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# THE SYMPLECTIC NATURE OF THE SPACE OF PROJECTIVE CONNECTIONS ON RIEMANN SURFACES

SHINGO KAWAI

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-01, JAPAN
(京大数理研　河井 真吾)

A projective structure on a Riemann surface $X$ is given by selecting a special complex analytic coordinate covering of $X$ such that the coordinate transition functions are linear fractional transformations. Such a coordinate covering is in general realized as the local solutions of a certain kind of Schwarzian equation on $X$ which is described as a projective connection on that surface. The multivaluedness of the solutions of such a Schwarzian equation, which is represented as a homomorphism from the fundamental group of $X$ into the group PSL(2,**C**) of linear fractional transformations, is called the monodromy representation of the corresponding projective structure (or the projective connection).

By allowing the complex structure on $X$ to vary, we naturally obtain the monodromy mapping from the space of projective connections (or structures) on varying (compact) Riemann surfaces to the space of representation classes of the fundamental group of $X$ into PSL(2,**C**). Although there are various aspects of study on the monodromy mapping, we shall concentrate in this report on the symplectic-geometric properties of that mapping. Before going into the details, let us begin by clarifying our motivation for this investigation. To make the exposition simple and explicit, we start with the case of genus one.

Let $X$ be a compact Riemann surface of genus one, and $H = \{\tau \in \mathbf{C};\ \mathrm{Im}\,\tau > 0\}$ the upper half-plane. One can represent $X$ as the quotient $\mathbf{C}/L(1,\tau)$ for some $\tau \in H$, where $L(1,\tau)$ denotes the usual lattice in the complex plane $\mathbf{C}$ generated by the periods 1 and $\tau$. Then linear ordinary differential equations on $X$ are represented as those on $\mathbf{C}$ whose coefficients are doubly periodic functions. Let us consider a Fuchsian equation (on $\mathbf{C}$) of the form

$$\frac{d^2y}{dz^2} = q(z)\,y\,, \tag{1}$$

where

$$q(z) = k + \sum_{i=1}^{m} \left\{ H_i\,\zeta(z - t_i\,,\,\tau) + \frac{1}{4}\,(\theta_i^2 - 1)\,\wp(z - t_i\,,\,\tau) \right\},$$

$$\sum_{i=1}^{m} H_i = 0\,.$$

Here $\zeta(z,\,\tau)$ and $\wp(z,\,\tau)$ denote respectively Weierstrass' $\zeta$-function and $\wp$-function with fundamental periods 1 and $\tau$; thus the Laurent expansion of the function $q(z)$ at $z = t_i$ has the form

$$q(z) = \frac{\theta_i^2 - 1}{4\,(z - t_i)^2} + \frac{H_i}{z - t_i} + \text{higher terms},$$

and therefore equation (1) has its (regular) singularities at $z \equiv t_i \pmod{L(1,\tau)}$ with exponents $\frac{1}{2}(1 \pm \theta_i)$. We remark that the equations of this form are precisely the *meromorphic projective connections* of Fuchsian type on the Riemann surface $X$ (see [**4**]). (In this report we make a technical assumption that each singularity $z \equiv t_i \pmod{L(1,\tau)}$ is *not logarithmic.*)

Select a suitable fundamental parallelogram

$$F = \{ z \in \mathbf{C} ;\, z = z_0 + r_1 \cdot 1 + r_2 \cdot \tau ,\, 0 \le r_i \le 1 \}$$

with no singularity of (1) on the boundary. We assume for simplicity that the (distinct) points $t_i\,(i = 1, \dots, m)$ lie in the interior of $F$. Identifying the opposite sides of $F$ yields an explicit realization of the surface $X$. Let $\gamma_i\,(i = 1, \dots, m)$ be a loop in $F \setminus \{ t_1, \dots, t_m \}$ with base point $z_0$ encircling the point $t_i$ once counterclockwise, and $l_1$ (or $l_\tau$) the directed segment from $z_0$ to $z_0 + 1$ (or $z_0 + \tau$); these segments also represent loops in $X$ with base point $[z_0]$, where $[z]$ denotes the congruence class of a point $z \in \mathbf{C}$. One observes that the homotopy classes of the loops $\gamma_i$ and the paths $l_1$, $l_\tau$ form a set of generators of the fundamental group $\pi_1( X \setminus \{ [t_1], \dots, [t_m] \}, [z_0])$.

Let us take a basis $(y_1, y_2)$ in the space $V$ of solutions of (1) in a small neighborhood of $z_0$. Analytic continuation of the functions $y_1, y_2$ along each loop $\gamma_i$ gives another basis $(\hat{y}_1, \hat{y}_2)$ in $V$, so it determines an invertible matrix $\chi(\gamma_i) \in \mathrm{GL}(2, \mathbf{C})$ such that

$$(\hat{y}_1, \hat{y}_2) = (y_1, y_2)\, \chi(\gamma_i) .$$

Similarly, analytic continuation of $y_1, y_2$ along the path $l_1$ (or $l_\tau$) yields functions $\hat{y}_1, \hat{y}_2$ on a small neighborhood of the point $z_0 + 1$ (or $z_0 + \tau$) and determines a matrix $\chi(l_1)$ (or $\chi(l_\tau)$) $\in \mathrm{GL}(2, \mathbf{C})$ such that

$$\begin{aligned} (\hat{y}_1(z+1), \hat{y}_2(z+1)) &= (y_1(z), y_2(z))\, \chi(l_1) \\ (\text{or } (\hat{y}_1(z+\tau), \hat{y}_2(z+\tau)) &= (y_1(z), y_2(z))\, \chi(l_\tau)) . \end{aligned}$$

These matrices depend only on the homotopy classes of the loops and paths; and thus one obtains the *monodromy representation* (or simply *monodromy*)

$$\chi \colon \pi_1( X \setminus \{ [t_1], \dots, [t_m] \}, [z_0]) \to \mathrm{GL}(2, \mathbf{C}) \tag{2}$$

of equation (1). (It follows from the special form of (1) that the Wronskian of any basis ($y_1$, $y_2$) is constant; hence the image group of the homomorphism (2) is actually a subgroup of $\mathrm{SL}(2, \mathbf{C})$, the complex Lie group of $2 \times 2$ matrices of determinant one.) If we take another base point or another basis of local solutions, representation (2) turns into a conjugate one. The monodromy of equation (1) is thus defined up to this equivalence.

Consider now a (small) *deformation* of equation (1). Here we assume that the local monodromy around each singular point remains *constant*. In other words, introducing a complex parameter $s$ varying in the unit disk $\Delta$, we consider (small) variations

(3)

$$k = k(s), \ H_i = H_i(s) \left( \sum_{i=1}^{m} H_i(s) = 0 \right), \ t_i = t_i(s), \ \tau = \tau(s)$$

of the parameters of (1). The condition for the local monodromy representations to be constant is just that the parameters $\theta_i$ are to be fixed. (To be precise, however, if some of the singularities are apparent, there appear additional conditions. See [**4**].) In particular, if the deformation (3) does not change the (global) monodromy (2) as well up to conjugacy, it is called a *monodromy preserving deformation*.

In general, monodromy preserving deformations are described in terms of a completely integrable system of partial differential equations on the space of deformation parameters; such a system is called a *deformation equation*. Early in this century, R. Fuchs, L. Schlesinger and R. Garnier considered monodromy preserving deformations of second order (or systems of first order) linear ordinary differential equations on the Riemann sphere $\mathbf{P}^1$. What they derived as deformation equations included the Painlevé equations I–VI as special instances. Over fifty years later, K. Okamoto started an extensive study on monodromy preserving deformations in early 1970s. On one hand he treated that kind of problem on a torus (genus one) and derived equations that can be viewed as generalizations of the Painlevé equations (see [**8**]–[**10**]). On the other hand, studying the genus zero case again, he was led to the crucial discovery [**11**] that the monodromy preserving deformations of a second order equation on $\mathbf{P}^1$ can be described as a completely integrable Hamiltonian system on the space of deformation parameters. (Later he verified this also for the genus one case [**12**], [**13**].) The generalization of that observation to the case of higher genus was carried out by K. Iwasaki. In [**4**] Iwasaki considered a certain space of meromorphic projective connections of Fuchsian type on a compact Riemann surface of arbitrary genus. By establishing a suitable parametrization of that space, he gave an explicit description of a closed 2-form corresponding to the fundamental 2-form of the desired Hamiltonian system. Furthermore he later found [**5**] that the closed 2-form above coincides precisely with the *pullback* of the natural symplectic form on the space of monodromy representations by the monodromy mapping. That work is fundamental in the sense that it provided a geometric principle of treating monodromy preserving deformations; indeed since symplectic forms are nondegenerate, it follows that the monodromy preserving deformations are completely described by the pulled-back (degenerate) symplectic form (under the condition that the differential of the monodromy mapping is surjective).

In the studies mentioned so far, the underlying Riemann surfaces had been fixed. Generalizing this situation further, one can consider a deformation (of a differential equation) such that the underlying Riemann surface itself varies. The main purpose of our current study has been to obtain a more unified perspective by applying Iwasaki's general principle to that type of situation. We first studied in [**6**] the genus one case; specifically we treated equations of the form (1) and considered deformations of the form (3). Applying the "pulling-back" principle, we obtained the following.

THEOREM 1. *The Monodromy preserving deformations of equation* (1) *are described by the closed* 2*-form*

(4)

$$2\sum_{i=1}^{m} dH_i \wedge dt_i + \frac{1}{\pi\sqrt{-1}}\, dk \wedge d\tau$$
$$- \frac{\eta_1(\tau)}{\pi\sqrt{-1}} \sum_{i=1}^{m} (\, t_i\, dH_i \wedge d\tau + H_i\, dt_i \wedge d\tau\,)\,.$$

Remarks are in order here. The term $\eta_1(\tau)$ denotes the complex constant given by $\zeta(z+1,\, \tau) - \zeta(z,\, \tau) = \eta_1(\tau)$. The closedness of the 2-form (4) can immediately be verified by rewriting the third term as $-\frac{\eta_1(\tau)}{\pi\sqrt{-1}} \sum_{i=1}^{m} d(H_i\, t_i) \wedge d\tau$, because $\eta_1(\tau)$ depends only on $\tau$. It is natural to ask how this 2-form is altered under canonical transformations of the parameters of (1); it turns out that the 2-form is *invariant* under certain changes of the parameters. Finally, it should be observed that if we consider the monodromy preserving deformations of (1) on a *fixed* torus, the resulting 2-form would be

$$2\sum_{i=1}^{m} dH_i \wedge dt_i\,;$$

indeed this 2-form was obtained by Okamoto [**11**]–[**13**] and Iwasaki [**4**] (for the case of arbitrary genus).

Having finished clarifying our motivation and reviewing the result of [**6**], we get back to the main topic of this report. As mentioned earlier, we consider next the space of (holomorphic) projective connections on varying compact Riemann surfaces (of genus $g \geq 2$). Although we restrict ourselves to the holomorphic connections, our result will provide an *intrinsic* description of the desired pulled-back symplectic structure. Let us first recall the basic terminology to be used.

Let $X$ be a compact Riemann surface of genus $g \geq 2$, and $p\colon H \to X$ a universal covering of $X$ with covering transformation group $\Gamma$. By the uniformization theorem, one can take $H$ to be the upper half-plane and $\Gamma \subset \mathrm{PSL}(2,\mathbf{R})$ a strictly hyperbolic Fuchsian group acting on $H$ by linear fractional transformations. A holomorphic function $q\colon H \to \mathbf{C}$ is called a (holomorphic) *quadratic differential for* $\Gamma$ if

$$q(\gamma z)\,\gamma'(z)^2 = q(z) \quad \text{for all} \quad \gamma \in \Gamma\,,\ z \in H\,.$$

The space $A_2(H,\Gamma)$ of all quadratic differentials for $\Gamma$ can canonically be identified with the space of holomorphic (2,0)-forms on $X$ and therefore turns out to be a $(3g-3)$-dimensional complex vector space (by Riemann-Roch). For a quadratic differential $q \in A_2(H,\Gamma)$, consider the differential equation

$$S(f)(z) = q(z) \tag{5}$$

on $H$, where $S(f) = (f''/f')' - 1/2\,(f''/f')^2$ denotes the Schwarzian derivative of the function $f$. Any solution $f$ of (5) turns out to be a *locally biholomorphic* (or locally schlicht) mapping from $H$ into the Riemann sphere $\widehat{\mathbf{C}}$, and there arises a homomorphism $\rho\colon \Gamma \to \mathrm{PSL}(2,\mathbf{C})$ such that

$$f(\gamma z) = \rho(\gamma) f(z) \quad \text{for all} \quad \gamma \in \Gamma\,,\ z \in H\,; \tag{6}$$

here $\mathrm{PSL}(2,\mathbf{C})$ is the group of linear fractional transformations acting on $\widehat{\mathbf{C}}$. The mapping $f$ can be viewed as describing a special complex analytic coordinate covering of the Riemann surface $X$, in the sense that the coordinate transition functions of the covering are linear fractional transformations; thus we say that $f$ determines a *projective structure* on $X$, and we call $\rho$ the *monodromy representation* determined by $f$.

Since the most general solution of (5) has the form $A \circ f$ for some $A \in \mathrm{PSL}(2,\mathbf{C})$ and the corresponding homomorphism can be written as $\gamma \mapsto A\rho(\gamma)A^{-1}$, it follows that each quadratic differential determines an *equivalence class* of projective structures on $X$ and a conjugacy class of representations $\Gamma \to \mathrm{PSL}(2,\mathbf{C})$. Conversely any local biholomorphism $f$ satisfying (6) yields an element of $A_2(H,\Gamma)$ via the identity (5), the element depending only on the equivalence class of $f$. Thus there is a canonical one-to-one correspondence between the space $A_2(H,\Gamma)$ and the set of equivalence classes of projective structures on $X$.

REMARK. In general, one can establish a natural one-to-one correspondence between the affine space of *projective connections* on a Riemann surface and the set of equivalence classes of projective structures on that surface. In the case above, since the Riemann surface $X$ has a fixed projective structure via the representation $X = H/\Gamma$, the set of projective connections can be identified with the set of quadratic differentials.

Let us turn next to *varying* the complex structure on the (marked) Riemann surface $X$. For this purpose we introduce the Teichmüller space $T(\Gamma)$ of the (marked) Fuchsian group $\Gamma$ and the universal Teichmüller curve $V(\Gamma)$, a natural fiber space over $T(\Gamma)$ with projection $\pi\colon V(\Gamma) \to T(\Gamma)$. To each point $\tau \in T(\Gamma)$ there are associated a quasidisk $H_\tau$ and a quasi-Fuchsian group $\Gamma_\tau$ (with invariant domain $H_\tau$) such that the fiber $\pi^{-1}(\tau)$ of the projection $\pi\colon V(\Gamma) \to T(\Gamma)$ above $\tau$ is precisely the marked Riemann surface $H_\tau/\Gamma_\tau$ represented by $\tau$. The crucial point here is that this construction of $V(\Gamma)$ (due to Bers) provides each fiber $\pi^{-1}(\tau)$ with a fixed projective structure via the representation $H_\tau/\Gamma_\tau$. Hence, just as in the discussion above, there arises a natural one-to-one correspondence between the set of equivalence classes of projective structures on $\pi^{-1}(\tau)$ and the space $A_2(H_\tau,\Gamma_\tau)$ of quadratic differentials on $H_\tau$ for $\Gamma_\tau$. Furthermore the spaces $A_2(H_\tau,\Gamma_\tau)$ for $\tau \in T(\Gamma)$ can be glued together to form a holomorphic vector bundle $Q \to T(\Gamma)$ of rank $3g-3$; thus the $(6g-6)$-dimensional total space $Q$ qualifies as the universal space of equivalence classes of projective structures on *varying* Riemann surfaces of genus $g$.

Recalling that each element $q \in A_2(H_\tau, \Gamma_\tau)$ determines a conjugacy class of representations $\Gamma_\tau \to \mathrm{PSL}(2,\mathbf{C})$, one obtains the *monodromy mapping*

$$F\colon Q \to \mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$$

via the canonical isomorphisms $\Gamma_\tau \cong \Gamma$, where $\mathrm{PSL}(2,\mathbf{C})$ acts as a group of transformations on $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))$ by inner automorphisms. The fundamental properties of the mapping $F$ are: (i) although the set $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ has in general rather complicated singularities, the points in $\mathrm{Im}F$ are regular points of that space, and (ii) the mapping $F$ is a *local biholomorphism* from the space $Q$ onto an open subset of $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ (see [**1**]–[**3**]).

As explained earlier, our purpose here is to describe that symplectic structure on $Q$ which is given by pulling back the natural symplectic structure $\omega_{\mathrm{PSL}}$ on $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ via the monodromy mapping $F$. However, since the space $Q$ can be viewed as the total space of the holomorphic cotangent bundle $T^*T(\Gamma)$ of the Teichmüller space $T(\Gamma)$, it follows that there is defined a canonical symplectic structure $\omega_Q$ on $Q$. Our mail result [**7**] then asserts that the desired pulled-back symplectic structure on $Q$ is precisely the canonical $\omega_Q$ (up to a constant factor).

THEOREM 2. *The mapping* $F\colon Q \to \mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ *preserves the symplectic structure up to the constant* $\pi$, *that is,*

$$\pi\ F^*\omega_{\mathrm{PSL}} = \omega_Q\,.$$

To be more precise, the theorem can be restated as follows. Let $P \to T_g$ be the (holomorphic) affine bundle of projective connections on marked genus $g$ Riemann surfaces. Selecting a point $\tau_0 \in T_g$ and representing it in the form $H/\Gamma$ allows us to identify $T_g$ with $T(\Gamma)$. By using Bers' construction of the universal Teichmüller curve $\pi\colon V(\Gamma) \to T(\Gamma)$, we obtain a *holomorphic* cross-section of the bundle $P \to T_g$; and there then arises a natural commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{F}} & \mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C}) \\ {\scriptstyle B}\downarrow & & \| \\ Q & \xrightarrow[F]{} & \mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C}), \end{array}$$

where $B\colon P \to Q$ is the (biholomorphic) mapping that identifies the projective connections on each fiber $\pi^{-1}(\tau)$ with the vector space $A_2(H_\tau, \Gamma_\tau)$. Our main result can now be rewritten as

$$\pi\, \widetilde{F}^* \omega_{\mathrm{PSL}} = B^* \omega_Q\,.$$

We emphasize here that the choice of the cross-section made above is crucial for describing the pulled-back symplectic structure $\widetilde{F}^*\omega_{\mathrm{PSL}}$ in this way; for instance, we cannot use that cross-section which is given by applying the usual uniformization theorem to each element of $T_g$ because it is *not* even *holomorphic.* In particular, since the zero-section of a cotangent bundle determines a Lagrangian immersion with respect to the canonical symplectic structure, we have the following corollary.

COROLLARY 3. *The cross-section of the bundle $P \to T_g$ given by Bers' construction of the universal Teichmüller curve $\pi\colon V(\Gamma) \to T(\Gamma)$ determines a Lagrangian immersion with respect to the pulled-back symplectic structure $\widetilde{F}^*\omega_{\mathrm{PSL}}$.*

It should be noted here that the cross-section above depends on the choice of the base point $\tau_0 \in T_g$; thus we have actually obtained a family of Lagrangian immersions parametrized by the Teichmüller space $T_g$.

Passing to the space $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ via the mapping $\widetilde{F}$ yields another formulation of the corollary. The space $\mathrm{Hom}(\Gamma,\ \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$ contains as a subset the *quasiconformal deformation space* $QH(\Gamma)$ of $\Gamma$; this space consists of those representations $\Gamma \to \mathrm{PSL}(2,\mathbf{C})$ which can be written as $\gamma \mapsto w \circ \gamma \circ w^{-1}$ for some quasiconformal mapping $w$ of the Riemann sphere $\widehat{\mathbf{C}}$ onto itself. By a simple argument we find that $QH(\Gamma)$ can be put into a canonical one-to-one correspondence with $T(\Gamma) \times T(\Gamma) = T_g \times T_g$; and (the image of) the cross-section of the bundle $P \to T_g$ in the corollary then corresponds to a "Bers slice" $T_g \times \{*\}$ via the mapping $\widetilde{F}$. ( A change of the base point $\tau \in T_g$ gives another slice of $T_g \times T_g$.) With these remarks in mind, we immediately obtain the following.

COROLLARY 4. *The Bers slices of the quasiconformal deformation space $QH(\Gamma)$ are Lagrangian submanifolds of the space* $\mathrm{Hom}(\Gamma,\, \mathrm{PSL}(2,\mathbf{C}))/\mathrm{PSL}(2,\mathbf{C})$.

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