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Residue formulas for singular foliations defined by meromorphic functions on surfaces

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1 Baum-Bott residue of singular foliation on surfaces

In this artile, let X be a two dimensional complex manifold (complex surface). A dimension one singular foliation is $\mathcal{E}$ on X is defined by a system $\{(U_\alpha, v_\alpha)\}$, where $\{U_\alpha\}$ is an open covering X and v_α is a holomorphic vector field on U_α for each α, such that $v_\beta = e_{\alpha\beta} v_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomorphic function $e_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.

Let $S(v_\alpha)$ be a zero-set of v_α on U_α. The condition $v_\beta = e_{\alpha\beta} v_\alpha$, we have $S(v_\alpha) = S(v_\beta)$ on $U_\alpha \cap U_\beta$. Therefore we can define the singular set $S(\mathcal{E})$ of $\mathcal{E}$ by $S(\mathcal{E}) = \cup_\alpha S(v_\alpha)$. We say $\mathcal{E}$ is reduced if $S(\mathcal{E})$ consists of only isolated points. Since $\{e_{\alpha\beta}\}$ satisfies the cocyle condition, $e_{\alpha\beta} = e_{\alpha\gamma} e_{\gamma\beta}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, it defines a line bundle E.

A singular foliation can also be defined in terms of holomorphic 1-forms. A codimension one singular foliation $\mathcal{F}$ on X is defined by a system $\{(U_\alpha, \omega_\alpha)\}$, where ω_α is a holomorphic 1-form on U_α for each U_α such that $\omega_\beta = f_{\alpha\beta}\omega_\alpha$ on $U_\alpha \cap U_\beta$ for some non-vanishing holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.

Similarly to the case of vector field, we can define the singular set $S(\mathcal{F})$ by $S(\mathcal{F}) = \cup_\alpha S(\omega_\alpha)$, where $S(\omega_\alpha)$ is the zero-set of ω_α on U_α. We say $\mathcal{F}$ is reduced if $S(\mathcal{F})$ consists of only isolated points. A line bundle F is determined by the cocyle $\{f_{\alpha\beta}\}$.

These two definitions are equivalent as long as we consider reduced foliations. There is a natural one-to-one correspondence as folowing.

$$\mathcal{E} = \{(U_\alpha, v_\alpha)\} \quad \underset{\langle v_\alpha, \omega_\alpha\rangle = 0}{\overset{\text{annihilator}}{\longleftrightarrow}} \quad \mathcal{F} = \{(U_\alpha, \omega_\alpha)\}$$

In this correspondence, $S(\mathcal{F}) = S(\mathcal{E})$, the integral curves of v_α are equal to the solution of $\omega_\alpha = 0$ (See [Sw]). Hence we consider only reduced foliations in what follows.

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Let $\mathcal{E}$ be a one dimensional reduced singular foliation. For each point $p \in S(\mathcal{E})$ and a homogeneous and symmetric polynomial ψ in degree two, we have the Baum-Bott residue $\mathrm{Res}_{\psi}(\mathcal{E},p) \in \mathbf{C}$ as following.

Suppose $(U_{\alpha},(x,y))$ is a coordinate neighborhood with the origin p, and p is the isolated zero of the vector field $v = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ on U, where v defines $\mathcal{E}$ on U. Let A be the Jacobian $\frac{\partial(a,b)}{\partial(x,y)}$, $\sigma_1 = X_1 + X_2$, $\sigma_2 = X_1X_2$, i.e. the elementary symmetric functions in two variables. We set

$$\sigma_1(A) = \mathrm{trace}A, \quad \sigma_2(A) = \det A.$$

ψ can be written as $\psi = \tilde{\psi}(\sigma_1,\sigma_2)$ by some polynomial $\tilde{\psi}$. We set $\psi(A) = \tilde{\psi}(\sigma_1(A),\sigma_2(A))$. Then the Baum-Bott residue $\mathrm{Res}_{\psi}(\mathcal{E},p)$ is given by the integral

$$\mathrm{Res}_{\psi}(\mathcal{E},p) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\Gamma} \frac{\psi(A)dx \wedge dy}{ab},$$

where $\Gamma = \{(x,y) \in U \mid |\, a(x,y)| = |\, b(x,y)| = \varepsilon\}$ for a sufficientry small positive number ε and is oriented $\deg a \wedge \deg b > 0$. In particular when $\psi = \sigma_2$, the residue $\mathrm{Res}_{\psi}(\mathcal{E},p)$ is equal to $(a,b)_p$, the index of v at p. If v is global, we get Poincaré-Hopf formula. We denote by TX the holomorphc tangent bundle of X. The following theorem is known. (See [BB].)

Theorem 1.1 (Baum-Bott) *If X is compact, we have*

$$\sum_{p \in S(\mathcal{E})} \mathrm{Res}_{\psi}(\mathcal{E},p) = \psi(TX - E) \frown [X],$$

where, denoting by $c_1 = c_1(TX - E)$ and $c_2 = c_2(TX - E)$ the first and second Chern classes of the virtual bundle of $TX - E$, we set $\psi(TX - E) = \tilde{\psi}(c_1,c_2)$.

Let $\mathcal{F}$ and F be the codimendion one foliation corresponding to $\mathcal{E}$ and the line bundle associated with $\mathcal{F}$ respectively. We have following lemma and proposition. For line bundles L_1 and L_2, we denote $c_1(L_1)c_1(L_2) \frown [X]$ by $L_1 \cdot L_2$

Lemma 1.2 *$F = E \otimes K$, where K is a cannonical bundle of X.*

Propsition 1.3 *If X is compact,we have*

$$\begin{aligned} \sum_{p \in S(\mathcal{F})} \mathrm{Res}_{\sigma_1^2}(\mathcal{E},p) &= F^2 \\ \sum_{p \in S(\mathcal{F})} \mathrm{Res}_{\sigma_2}(\mathcal{E},p) &= \chi(X) - K \cdot F + F^2, \end{aligned}$$

where $\chi(X)$ is Euler number of X.

2 Singular foliations defined by meromorphic functions

Let φ be a meromorphic function on X. Take a coordinate covering $\mathcal{U} = \{U_\alpha\}$ of X such that on each U_α, the differential $d\varphi$ of φ is writen as $d\varphi = \varphi_\alpha \omega_\alpha$ where ω_α is a holomorphic 1-form with isolated zeros on U_α and φ_α is a meromorphic function on U_α. Then the system $\{(U_\alpha, \omega_\alpha)\}$ determines a singular foliation $\mathcal{F}$ which is reduced and codimension one. The associated line bundle F is defined by the cocycle $\{f_{\alpha\beta}\}$, where $f_{\alpha\beta} = \frac{\varphi_\alpha}{\varphi_\beta}$. The leaves of $\mathcal{F}$ are the level sets of φ.

Let $D^{(0)}$, $D^{(\infty)}$ be a zero and pole divisor of φ, respectively. $D^{(0)} = \sum_{j=1}^{s} n_j D_j^{(0)}$ and $D^{(\infty)} = \sum_{i=1}^{r} m_i D_i^{(\infty)}$ are irreducible decompositions. We denote by $|D|$ the support of D and by $[D]$ the line bundle determined by D.

Lemma 2.1 *If the critical points of φ in $X - |D^{(\infty)}|$ are all isolated, then we have $F = [-\sum_{i=1}^{r}(m_i+1)D_i^{(\infty)}]$.*

Under the assumption of this lemma,

$$\begin{aligned} S(\mathcal{F}) \cap (X - |D^{(\infty)}|) &= \{\text{the critical point of } \varphi\ \} \\ S(\mathcal{F}) \cap |D^{(\infty)}| &\supset D^{(0)} \cap D^{(\infty)} \quad (\text{indeterminacies of } \varphi) \\ & \qquad D_i^{(\infty)} \cap D_j^{(\infty)} \quad (\text{singularities of } D^{(\infty)})\ . \end{aligned}$$

Hereafter we asuume that the critical point of φ in $X - |D^{(\infty)}|$ are all isolated. We denote by $\mathcal{E}$ the dimension one foliation corresponding to $\mathcal{F}$, which is an annihilator of $\mathcal{F}$.

Lemma 2.2 *For the singular point p of $\mathcal{E}$ in $X - |D^{(\infty)}|$, we have*

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0, \quad \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\varphi),$$

where $\mu_p(\varphi)$ is the Milnor number of φ at p.

In what follows, for divisors D_1 and D_2, we denote by $(D_1, D_2)_p$ the intersection number at p and by $D_1 \cdot D_2$ the total intersection number.

Lemma 2.3 *For the singular point p of $\mathcal{E}$ in $|D^{(\infty)}|$, we have*

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = \sum_{i=1}^{r} \frac{(m_i+1)^2}{m_i}(D^{(0)}, D^{(\infty)})_p - \sum_{1 \le i \le j \le r} \frac{(m_i - m_j)^2}{m_i m_j}(D_i^{(\infty)}, D_j^{(\infty)})_p.$$

Thus if p is not an intersection point of $D^{(0)}$ and $D_i^{(\infty)}$ or of $D_i^{(\infty)}$ and $D_j^{(\infty)}$ which is $m_i \neq m_j$ then $\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = 0$.

Set $D=\sum_{i=1}^{r}(m_i+1)D^{(\infty)}$ which may be called the pole divisor of $d\varphi$. From the above (2.2) and (2.3), we get following.

Propsition 2.4 *Let φ be a meromorphic function on a compact complex surface X. If the critical points of φ in $X-|D^{(\infty)}|$ are all isolated, we have*

$$D^2=\sum_p\left(\sum_{i=1}^{r}\frac{(m_i+1)^2}{m_i}(D^{(0)},D^{(\infty)})_p-\sum_{1\le i\le j\le r}\frac{(m_i-m_j)^2}{m_im_j}(D_i^{(\infty)},D_j^{(\infty)})_p\right)$$

$$\sum_{p\in S(\mathcal{E})\cap(X-|D|)}\mu_p(\varphi)+\sum_{p\in S(\mathcal{E})\cap|D|}\mathrm{Res}_{\sigma_2}(\mathcal{E},p)=\chi(X)+D^2+K\cdot D$$

Remark 2.5 *We call the quantity $\frac{1}{2}(D^2+K\cdot D)+1$ the "virtual genus" of a divisor of D (See [K]). Then we may define the "virtual euler number" of a divisor D by $\chi'(D)=-(D^2+K\cdot D)$. (c.f. $\chi(X)=2-2g(X)$) With this the second equation of (2.4) is written as*

$$\sum_{p\in S(\mathcal{E})\cap(X-|D|)}\mu_p(\varphi)+\sum_{p\in S(\mathcal{E}\cap|D|)}\mathrm{Res}_{\sigma_2}(\mathcal{E},p)=\chi(X)-\chi'(D)$$

3 Foliations arising from polynomials

Let $f(x,y)$ be a polynomial of degree d with complex coefficients. Consider the rational function φ_0 on $\mathbf{P}^2=\{[\zeta_0,\zeta_1,\zeta_2]\}$ given by

$$\varphi_0(\zeta_0,\zeta_1,\zeta_2)=\frac{\tilde{f}(\zeta_0,\zeta_1,\zeta_2)}{\zeta_0^d},$$

where $\tilde{f}(\zeta_0,\zeta_1,\zeta_2)$ is a homogenized polynomial of f. Suppose that the critical points of f are all isolated. Thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are relatively prime and f is reduced.

We denote by $\mathcal{F}$ the singular foliation on $\mathbf{P}^2$ defined by φ_0. The pole divisor of φ_0 is dL_∞, where L_∞ is the infinite line $\{\zeta_0=0\}$. Thus the line bundle F corresponding to $\mathcal{F}$ is given by $F=[-(d+1)L_\infty]$.

Let $U_i=\{\zeta_i\neq 0\}\subset\mathbf{P}^2$ $(i=1,2,3)$. On the finite part $U_0=\mathbf{C}^2\subset\mathbf{P}^2$, $\mathcal{F}$ is defined by df. By assumption the critical points of f are all isolated, we have $S(\mathcal{F})\cap U_0=C(f)$, the set of critical points of f on U_0. Now we consider in the infinite part of $\mathbf{P}^2$. We work on U_2 however it is similar on U_1. We can assume that $f_d(x,y)$ is not divisible by y, where $f_d(x,y)$ is a homogeneous piece of degree d of f. Then $S(\mathcal{F})\cap L_\infty\subset U_2$. We take $(u,v)=\left(\frac{\zeta_0}{\zeta_2},\frac{\zeta_1}{\zeta_2}\right)$ as a coordinate system on U_2. The function φ_0 is written as $\varphi_0(u,v)=\frac{\hat{f}(u,v)}{u^d}$ on U_2, where $\hat{f}(u,v)=\tilde{f}(u,v,1)$. On U_2, $\mathcal{F}$ is defined by

$$\omega=\left(u\frac{\partial\hat{f}}{\partial u}-d\cdot f\right)du+u\frac{\partial\hat{f}}{\partial v}dv.$$

Now since $S(\mathcal{F}) \cap L_\infty = \{u = f_d(v,1) = 0\}$, the set of intersection points $D^{(0)}$ and $D^{(\infty)}$, and $D^{(\infty)}$, by (2.3), we have

$$\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = \frac{(d+1)^2}{d}\mathrm{m}_p(f), \quad p \in S(\omega) \cap L_\infty,$$

where $\mathrm{m}_p(f) = (D^{(0)}, L_\infty)_p$. Since $\sum \mathrm{m}_p(f) = d$, the formula $\sum \mathrm{Res}_{\sigma_1^2} = F^2$ is a tautology.

The foliation $\mathcal{E}$ corresponding to $\mathcal{F}$ is defined by the vector field $u\frac{\partial \hat{f}}{\partial v}\frac{\partial}{\partial u} - \left(u\frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right)\frac{\partial}{\partial v}$ on U_2. For the singular point p of $\mathcal{E}$ in $L_\infty \cap U_2$, we can calculate $\mathrm{Res}_{\sigma_2}(\mathcal{E}, p)$ as following.

$$\begin{aligned}
\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) &= \left(u\frac{\partial \hat{f}}{\partial v}, u\frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right)_p \\
&= (u, \hat{f})_p + \left(u\frac{\partial \hat{f}}{\partial v}, u\frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right)_p \\
&= \mathrm{m}_p(f) + I_2,
\end{aligned}$$

where $I_2 = \left(u\frac{\partial \hat{f}}{\partial v}, u\frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right)_p$. In order to calculate I_2, let $\frac{\partial \hat{f}}{\partial v} = h_1^{m_1} h_2^{m_2} \cdots h_l^{m_l}$ be a irreducible decomposition at p and $\pi(t) = (u(t), v(t))$ a uniformalization of $h_i = 0$. Now if we write

$$\hat{f}(\pi(t)) = \sum_{n \ge q_i} a_n t^n, \quad \frac{\partial \hat{f}}{\partial u}(\pi(t)) = \sum_{n \ge r} b_n t^n, \quad u(t) = \sum_{n \ge s} c_n t^n$$

with a_{q_i}, b_r, $c_s \neq 0$. From $\frac{d\hat{f}}{dt}(\pi(t)) = \frac{\partial \hat{f}}{\partial u}(\pi(t))\frac{du}{dt}$,

$$q_i = r + s, \quad n a_n = \sum_{k=s}^{n-r} k c_k b_{n-k} \quad (n \ge q_i).$$

Thus we may write

$$\left(u\frac{\partial \hat{f}}{\partial u} - d \cdot \hat{f}\right)(\pi(t)) = \sum_{n \ge q_i}\left(\sum_{k=s}^{n-r} c_k b_{n-k} - d a_n\right) t^n.$$

We denote the order of this power series by $q_i + \delta_i$. Since $q_i = \left(h_i, u\frac{\partial \hat{f}}{\partial u}\right)_p$, we have

$$\begin{aligned}
I_2 &= \sum_{i=1}^{l} m_i q_i + \sum_{i=1}^{l} m_i \delta_i = \left(\frac{\partial \hat{f}}{\partial v}, u\frac{\partial \hat{f}}{\partial u}\right)_p + \delta_p \\
&= \left(\frac{\partial \hat{f}}{\partial v}, u\right)_p + \left(\frac{\partial \hat{f}}{\partial v}, \frac{\partial \hat{f}}{\partial u}\right)_p + \delta_p = \mu_p(\hat{f}) + \mathrm{m}_p(f) - 1 + \delta_p,
\end{aligned}$$

where $\delta_p = \sum_{i=1}^{l} m_i \delta_i$. The number δ_p is reffered to as the "value of a jump in Milnor number at ∞ " by D.T.Lê. In general $\delta_p = 0$. Thus we have

$$\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\hat{f}) + 2\mathrm{m}_p(f) - 1 + \delta_p$$

Since $\chi(\mathbf{P}^2) = 3$, $K_{\mathbf{P}^2} = -3L_\infty$, $D = (d+1)L_\infty$, $L_\infty^2 = 1$, $\sum \mathrm{m}_p(f) = d$, we have the following formula.

Theorem 3.1

$$\sum_{p \in C(f)} \mu_p(f) + \sum_{i=1}^{k} (\mu_{p_i}(\hat{f}) + \delta_{p_i} - 1) = d^2 - 3d + 1,$$

where, letting $f_d(x, y) = \Pi_{i=1}^{k} (b_i x - a_i y)^{d_i}$, $p_i = [0, a_i, b_i]$, $\mathrm{m}_{p_i}(f) = d_i$.

This formula is also obtained by D.T.Lê in the case f has no critical points. (not published.)

Next we consider the compactification $\pi : X \to \mathbf{P}^2$ of f as constructed by D.T.Lê and C.Webber (See [LW]). The set $A(f)$ of atypical values of f is expressed as $A(f) = D(f) \cup I(f)$, where $D(f)$ is the set of critical values of f and $I(f)$ is determined by the behavior of f at infinity. Then the compactification $\pi : X \to \mathbf{P}^2$ is obtained from $\mathbf{P}^2$ by a finite sequence of blowing up "points at infinity" and have following properties.

(1) X is a compact complex surface and π is a proper holomorphic map inducing a biholomorphic map of $X - \pi^{-1}(L_\infty)$ onto $\mathbf{P}^2 - L_\infty = \mathbf{C}^2$.

(2) $\pi^{-1}(L_\infty)$ is a union of projective lines with normal crossings.

(3) The meromorphic function $\varphi = \varphi_0 \circ \pi$ does not have indeterminacy points, where $\varphi_0 = \frac{\tilde{f}}{\zeta_0^d}$. Thus we may think of $\varphi : X \to \mathbf{P}^1$ as a holomorphic map.

(4) For $\lambda \in \mathbf{C} - I(f)$, π gives an imbedded resolution of the singularities of the curve $C_\lambda : \tilde{f} - \lambda \zeta_0^d = 0$ on L_∞.

Moreover, if we denote by $\mathcal{A}$ and $\mathcal{A}_\infty$, respectively, the intersection graphes of the divisor $\pi^{-1}(L_\infty)$ and the pole divisor of φ,

(5) $\mathcal{A}$ is a connected tree and $\mathcal{A}_\infty$ is a connected sub-t ree of $\mathcal{A}$.

(6) Each connected component of $\mathcal{A} - \mathcal{A}_\infty$ is a bamboo which contains a unique dicritical component (a component of $\pi^{-1}(L_\infty)$ on which φ is not constant).

Let $\mathcal{E}$ be the foliation on X which is determined by φ and $D^{(\infty)} = \sum_{i=1}^{r} m_i D_i^{(\infty)}$ be the pole divisor of φ. We assume all the critical points of φ are isolated. Then there are two types of singularities of $\mathcal{E}$.

(a) critical ponits of φ on $X - |D^{(\infty)}|$,

(b) intersection points in $D^{(\infty)}$.

For the type (a) singularity p, $\mathrm{Res}_{\sigma_1}(\mathcal{E}, p) = 0$ and $\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \mu_p(\varphi)$ as before. For the type (b) singularity p, $\mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = -\frac{(m_i - m_j)^2}{m_i m_j}$ if p is an intersection point of $D_i^{(\infty)}$ and $D_j^{(\infty)}$. On the neighborhood of the type (b) singularity p, we can write $\mathcal{E} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. Then $\mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = (x, y)_p = 1$. Again we set $D = \sum_{i=1}^{r}(m_i + 1)D_i^{(\infty)}$, then $\sum \mathrm{Res}_{\sigma_1^2}(\mathcal{E}, p) = F^2$ becomes

$$D^2 = -\sum_{1 \leq i < j \leq r} \frac{(m_i - m_j)^2}{m_i m_j}\delta_{ij}, \quad \delta_{ij} = \begin{cases} 0 & \text{when } D_i^{(\infty)} \text{ meets } D_j^{(\infty)} \\ 1 & \text{othewise} \end{cases}.$$

We recall $D(f) \subset A(f)$, then $\sum \mathrm{Res}_{\sigma_2}(\mathcal{E}, p) = \chi(X) + K \cdot D + D^2$ becomes

$$\sum_{\lambda \in A(f)} \mu(X_\lambda) + l = \chi(X) - \chi'(D),$$

where $\mu(X_\lambda)$ is a total Milnor number of $X_\lambda = \{\varphi = \lambda\}$ and l is the number of intersection points of $D^{(\infty)}$. The last equation may be thought of as a "Milnor number formula " in the presence of multiple fibers. In fact, we assume that $D^{(\infty)}$ is reduced. We obtain the Milnor number formula in the two domensional case. (See also [TT])

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