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Periodic commuting squares

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Introduction

In his paper [3], Jones introduced an index for a pair of type II_1 factors and showed that the index value less than 4 is equal to $4\cos^2(\pi/n)$ for some integer $n \geq 3$. Since then the interests of study in the theory of operator algebras have been gradually extended from a single factor to a pair of factors. Pimsner-Popa [6] showed for a pair of factors $N \subset M$ with finite index, the existence of a special orthonormal basis, called Pimsner-Popa basis, of M as an N-module. Kosaki [4] extended index theory to arbitary factors and gave the definition of an index depending on a conditional expectation. In the case of C*-algebras, Watatani defined an index by using a quasi-basis.

However it is not easy to calculate explicitly the index even for a pair of II_1 factors from the definition itself or from such a basis. So many index formulas were given by Pimsner-Popa [6], Wenzl [12], Ocneanu [5] and the present author [9] respectively. In the preceding paper [9], we treat a pair of factors $N \subset M$ generated by the increasing sequences $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ of finite direct sums of II_1 factors such that the diagram

(A)
$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square for any $n \in \mathbb{N}$, and obtained the following

Theorem. *Let* $\{M_n\}_{n\in\mathbb{N}}$ *and* $\{N_n\}_{n\in\mathbb{N}}$ *be increasing sequences of finite direct sums of* II_1 *factors such that the diagram (A) is a commuting square for any* $n \in \mathbb{N}$. *Set* $M = (\bigcup M_n)''$ *and* $N = (\bigcup N_n)''$. *If a certain periodicity condition (Condition I in 1.4 below) holds, then there exists* $n_0 \in \mathbb{N}$ *such that*

$$[M : N] = [M_n : N_n] \qquad \text{for } n \geq n_0.$$

In this note we study commuting squares which generate increasing sequences satisfying the above periodicity condition.

Let us explain more exactly, let a diagram

$$\text{(C)} \qquad \begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$$

be a commuting square of finite direct sums of finite factors. By iterating the basic construction, we get projections $e_n = e_{B_{n-1}}$, and finite von Neumann algebras $B_{n+1} = \langle B_n, e_n\rangle$ and put $A_{n+1} = (A_n \cup \{e_n\})''$ for $n \in \mathbb{N}$.

Definition 2.1. A commuting square (C) is periodic if, for any $n \in \mathbb{N}$,

(i) trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2, and

(ii) $T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive.

We give a neccesary and sufficient condition for a commuting square to be periodic.

Theorem 2.1. *A commuting square (C) is periodic if and only if there exists a positive constant* λ *such that* $F_{A_0}^{A_1} = \lambda I_n$ *and* $F_{B_0}^{B_1} = \lambda I_m$, *where* $n = \dim_{\mathbb{C}} Z(A_0)$, $m = \dim_{\mathbb{C}} Z(B_0)$ *and* I_n *is the identity matrix in* $M_n(\mathbb{C})$.

Moreover increasing sequences constructed from a periodic commuting square satisfy the periodicity condition.

Futhermore we consider a periodic commuting square, in which only one von Neumann algebra among the four is not a factor, and show properties of such squares.

Theorem 3.2. *Let $N \subset M \subset L$ be II_1 factors such that $[L : M] = [M : N] = 2$, and K be a nonfactor intermediate von Neumann algebra for $N \subset L$. Suppose that the diagram*

$$\begin{matrix} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{matrix}$$

is a periodic commuting square. Then there exists an outer action α of $\mathbb{Z}_2$ on N such that

$$\begin{matrix} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{matrix} \cong \begin{matrix} N & \subset & N \rtimes_\alpha \mathbb{Z}_2 \\ \cap & & \cap \\ (N \cup \{\mu\})'' & \subset & N \rtimes_\alpha \mathbb{Z}_2 \rtimes_{\widehat{\alpha}} \widehat{\mathbb{Z}_2} \end{matrix},$$

where μ is the implementing unitary for $\widehat{\alpha}$.

1. Preliminaries

1.1. Inclusions of von Neumann algebras. Let $M = \bigoplus_{j=1}^{m} M_j$ be a finite direct sums of finite factors and $\{q_j ; j = 1, \cdots, m\}$ the corresponding minimal central projections. Since the normalized trace on a factor is unique, a trace tr on M is specified by a column vector $\vec{s} = (\mathrm{tr}(q_1) \cdots \mathrm{tr}(q_m))^t$, called the trace vactor.

Let $N = \bigoplus_{i=1}^{n} N_i \subset M$ be another finite direct sum of finite factors having the same identity and $\{p_i ; i = 1, \cdots, n\}$ the corresponding minimal central projections. We assume that the trace on N is the restriction of the trace tr and denote by $\vec{t}$ the trace vector for N.

The inclusion $N \subset M$ is represented by two matrices, one is the index matrix and the other is the trace matrix. The index matrix $\Lambda_N^M = (\lambda_{ij})$ is defined by

$$\lambda_{ij} = \begin{cases} [M_{p_i q_j} : N_{p_i q_j}]^{1/2} & \text{if } p_i q_j \neq 0, \\ 0 & \text{if } p_i q_j = 0, \end{cases}$$

and the trace matrix $T_N^M = (t_{ij})$ is defined by $t_{ij} = \mathrm{tr}_{M_j}(p_i q_j)$, where tr_{M_j} is the normalized trace on M_j. The following properties are easy consequences of the definitions.

(1.1) $$\lambda_{ij} \in \{0\} \cup \{2\cos(\pi/n) \; ; n \geq 3\} \cup [2, \infty].$$

(1.2) The trace matrix T_N^M is column-stochastic, i.e., $t_{ij} \geq 0$ and $\sum_{i=1}^{n} t_{ij} = 1$ for all j.

(1.3) The equality $\vec{t} = T_N^M \vec{s}$ holds.

(1.4) If $N \subset M \subset L$ are finite direct sums of finite factors, then $T_N^L = T_N^M T_M^L$.

1.2. Basic construction. Now we suppose that N is of finite index in M in the sense of [2], i.e., there is a faithful representation π of M on a Hilbert space such that $\pi(N)'$ is finite. Then the algebra $\langle M, e_N \rangle$ obtained by the basic construction for $N \subset M$ is a finite direct sum of finite factors and the corresponding minimal central projections are $J_M p_1 J_M, \cdots, J_M p_n J_M$, where J_M is the canonical conjugation on $L^2(M, \mathrm{tr})$. The following properties comes from the definitions:

(1.4) $e_N x e_N = E_N(x) e_N$ for $x \in M$,

(1.5) $e_N J_M p_i J_M = e_N p_i$ for all i.

We now list up some of properties concerning the index matrix and the trace matrix for $M \subset \langle M, e_N \rangle$:

(1.6) $$\Lambda_M^{\langle M, e_N \rangle} = (\Lambda_N^M)^t,$$

(1.7) $$T_M^{\langle M, e_N \rangle} = \tilde{T}_N^M F_N^M,$$

where $(\tilde{T}_N^M)_{ji} = \begin{cases} t_{ij}^{-1} \lambda_{ij}^2 & p_i q_j \neq 0, \\ 0 & p_i q_j = 0, \end{cases}$ $F_N^M = \mathrm{diag}(\varphi_1, \cdots, \varphi_n)$, $\varphi_i = (\sum_j (\tilde{T}_N^M)_{ji})^{-1}$,

(1.8) for any trace Tr on $\langle M, e_N \rangle$, $\mathrm{Tr}(e_N J_M p_i J_M) = \varphi_i \mathrm{Tr}(J_M p_i J_M)$.

The index $[M : N]$ is defined as follows:

(1.9) $[M : N] = r(\tilde{T}_N^M T_N^M)$, where $r(T)$ is the spectral radius of T.

1.3. Markov traces. A trace tr is called a Markov trace of modulus β for the pair $N \subset M$, if there exists a trace Tr on $\langle M, e_N \rangle$ such that tr is the restriction of Tr and $\beta \mathrm{Tr}(x e_N) = \mathrm{tr}(x)$ for $x \in M$. The following are important properties of Markov traces.

(1.10) The trace tr is a Markov trace of modulus β if and only if $\tilde{T}_N^M T_N^M \vec{s} = \beta \vec{s}$.

(1.11) If inclusion $N \subset M$ is connected, i.e., $Z(N) \cap Z(M) = \mathbb{C}$, there exists a unique normalized Markov trace for $N \subset M$. Moreover it is faithful and has modulus $[M : N]$.

1.4. Index formula. We consider two increasing sequences $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ of finite direct sums of finite factors. Assume that the traces on N_n and M_{n+1} are restrictions of the one on M_{n+1} and that the diagram

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square, i.e., $E_{N_n}^{M_n} E_{M_n}^{M_{n+1}} = E_{N_n}^{N_{n+1}} E_{N_{n+1}}^{M_{n+1}}$.

We deal with the following condition.

Condition I (Periodicity): There exist $n_0 \geq 1$ and $p \geq 1$ such that for any $n \geq n_0$,

(1) $T_{N_n}^{N_{n+1}}$, $T_{M_n}^{M_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic modulo p, and

(2) $T_{N_n}^{N_{n+p}}$ and $T_{M_n}^{M_{n+p}}$ are primitive.

Now we put $M = (\bigcup M_n)''$ and $N = (\bigcup N_n)''$. If Condition I holds, then

(1.12) M and N are II_1 factors,

and for all $n \geq n_0$

$$[M : N] = [M_n : N_n], \tag{1.13}$$

$$(M_n \cup \{e_N\})'' \cong \langle M_n, e_{N_n} \rangle. \tag{1.14}$$

2. Periodic commuting squares

Let a diagram

$$\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array} \tag{C}$$

be of finite direct sums of finite factors, and suppose that all indices of inclusions are finite and that the diagram is a commuting square with respect to a Markov trace tr on B_1 for $B_0 \subset B_1$.

By iterating the basic construction, we get projections $e_n = e_{B_{n-1}}$ and finite von Neumann algebras $B_{n+1} = \langle B_n, e_n \rangle$ and then put $A_{n+1} = (A_n \cup \{e_n\})''$ for $n \in \mathbb{N}$.

Definition 2.1. A commuting square (C) is periodic if for any $n \in \mathbb{N}$

(i) trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2, and

(ii) $T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive.

Remark 2.1. If a commuting square (C) is periodic, then for any $n \in \mathbb{N}$ a commuting square

$$\begin{array}{ccc} A_n & \subset & B_n \\ \cap & & \cap \\ A_{n+1} & \subset & B_{n+1} \end{array}$$

is periodic. Moreover by Theorem 2.3 of [7] we see that a commuting square $\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_n & \subset & B_n \end{array}$ is periodic for any $n \in \mathbb{N}$.

Remark 2.2. If a commuting square (C) is periodic, then it holds that $\dim_{\mathbb{C}} Z(A_0) = \dim_{\mathbb{C}} Z(A_2)$. By [9], this is equivalent to $A_2 \cong \langle A_1, e_{A_0}\rangle$, and the map $\theta\colon \langle A_1, e_{A_0}\rangle \longrightarrow A_2$, defined by $\theta(\sum_{i=1}^{n} x_i e_{A_0} y_i) = \sum_{i=1}^{n} x_i e_{B_0} y_i$ for x_i, $y_i \in A_1$, is a $*$-isomorphism. So it follows that the central support of e_{B_0} in A_2 is equal to 1, and hence the commuting square $\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$ is nondegenerate, i.e., $\overline{\mathrm{sp} A_1 B_0} = B_1$, where $\mathrm{sp}A$ denotes the linear span of A.

Example 2.1. Let $N \subset M$ be II_1 factors with finite index and $L = (N \cup \{e_N\})''$. If $[M : N] \geq 2$, then L has a cannonical decomposition as a direct sum of two II_1 factors. The diagram

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ L & \subset & \langle M, e_N\rangle \end{array}$$

is a commuting square, and it is periodic if and only if $[M : N] = 1$ or 2.

Lemma 2.1. *Assume that trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2 for any $n \in \mathbb{N}$. Then the following are equivalent:*

(i) *$T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive for any $n \in \mathbb{N}$;*

(ii) $Z(A_0) \cap Z(A_1) = Z(B_0) \cap Z(B_1) = \mathbb{C}$, *i.e., inclusions* $A_0 \subset A_1$ *and* $B_0 \subset B_1$ *are connected.*

In the following of this section, we assume that all inclusions are connected.

Lemma 2.2. *Let* tr *be a normalized Markov trace on* B_1 *for* $B_0 \subset B_1$ *and* $\{p_i; i = 1, \cdots, n\}$ *minimal central projections of* A_0, *and set* $\varphi_i = (F_{A_0}^{A_1})_{ii}$ *for* $i = 1, \cdots, n$. *Then the following are equivalent:*

(i) $A_2 \cong \langle A_1, e_{A_0} \rangle$;

(ii) $[B_1 : B_0] = \sum_{i=1}^{n} \varphi_i^{-1} \mathrm{tr}(p_i)$.

Proposition 2.1. *Let* $A_2 = (A_1 \cup e_{B_0})''$ *and* $B_2 = \langle B_1, e_{B_0} \rangle$, *and suppose that* A_2 *is* $*$*-isomorphic to* $\langle A_1, e_{A_0} \rangle$. *Then we have*

(i) $[A_1 : A_0] = [B_1 : B_0]$,

(ii) $T_{A_2}^{B_2} = (F_{A_0}^{A_1})^{-1} T_{A_0}^{B_0} F_{B_0}^{B_1}$,

(iii) $\Lambda_{A_2}^{B_2} = \Lambda_{A_0}^{B_0}$.

Now we obtain a neccesary and sufficient condition for a commuting square to be periodic

Theorem 2.1. *A commuting square (C) is periodic if and only if there exists a positive constant* λ *such that* $F_{A_0}^{A_1} = \lambda I_n$ *and* $F_{B_0}^{B_1} = \lambda I_m$, *where* $n = \dim_{\mathbb{C}} Z(A_0)$, $m = \dim_{\mathbb{C}} Z(B_0)$ *and* I_n *is the identity matrix in* $M_n(\mathbb{C})$. *Moreover, in this case, the constant* λ *is equal to* $[B_1 : B_0]^{-1}$.

Corollary 2.1. *Let a diagram*

$$\begin{array}{ccccc} A_0 & \subset & B_0 & \subset & C_0 \\ \cap & & \cap & & \cap \\ A_1 & \subset & B_1 & \subset & C_1 \end{array}$$

consist of commuting squares. If the two small commuting squares are periodic, then the big commuting square is periodic.

The following theorem is one of main results of this section.

Theorem 2.2. *Let* $\{e_n = e_{B_{n-1}};\ n \in \mathbb{N}\}$ *be projections and* $\{B_{n+1} = \langle B_n, e_n\rangle;\ n \in \mathbb{N}\}$ *finite von Neumann algebras obtained by iterating the basic construction, and put* $A_{n+1} = (A_n \cup \{e_n\})''$ *for* $n \in \mathbb{N}$. *If the commuting square (C) is periodic, then two increasing sequences* $\{A_n\}_{n=0,1,2,\cdots}$ *and* $\{B_n\}_{n=0,1,2,\cdots}$ *satisfy Condition I.*

Corollary 2.2. *If a commuting square (C) is periodic, then* $[B_1 : A_1] = [B_0 : A_0]$.

Proposition 2.2. *Set* $C_1 = \langle B_1, e_{A_1}\rangle$ *and* $C_0 = (B_0 \cup \{e_{A_1}\})''$. *If the commuting square (C) is periodic, then* $C_0 \cong \langle B_0, e_{A_0}\rangle$.

The periodic commuting squares have the symmetry as below.

Theorem 2.3. *Let*

$$\text{(C)} \qquad \begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$$

be a diagram of finite direct sums of finite factors such that any inclusions are connected and indices are finite. Assume that this diagram is a periodic commuting square with respect to a Markov trace on B_1 *for* $B_0 \subset B_1$, *then the commuting square*

$$\text{(C}'\text{)} \qquad \begin{array}{ccc} A_0 & \subset & A_1 \\ \cap & & \cap \\ B_0 & \subset & B_1 \end{array}$$

is periodic.

3. Examples

In this section, we give some examples of periodic commuting squares and the classification of particular ones.

Proposition 3.1. *Let* N *be a* II_1 *factor,* G *a finite abelian group of outer automorphism of* N *and* $N \rtimes G$, $N \rtimes G \rtimes \widehat{G}$ *be crossed products. Further set* $K = (N \cup \{\mu_\gamma; \gamma \in \widehat{G}\})''$, *where* μ_γ *is the implementing unitary for* $\gamma \in \widehat{G}$. *Then the diagram*

$$\begin{array}{ccc} N & \subset & N \rtimes G \\ \cap & & \cap \\ K & \subset & N \rtimes G \rtimes \widehat{G} \end{array}$$

is a periodic commuting square.

Let $N \subset M \subset L$ be II_1 factors with finite indices and K a nonfactor intermediate von Neumann algebra for $N \subset L$. Now suppose that the diagram

$$
\text{(D)} \qquad \begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}
$$

is a commuting square. Then a necessary and sufficient condition for the above diagram to be periodic is given by the next proposition.

Proposition 3.2. *Let $\{p_i; i = 1, \cdots, n\}$ be minimal central projections of K and* tr *a normalized trace on L. Then the commuting square (D) is periodic if and only if for any i*

$$[K_{p_i} : N_{p_i}] = [L : M]\mathrm{tr}(p_i) \text{ and } [L_{p_i} : K_{p_i}] = [M : N]\mathrm{tr}(p_i).$$

We see from the preceding theorem that trace matrices and index matrices for inclusions in a periodic commuting square such as (D) are expressed by means of indices $[L : M]$, $[M : N]$ and the vector $\vec{t} = (\mathrm{tr}(p_1), \cdots, \mathrm{tr}(p_n))$. In the following we assume that $\mathrm{tr}(p_1) \leq \cdots \leq \mathrm{tr}(p_n)$.

Theorem 3.1. *Let $N \subset M \subset L$ be II_1 factors such that indices $[L : M]$ and $[M : N]$ are less than 4, and K a nonfactor intermediate von Neumann algebra for $N \subset L$. Suppose that a diagram*

$$
\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}
$$

is a periodic commuting square. Then

(i) $[M : N] = [L : M]$,

(ii) *the pair $([M : N]; \vec{t})$ is one of the following:*

$$\left(2; \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right), \left(3; \left(\tfrac{1}{3}, \tfrac{2}{3}\right)\right), \left(3; \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)\right), \left(4\cos^2\tfrac{\pi}{10}; \left(\frac{1}{4\cos^2\frac{\pi}{10}}, \frac{\cos^2\frac{\pi}{5}}{\cos^2\frac{\pi}{10}}\right)\right).$$

Remark 3.1. The periodic commuting square in Proposition 3.1 corresponds to $([M : N];\vec{t}) = \left(|G|;\left(\frac{1}{|G|},\cdots,\frac{1}{|G|}\right)\right)$.

In the rest of this section we consider the classification of periodic commuting squares

$$\text{(E)} \qquad \begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}$$

corresponding to $([M : N];\vec{t}) = (2;(\frac{1}{2},\frac{1}{2}))$.

Since $N' \cap L \supset Z(K) \cong \mathbb{C} \oplus \mathbb{C}$ and $[L : N] = 4$, there exist a II_1 factor P and an automorphism α of P such that $(N \subset L) \cong (P_\alpha \subset P \otimes M_2(\mathbb{C}))$, where $P_\alpha = \left\{\begin{pmatrix} x & 0 \\ 0 & \alpha(x)\end{pmatrix}; x \in P\right\}$. By Theorem 5.4 of [10], we may assume that α is outer and $\alpha^2 = id$. Moreover it follows that $(N \subset M \subset L) \cong (P_\alpha \subset Q \subset P \otimes M_2(\mathbb{C}))$, where $Q = \left\{\begin{pmatrix} x & y \\ \alpha(y) & \alpha(x)\end{pmatrix}; x, y \in P\right\} \cong P \rtimes \mathbb{Z}_2$. On the other hand, by Remark 5.5 of [10] we have that

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array} \cong \begin{array}{ccc} P_\alpha & \subset & Q \\ \cap & & \cap \\ S & \subset & P \otimes M_2(\mathbb{C}) \end{array}$$

$$\cong \begin{array}{ccc} P & \subset & P \rtimes_\alpha \mathbb{Z}_2 \\ \cap & & \cap \\ (P \cup \{\mu\})'' & \subset & P \rtimes_\alpha \mathbb{Z}_2 \rtimes_{\widehat{\alpha}} \widehat{\mathbb{Z}_2} \end{array}$$

where $S = \left\{\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}; x, y \in P\right\}$ and μ is the implementing unitary for $\widehat{\alpha}$. Therefore the next theorem follows, which asserts that the periodic commuting square (E) is written in the form of the one in Proposition 3.1.

Theorem 3.2. *Let $N \subset M \subset L$ be II_1 factors such that $[L : M] = [M : N] = 2$, and K a nonfactor intermediate von Neumann algebra for $N \subset L$. Suppose that the diagram (E) is a periodic commuting square. Then there exists an outer action of* $\mathbb{Z}_2$

on N such that

$$\begin{array}{ccccccc} N & \subset & M & & N & \subset & N \rtimes_\alpha \mathbb{Z}_2 \\ \cap & & \cap & \cong & \cap & & \cap \\ K & \subset & L & & (N \cup \{\mu\})'' & \subset & N \rtimes_\alpha \mathbb{Z}_2 \rtimes_{\widehat{\alpha}} \widehat{\mathbb{Z}_2}, \end{array}$$

where μ is the implementing unitary for $\widehat{\alpha}$.

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