the strict reduction of $\mathcal{F}$ consists of :*

a) *the connected components c of $\tilde{\Sigma} := \tilde{\Sigma}^h$, and*

b) *the connected components α of $\tilde{\mathcal{D}} - \tilde{\Sigma}$.*

A critical element of type a) resp. b) has dimension 0, resp. 1.

Definition 2.3.4 *Two critical elements are adjacent if their closures intersect.*

Definition 2.3.5 *A distinguished covering $\mathcal{U}$ of $\tilde{\mathcal{D}}$ is a covering by open sets $(U_\alpha)_{\alpha \in I}$ where :*

a) *$U_\alpha := \alpha$ if $dim(\alpha) = 1$,*

b) *U_α is the intersection of a small tubular neighbourhood of α in $\tilde{\mathcal{M}}$ with $\tilde{\mathcal{D}}$, if $dim(\alpha) = 0$.*

In particular, a distinguished covering has the properties :

Remark 2.3.6 i) $U_\alpha \cap U_\beta \cap U_\gamma = \emptyset$ if $\alpha, \beta, \gamma, \in I$ and $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$

ii) $U_\alpha \cap U_\beta \cap \tilde{\Sigma} = \emptyset$ if $\alpha, \beta \in I$, $\alpha \neq \beta$,

iii) each $U_\alpha \cap U_\beta$ is a Stein open set.

Definition 2.3.7 *We can associate to $\mathbb{A}(\mathcal{F})$ its dual tree $\mathbb{A}^*(\mathcal{F})$:*

- *each vertex corresponds by a 1-1 map to an irreducible component D of $\tilde{\mathcal{D}}$;*

- *two vertices are connected by an edge if the corresponding irreducible components of $\widetilde{\mathcal{D}}$ intersect;*
- *for each component Σ of $\widetilde{\Sigma}$ contained in an irreducible component D, we attach an arrow to the vertex corresponding to D;*
- *the weight at the vertex corresponding to D is the Chern class of the normal bundle of D in $\widetilde{\mathcal{M}}$.*

3 The sheaves of basic and transverse vector fields.

Let $\mathcal{F}$ be a foliation defined by a germ at the origin of $\mathbb{C}^2$ of a formal, nondicritical differential 1-form ω. Let $\widetilde{E} : \widetilde{\mathcal{M}} \to \mathbb{C}^2$ be the map of strict reduction of its singularities, $\widetilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$ by $\widetilde{E}$ and let us denote the sheaf $\widehat{\mathcal{O}}^{\widetilde{\mathcal{D}}}_{\widetilde{\mathcal{M}}}$ (2.1.1) over $\widetilde{\mathcal{D}}$, of functions which are transversally formal along $\widetilde{\mathcal{D}} := \widetilde{E}^{-1}(0)$ by $\widehat{\mathcal{O}}$.

We consider the sheaf $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}} \subset \widehat{\mathcal{O}}$ over $\widetilde{\mathcal{D}}$ of germs of first integrals of $\widetilde{\mathcal{F}}$ which are transversally formal : $f \in \widehat{\mathcal{O}}$ and $df \in \Lambda_{\widetilde{\mathcal{F}}}$. The aim of this paper is to compute the cohomology of a distinguished covering (2.3.5) $\mathcal{U}$ of $\widetilde{\mathcal{D}}$ with values in the sheaf

$$\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}} \subset \widehat{\mathcal{X}}^{\widetilde{\mathcal{D}}}_{\widetilde{\mathcal{M}}}$$

of transversally formal basic vector fields. By *basic vector field*, we mean a vector field leaving $\widetilde{\mathcal{F}}$ invariant and which is tangent to $\widetilde{\mathcal{D}}$. This sheaf admits only a structure of an $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$-module; but the sheaf

$$\widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}} \subset \widehat{\mathcal{X}}^{\widetilde{\mathcal{D}}}_{\widetilde{\mathcal{M}}}$$

of transversally formal vector fields which are tangent to $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{D}}$ is clearly a locally free $\widehat{\mathcal{O}}$- module of rank one. Thus, the computation of $H^1(\mathcal{U}; \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}})$ is mainly a geometrical problem. It follows from the theorem of Andreotti-Grauert that its dimension is finite, and it has been computed in [7].

The space $H^1(\mathcal{U}; \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}})$ has a dynamical nature. To split the problem, according to the two different types of difficulties we have, we make the following definition :

Definition 3.0.8 *We define the sheaf of transverse vector fields to be the quotient sheaf given by the short exact sequence of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$-modules :*

$$0 \longrightarrow \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}} \longrightarrow \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}} \longrightarrow \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}} \longrightarrow 0\,. \tag{4}$$

We are now going to describe the sheaf $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$.

Remark 3.0.9 If W is an open set of the distinguished covering $\mathcal{U}$ of $\widetilde{\mathcal{D}}$, then

$$0 \longrightarrow \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}}(W) \longrightarrow \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}(W) \longrightarrow \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}(W) \longrightarrow 0\,.$$

We shall denote the class of $Z \in \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}(W)$ in $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}(W)$ by $\{Z\}$.

Let us fix a critical element α of dimension 1 and a point $m \in U_\alpha$.

Proposition 3.0.10 *The restriction of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ to U_α is locally free of rank 1 over $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$.*

Proof. At each point m of U_α we choose transversally formal coordinates (z_1, z_2) of $\widetilde{\mathcal{D}}$ such that $\Lambda_{\widetilde{\mathcal{F}},m} = \widehat{\mathcal{O}}_{\mathbb{C},m}\, dz_2$, where $(z_2 = 0)$ is the equation of a component of $\widetilde{\mathcal{D}}$. We have : $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}},m} = \mathbb{C}\,[[z_2]]$, $\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}},m} = \widehat{\mathcal{O}}_{\mathbb{C},m}\frac{\partial}{\partial z_1} + \mathbb{C}\,[[z_2]]\, z_2\frac{\partial}{\partial z_2}$ and

$$\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}},m} = \mathbb{C}\,[[z_2]] \cdot \left\{ z_2\frac{\partial}{\partial z_2} \right\}$$

which leads to the conclusion. □

Let T_m be a germ of a smooth curve transverse to U_α at a point $m \in \alpha$. The restriction of z_2 to T_m is a formal coordinate on T_m and the above lemma allows us to identify $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}},m}$ with the ring $\widehat{\mathcal{O}}_{T_m}$ of formal series on T_m, and $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}},m}$ with the module $\widehat{\mathcal{X}}_{T_m}$ of formal vector fields on T_m. The continuation of first integrals along paths in U_α is given by the holonomy, and one can easily check that

Proposition 3.0.11 *Let $W \subset U_\alpha$ be a connected open neighbourhood of m in U_α. A formal vector field $Z \in \widehat{\mathcal{X}}_{T_m}$ (resp. a germ of a formal power series $f \in \widehat{\mathcal{O}}_{T_m}$) induces a (unique) global section $Z^{ext} \in H^0(W; \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$ (resp. $f^{ext} \in H^0(W; \widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}})$) if and only if the flow of Z commutes with the action (resp. f is invariant under the action) of the holonomy group $\widetilde{\mathcal{F}}$ generated by $\pi_1(W; m)$.*

We now describe the fibre of $\widehat{\mathcal{T}}_{\tilde{\mathcal{F}}}$ at a singular point. It is well known (see [4][page 143]) that any germ of a strictly reduced 1-form admits a basic vector field, which is unique up to multiplication by a first integral. To be more precise, let us consider a singular point c of $\tilde{\mathcal{F}}$ on an irreducible component D of $\widetilde{\mathcal{D}}$ and let us take normalizing coordinates at c, i.e transversally formal coordinates (z_1, z_2) at this point under which $D = (z_2 = 0)$ and $\tilde{\mathcal{F}}$ has a formal normal form. We denote the holonomy of $\tilde{\mathcal{F}}$ induced by the loop in D around $c = (0,0)$: $z_1(\theta) = e^{i\theta}, z_2(\theta) = 0$, $\theta \in [0, 2\pi]$ by h. We have

$$\widehat{\mathcal{T}}_{\tilde{\mathcal{F}},c} = \widehat{\mathcal{O}}_{\tilde{\mathcal{F}},c} \cdot \{Z_c\} ,$$

and according to the cases ([4],[5],[6]):

1. If $\tilde{\mathcal{F}}$ is defined by $\omega_c := \lambda_1 z_1\, dz_2 + \lambda_2 z_2\, dz_1$ with $\lambda_1\lambda_2 \neq 0$, $\lambda_2/\lambda_1 \notin \mathbb{Q}_{\leq 0}$, then
   $a)$ if $\lambda_2/\lambda_1 \notin \mathbb{Q}_{\geq 0}$, $\widehat{\mathcal{O}}_{\tilde{\mathcal{F}},c} = \mathbb{C}$,
   $b)$ if $\lambda_2/\lambda_1 = p/q$, $p, q \in \mathbb{N}$, $(p,q) = 1$, $\widehat{\mathcal{O}}_{\tilde{\mathcal{F}},c} = \mathbb{C}\,[[z_1^p z_2^q]]$,
   and in both cases one has :
   $\{Z_c\} = 1/2\{\lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2}\} = \{\lambda_1 z_1 \frac{\partial}{\partial z_1}\} = \{\lambda_2 z_2 \frac{\partial}{\partial z_2}\}$
   and $h(z_2) = e^{-2i\pi\lambda_2/\lambda_1} z_2$.

2. If $\tilde{\mathcal{F}}$ is defined by $\omega_c := q\, z_1(1+\zeta\,(z_1^p z_2^q)^k)\, dz_2 + p\, z_2(1+(\zeta-1)\,(z_1^p z_2^q)^k)\, dz_1$, with $p, q, k \in \mathbb{N}^*$, $(p,q) = 1$, $\zeta \in \mathbb{C}$, then
   $\widehat{\mathcal{O}}_{\tilde{\mathcal{F}},c} = \mathbb{C}$, $\quad h(z_2) = e^{\frac{-2i\pi p}{q}} \exp\, (2i\pi \frac{p z_2^{qk+1}}{q(1+\zeta z_2^{qk})} \frac{\partial}{\partial z_2})$, and
   $\{Z_c\} = \left\{-q z_1 \frac{\partial}{\partial z_1} + p z_2 \frac{\partial}{\partial z_2}\right\}$
   which is also equal to $\left\{\frac{(z_1^p z_2^q)^k}{1+\zeta(z_1^p z_2^q)^k} z_2 \frac{\partial}{\partial z_2}\right\} = \left\{\frac{(z_1^p z_2^q)^k}{1+(\zeta-1)(z_1^p z_2^q)^k} z_1 \frac{\partial}{\partial z_1}\right\}$.

3. If $\tilde{\mathcal{F}}$ is defined by $\omega_c := (\zeta z_2^p - p) z_1\, dz_2 + z_2^{p+1}\, dz_1$, $\zeta \in \mathbb{C}$, then
   $\widehat{\mathcal{O}}_{\tilde{\mathcal{F}},c} = \mathbb{C}$, $\quad h$ is never periodic .
   $\{Z_c\} = 1/2\{\frac{z_2^{p+1}}{(\zeta z_2^p - p)} \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_1}\} = \{\frac{z_2^{p+1}}{(\zeta z_2^p - p)} \frac{\partial}{\partial z_2}\} = \{z_1 \frac{\partial}{\partial z_1}\}$.

4. If $\tilde{\mathcal{F}}$ is defined by $\omega_c := z_1^{p+1}\, dz_2 + (\zeta z_1^p - p) z_2\, dz_1$, $\zeta \in \mathbb{C}^*$, $\widehat{\mathcal{O}}_{\tilde{\mathcal{F}}}$ and $\{Z_c\}$ have the same expressions as above but now h is periodic if and only if $\zeta \in \mathbb{Q}$.

Using the previous proposition, one can easily construct in case 4), when $\zeta \in \mathbb{Q}$, a non-constant section of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$ and a non-zero section of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ on a neighbourhood of $D - \{c\}$ which do not extend to the point c. To exclude this case, one gives the following definition :

Definition 3.0.12 *A singularity of $\widetilde{\mathcal{F}}$ at a point $c \in \widetilde{\mathcal{D}}$ is a resonant saddle-node along $\widetilde{\mathcal{D}}$ if there exists a system of transversally formal coordinates (z_1, z_2) at c such that $(z_2 = 0)$ is the local equation of a component of $\widetilde{\mathcal{D}}$, and $\widetilde{\mathcal{F}}$ is defined by*

$$\omega_c := (\zeta z_1^p - p) z_2 \, dz_1 + z_1^{p+1} \, dz_2 \,, \quad \zeta \in \mathbb{Q} \,.$$

When no singular point of $\widetilde{\mathcal{F}}$ is of this type we shall say that $\widetilde{\mathcal{F}}$ is *without resonant saddle-node.*

Lemma 3.0.13 *Let α and c be two adjacent critical elements of dimension 1 and 0 respectively. If c is not a resonant saddle-node for $\widetilde{\mathcal{F}}$ along the component of $\widetilde{\mathcal{D}}$ corresponding to α, then :*

1. Every section $f_{\alpha c}$ of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$ over $U_{\alpha c} := U_\alpha \cap U_c$ can be extended in a unique way to a section of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$ over U_c.

2. Every section $X_{\alpha c}$ of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ over $U_{\alpha c} := U_\alpha \cap U_c$ can be extended in a unique way to a section of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ over U_c.

Proof. Let us begin by proving the second part of the lemma. We consider a curve T_m transverse to the divisor at a point $m \neq c$ of a small "disc" $W \subset U_c$ centred on c on which the section $\{Z_c\}$ described above is globally defined. By (3.0.11), this section induces a formal vector field X_m on T_m, invariant under the holonomy map h_m relative to a loop γ_m generating $\pi_1(W - \{c\}; m)$. By studying each case in the normalizing coordinates (z_1, z_2) at c where $T_m = \{z_1 = \varepsilon\}$, one deduces the following expression for X_m :

- case 1.a) : $X_m = \mu \, z_2 \dfrac{\partial}{\partial z_2}, \quad \mu \in \mathbb{C},$
- case 1.b) : $X_m = f(z_1^p z_2^q) \, z_2 \dfrac{\partial}{\partial z_2}, \quad f(x) \in \mathbb{C}[[x]],$
- case 2. : $X_m = \mu \dfrac{z_2^{qk+1}}{1 + \zeta z_2^{qk}} \dfrac{\partial}{\partial z_2}, \quad \mu \in \mathbb{C}.$

In the first two cases it follows by direct computation. In the last case, one uses the following classical lemma

Sublemma 3.0.14 *Every formal diffeomorphism ϕ of $(\mathbb{C},0)$ commuting with*

$$H(z) := e^{\frac{2i\pi p}{q}} exp\,(Y_{p/q,\zeta}) \qquad Y_{p/q,\zeta} := 2i\pi \frac{z^{qk+1}}{1+\zeta z^{qk}}\frac{\partial}{\partial z}$$

where $p,q,k \in \mathbb{N}^*$, $(p,q)=1$, $\zeta \in \mathbb{C}$ can be written as :

$$\phi = e^{\frac{2i\pi k}{q}} exp\,(t\,Y_{p/q,\zeta}), \qquad k \in \mathbb{Z}, \quad t \in \mathbb{C}.$$

Proof of the sublemma. [2] Let us denote

$$L^k(z) := e^{\frac{2i\pi pk}{q}} z, \quad G^t(z) := \exp\,(2i\pi t \frac{z^{qk+1}}{1+\zeta z^{qk}}\frac{\partial}{\partial z}), \quad H^{k,t} := L^k \circ G^t.$$

The two formal diffeomorphisms L^1 and G^1 commute with each other and one has :

$$H(z) = H^{1,1}, \quad H^n = H^{n,n}, \quad H^{k,t} = H^{k+rq,t}, \quad k,n,r \in \mathbb{Z}.$$

By developing the commutativity relation, $\phi \circ H^{k,t} \circ \phi^{-1} \circ (H^{k,t})^{-1}(z)$, one gets a series $\sum P_j(k,t)z^j$ whose coefficients are polynomials in the variables k and t and satisfy the relations : $P_j(k+rq,t) = P_j(k,t)$. As ϕ commutes with H and thus with all of its iterates, one has $P_j(n,n)=0$ for any integer n, and therefore $P_j(k+rq,k)=0$ for all $k,r \in \mathbb{Z}$. This implies that for every $j \in \mathbb{N}$, P_j is identically zero. Thus ϕ commutes with $H^{k,t}$ and therefore with G^t also. On the other hand, every germ at 0 of a diffeomorphism ϕ of $\mathbb{C}$ can be formally decomposed as $\phi = \tilde{L} \circ \exp\,(\tilde{Y}_{p'/q',\zeta'})$, with

$$[\tilde{L}, \exp\,(\tilde{Y}_{p'/q',\zeta'})](z) = z, \tilde{Y}_{p'/q',\zeta'} \sim_{for} Y_{p'/q',\zeta'}, \tilde{L}(z) \sim_{for} e^{\frac{2i\pi p'}{q'}} z.$$

By exchanging the rôles of ϕ and H in the previous discussion, one gets that

[2]The proof we give here is due to D. Cerveau

the flows of $\widetilde{Y}_{p'/q',\zeta'}$ and of $Y_{p/q,\zeta}$ commute. If one sets $\widetilde{Y}_{p'/q',\zeta'} := a(z)\frac{\partial}{\partial z}$ one has

$$\frac{a'(z)}{a(z)} = \frac{qk+1}{z} + \frac{\partial}{\partial z}\left(\frac{1}{1+\zeta z^{qk}}\right).$$

We can easily see that $a(z) = \lambda \frac{z^{qk+1}}{1+\zeta z^{qk}} \frac{\partial}{\partial z}, \lambda \in \mathbb{C}$, which ends the proof. □ To complete the proof of the second part of the lemma, it is enough to remark that in all the cases, the vector field X_m is the restriction to T_m of the vertical representative of $\{Z_c\}$ multiplied by an element of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}(W)$.

The proof of the first part of the lemma can be done in the same way. The restriction of $f_{\alpha c}$ to T_m defines a series $f^0_{\alpha c} \in z\widehat{\mathcal{O}}_{T_m}$ invariant under the holonomy. When h_m is not periodic, $f^0_{\alpha c}(z) = z$ and the lemma is trivial. Using the above description of the strictly reduced cases, h_m is periodic only if $\widetilde{\mathcal{F}}$ admits a (transversally formal) first integral F at the point c. This first integral can be written in the normalizing coordinates (z_1, z_2) as $F(z_1, z_2) = z_1^p z_2^q$ and can be extended to the whole U_α using proposition (3.0.11). Moreover the restriction $f^0_{\alpha c}$ of $f_{\alpha c}$ to T_m is a formal series $l(z^q)z$ with $l \in \mathbb{C}[[z]]$. Since z^q is equal on $U_{\alpha c}$ to the restriction F^0 of F to T_m, by the uniqueness of the extension (3.0.11), one has $f_{\alpha c} = l_{\alpha c}(F)z$; which completes the proof. □

4 Cohomological spaces associated to a distinguished covering.

We keep the notation of the previous paragraph and we still denote a germ at the origin of $\mathbb{C}^2$ of a formal nondicritical foliation by $\mathcal{F}$, and the distinguished covering of the divisor $\widetilde{\mathcal{D}}$ of the strict reduction of $\mathcal{F}$ by $\mathcal{U}$. For any subset $\mathcal{D}'$ of $\widetilde{\mathcal{D}}$ we denote the covering of $\mathcal{D}'$ consisting of the open sets $\mathcal{U}$ which intersect $\mathcal{D}'$ by $\mathcal{U}(\mathcal{D}')$ and the neighbourhood of $\mathcal{D}'$ in $\widetilde{\mathcal{D}}$ obtained as the union of all the open sets in $\mathcal{U}(\mathcal{D}')$ by $\overset{\circ}{\mathcal{D}'}$. From the above lemma (3.0.13) we obtain that for any irreducible component D of $\widetilde{\mathcal{D}}$ one has :

$$H^1(\mathcal{U}(D)\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = 0\,. \tag{5}$$

Definition 4.0.15 *The valence $v(D)$ of an irreducible component D of $\widetilde{\mathcal{D}}$ is the number of singular points of $\widetilde{\mathcal{F}}$ on D.*

The irreducible components of valence greater than or equal to 3 will play a special rôle in our discussion. We shall denote the set of irreducible components of $\widetilde{\mathcal{D}}$ of valence ≥ 3 by $Comp(\widetilde{\mathcal{D}})$.

Definition 4.0.16 *A chain $\mathcal{C}$ in $\widetilde{\mathcal{D}}$ is either*

- *a connected component of the union of the irreducible components of $\widetilde{\mathcal{D}}$ having valence < 3,*
- *or the intersection point c of two elements of $Comp(\widetilde{\mathcal{D}})$.*

Definition 4.0.17 *The valence $v(\mathcal{C})$ of a chain $\mathcal{C}$ is the number of intersection points of $\mathcal{C}$ with $Comp(\widetilde{\mathcal{D}})$, or 2 if the chain is reduced to the intersection point of two elements of $Comp(\widetilde{\mathcal{D}})$.*

If the set $Comp(\widetilde{\mathcal{D}})$ is not empty, the valence of a chain is either 1 or 2. We shall denote the set of chains of $\widetilde{\mathcal{D}}$ (resp. having valence $\geq r$) by $Ch(\widetilde{\mathcal{D}})$ (resp. $Ch_r(\widetilde{\mathcal{D}})$).

Remark 4.0.18 If $\widetilde{\mathcal{F}}$ is without resonant saddle-node, any chain $\mathcal{C}$ of $\widetilde{\mathcal{D}}$ has the following properties :

1) Any germ $X \in \hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ of a transversal vector field (resp. any germ of a first integral $f \in \hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$) at a singular point of $\widetilde{\mathcal{F}}$ on $\mathcal{C}$ can be extended to a unique global section of $\hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ (resp. $\hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$) on $\mathring{\mathcal{C}}$,

2) Any section of $\hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ (resp. of $\hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$) over an open set of $\mathcal{U}(\mathcal{C})$, or over the intersection of two open sets of $\mathcal{U}(\mathcal{C})$, can be extended as a unique global section of $\hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ (resp. $\hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$) on $\mathring{\mathcal{C}}$,

3) $H^1(\mathcal{U}(\mathcal{C}); \hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = 0$ and $H^1(\mathcal{U}(\mathcal{C}); \hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}) = 0$.

We get the two first properties by combining propositions (3.0.11) and (3.0.13). They imply that $H^1(\mathcal{U}(\mathcal{C}); \hat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$ (resp. $H^1(\mathcal{U}(\mathcal{C}); \hat{\mathcal{O}}_{\widetilde{\mathcal{F}}})$) is equal to the cohomology of $\mathcal{U}(\mathcal{C})$ with coefficients in the $\mathbb{C}$-vector space $\mathcal{E}$ of global sections of $\hat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ (resp. $\hat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$). To prove 3) it is enough to solve explicitly the associated system of linear equations. One can do it directly, or one can also consider the nerve of this covering, to which we can associate a simplicial 1-chain whose geometric realization is a closed interval $J \subset \mathbb{R}$, and therefore

$$H^1(\mathcal{U}(\mathcal{C}); \mathcal{E}) = H^1(J; \mathcal{E}) = H^1(J; \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{E} = 0 .$$

Lemma 4.0.19 *Let D be an irreducible component of $\widetilde{\mathcal{D}}$ having no singularity of $\widetilde{\mathcal{F}}$ of resonant saddle-node type and let H_D be its holonomy group. Then :*

1. *There exists a non-constant section of $\widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$ over D if and only if H_D is finite.*
2. *There exists a non-zero section of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ over D if and only if H_D is abelian.*

Proof. The first equivalence is well known (see [8]).
Let us consider a section $X \in H^0(W\,;\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$. It induces (3.0.11) a formal vector field Z_m on T_m whose flow commutes with the action of H_D.

If Z_m is not linearizable, it is formally conjugate (see [6]), to a vector field of the form $Y_{p/q,\zeta} := 2i\pi \dfrac{z^{qk+1}}{1+\zeta z^{qk}} \dfrac{\partial}{\partial z}$ with $p,q,k \in \mathbb{N}^*$, $(p,q) = 1$, $\zeta \in \mathbb{C}$. By (3.0.14) H_D is then a subgroup of $\mathbb{Z} \times \mathbb{C}$.

If Z_m is linearizable, with eigenvalue λ not a root of unity, H_D is also linearizable.
We deduce from the previous study of all the reduced cases that if $\lambda = exp(2i\pi sp/q)$ with $s \in \mathbb{N}$, all the singularities of $\widetilde{\mathcal{F}}$ on D admit a non-constant first integral, and in particular every element of H_D is periodic. On the other hand, the commutativity hypothesis implies that in the coordinate z which linearizes Z_m, all the elements of H_D can be written as : $h(z) = z\,l(z^q)$. We deduce that $l(z) \equiv l(0)$, with $l(0)$ a root of unity.

In all cases we have shown that H_D is abelian. The converse (that we shall not need) can be proven in the same way, using (3.0.14) and going through all the cases. □

Theorem 4.0.20 *Let $\mathcal{F}$ be a formal nondicritical foliation at the origin of $\mathbb{C}^2$ such that the strictly reduced associated foliation $\widetilde{\mathcal{F}}$ and the divisor $\widetilde{\mathcal{D}}$ have the following properties :*

1. *$\widetilde{\mathcal{F}}$ is without resonant saddle-node (3.0.12) along $\widetilde{\mathcal{D}}$.*
2. *The holonomy group of each irreducible component of $\widetilde{\mathcal{D}}$ of valence ≥ 3 is non abelian.*

We denote the sheaf of first integrals vanishing at the points of $\widetilde{\mathcal{D}}$ *by* $\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}} \subset \widehat{\mathcal{O}}_{\widetilde{\mathcal{F}}}$. *Then for every distinguished covering* $\mathcal{U}$ *of* $\widetilde{\mathcal{D}}$ *one has :*

$$H^1(\mathcal{U};\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) \simeq \bigoplus_{v(\mathcal{C})=2} \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}(\mathcal{C}), \qquad H^1(\mathcal{U};\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) \simeq \bigoplus_{v(\mathcal{C})=2} \widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}(\mathcal{C}),$$

where the direct sums are taken over all the chains of $\widetilde{\mathcal{D}}$ *of valence 2.*

Proof. Let us consider the open covering of $\widetilde{\mathcal{D}}$

$$\mathcal{W} := \left\{ \overset{\circ}{D} \ / \ D \in Comp(\widetilde{\mathcal{D}}) \right\} \bigsqcup \left\{ \overset{\circ}{\mathcal{C}} \ / \ \mathcal{C} \in Ch(\widetilde{\mathcal{D}}) \right\} .$$

Every open set $W \in \mathcal{W}$ satisfies, by (5) and (4.0.18) :

$$H^1(\mathcal{U}(W)\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) = 0 \quad \text{and} \quad H^1(\mathcal{U}(W)\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = 0\,.$$

Thus we have ([2], chapter 4) :

$$H^1(\mathcal{U}\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) = H^1(\mathcal{W}\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) \quad \text{and} \quad H^1(\mathcal{U}\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = H^1(\mathcal{W}\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})\,.$$

From the above lemma, every section S of these sheaves over a divisor of valence ≥ 3 is zero and the system of cohomological equations can be split into independant equations :

$$H^1(\mathcal{W}\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = \bigoplus_{\mathcal{C}\in Ch_1(\widetilde{\mathcal{D}})} H^1(\mathcal{W}(\mathcal{C})\ \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}})$$

and

$$H^1(\mathcal{W}\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) = \bigoplus_{\mathcal{C}\in Ch_1(\widetilde{\mathcal{D}})} H^1(\mathcal{W}(\mathcal{C})\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}),$$

where $\mathcal{W}(\mathcal{C})$ is the family of 2 or 3 elements which consists of $\overset{\circ}{\mathcal{C}}$ and of the open sets $\overset{\circ}{D}$ corresponding to the components D of $\widetilde{\mathcal{D}}$ of valence ≥ 3 and intersecting $\mathcal{C}$. When $\mathcal{C}$ has valence 1, $\mathcal{W}$ has two elements and the cohomological equation reduces to : $S_{\overset{\circ}{D}\overset{\circ}{\mathcal{C}}} = S_{\overset{\circ}{\mathcal{C}}}$; it always has a solution by the extension lemma (3.0.13), and therefore

$$H^1(\mathcal{W}(\mathcal{C})\,;\,\widehat{\mathcal{O}}^0_{\widetilde{\mathcal{F}}}) = 0\,, \quad H^1(\mathcal{W}(\mathcal{C})\,;\,\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = 0\,, \quad \text{if} \quad v(\mathcal{C}) = 1\,.$$

When $\mathcal{C}$ has valence 2, $\mathcal{W}$ has three elements and the cohomological equations can be written as :

$$(\star)\quad \begin{cases} S_{\overset{\circ}{D}\overset{\circ}{\mathcal{C}}} = S_{\overset{\circ}{\mathcal{C}}} \\ S_{\overset{\circ}{\mathcal{C}}\overset{\circ}{D}'} = S_{\overset{\circ}{\mathcal{C}}} \end{cases}$$

Again using (3.0.13) we see that every cocycle $(S_{\overset{\circ}{D}\overset{\circ}{\mathcal{C}}}, S_{\overset{\circ}{D}'\overset{\circ}{\mathcal{C}}})$ is cohomologous to a unique cocycle $(S_{\overset{\circ}{D}\overset{\circ}{\mathcal{C}}}, 0)$. As $S_{\overset{\circ}{D}\overset{\circ}{\mathcal{C}}}$ can be extended in a unique way we have isomorphisms (well defined if we orient the dual tree $\mathbb{A}^*(\mathcal{F})$ considered as a graph),

$$H^1(\mathcal{W}(\mathcal{C})\,;\,\hat{\mathcal{O}}^0_{\tilde{\mathcal{F}}}) \simeq \hat{\mathcal{O}}^0_{\tilde{\mathcal{F}}}(\mathcal{C})\,, \quad H^1(\mathcal{W}(\mathcal{C})\,;\,\hat{\mathcal{T}}_{\tilde{\mathcal{F}}}) \simeq \hat{\mathcal{T}}_{\tilde{\mathcal{F}}}(\mathcal{C})\,.$$

Therefore the conclusion holds. □

This theorem shows the importance of the following class of formal foliations :

Definition 4.0.21 *We shall say that a formal foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ is of finite formal type (f.f.t.) if for a distinguished covering $\mathcal{U}$ of the exceptional divisor $\tilde{\mathcal{D}}$, the foliation $\tilde{\mathcal{F}}$ obtained after strict reduction of its singularities satisfies :*

$$dim_{\mathbb{C}} H^1(\mathcal{U};\, \hat{\mathcal{B}}_{\tilde{\mathcal{F}}}) \;<\; \infty\,,$$

where $\hat{\mathcal{B}}_{\tilde{\mathcal{F}}}$ is the sheaf over $\tilde{\mathcal{D}}$ of transversally formal basic vector fields.

We can give a finiteness criterium :

Definition 4.0.22 *We shall say that a formal foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ is non-degenerate if it satisfies the following conditions :*

1. *$\mathcal{F}$ is nondicritical.*
2. *$\tilde{\mathcal{F}}$ has no singularity of resonant saddle-node type along $\tilde{\mathcal{D}}$.*
3. *The holonomy group of each irreducible component of $\tilde{\mathcal{D}}$ of valence ≥ 3 is non abelian.*
4. *Every germ of a transversally formal first integral of $\tilde{\mathcal{F}}$ at a singular point which is the intersection of an irreducible component of $\tilde{\mathcal{D}}$ of valence ≥ 3 with a chain of valence 2, is constant.*

Theorem 4.0.23 *Every formal nondegenerate foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ is f.f.t. and satisfies :*

$$dim_{\mathbb{C}}\, H^1(\mathcal{U};\, \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}) = \tau\,(\mathcal{F}) + \sum_{c\,\in\sigma\,(\omega)} \frac{(\nu_c - 1)(\nu_c - 2)}{2}\,,$$

where :

- *$\sigma(\omega)$ is the disjoint union of all centres C^j , $j = 0, \ldots, h-1$, in the strict reduction tree of $\mathcal{F}$.*
- *For $c \in S^j$, ν_c is the algebraic multiplicity (2) at the point c of the strict transform $\widetilde{\mathcal{F}}^{(j)}$.*
- *$\tau\,(\mathcal{F})$ is the number of chains of $\widetilde{\mathcal{D}}$ of valence 2.*

Proof. We consider the long exact sequence associated to the short exact sequence (4) defining $\widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}$. By (4.0.19) there is no non-zero global section of $\widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}$ over the irreducible components of $\widetilde{\mathcal{D}}$ of valence greater than or equal to 3, so there is none on $\widetilde{\mathcal{D}}$ and $H^0(\mathcal{U};\, \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = 0$. On the other hand $H^2(\mathcal{U};\, \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}}) = 0$ as the three by three intersections of open sets of $\mathcal{U}$ are empty (2.3.6). Therefore the sequence

$$(6) \qquad 0 \longrightarrow H^1(\mathcal{U};\, \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}}) \longrightarrow H^1(\mathcal{U};\, \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}) \longrightarrow H^1(\mathcal{U};\, \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) \longrightarrow 0$$

is exact and

$$dim_{\mathbb{C}}\, H^1(\mathcal{U};\, \widehat{\mathcal{B}}_{\widetilde{\mathcal{F}}}) = dim_{\mathbb{C}}\, H^1(\mathcal{U};\, \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) + dim_{\mathbb{C}}\, H^1(\mathcal{U};\, \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}})\,.$$

The preceding theorem tells us that $dim_{\mathbb{C}}\, H^1(\mathcal{U};\, \widehat{\mathcal{T}}_{\widetilde{\mathcal{F}}}) = \tau\,(\mathcal{F})$. The remaining term is the dimension of $H^1(\mathcal{U};\, \widehat{\mathcal{X}}_{\widetilde{\mathcal{F}}})$ computed in [7]. □

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Jean-François Mattei
Laboratoire Emile Picard, UFR MIG Université P. Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex.
mattei@picard.ups-tlse.fr

Eliane Salem
current address :
Laboratoire Emile Picard, UFR MIG Université P. Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex.
salem@picard.ups-tlse.fr