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Quantum double construction for subfactors arising from periodic commuting squares

上智大学 数学科　後藤 聡史 (Satoshi Goto)

1 Introduction

V. F. R. Jones introduced index for subfactors in [11] and he found his celebrated polynomial invariant for knots by using the subfactor theory in [12]. This work has revealed an unexpected relation between subfactor theory and 3-dimensional topology. A. Ocneanu's paragroup theory (see [19], [20], [21], [15], [30] for the definition of a *paragroup*, for example) also has revealed a deep relation between subfactor theory and quantum group theory, 3-dimensional topology and rational conformal field theory etc. (See [1], [8], [9], [21], [22], [25], [29] etc. for these topics.)

A. Ocneanu introduced the *asymptotic inclusion* for a subfactor in [19], [20]. This asymptotic inclusion in subfactor theory is regarded as the right analogue of the quantum double construction of Drinfel'd [3] (see [24] and [9]). In [23] A. Ocneanu claimed that combinatorial data satisfying Moore-Seiberg axiom ([18]) in the rational conformal field theory can be constructed after passing to the asymptotic inclusion from a given paragroup (see [10, Section 13.5]). A. Ocneanu also says in [25] that if the fusion graph of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ is connected then the system of M_∞-M_∞ bimodules are braided and non-degenerate and that if we have a non-degenerate braided system of bimodules then we get a Reshetikhin-Turaev type invariant of 3-manifolds based on surgery ([27]). So the asymptotic inclusions are important for these reasons and others.

Most fundamental examples of the asymptotic inclusions are subfactors generated by the commuting squares of the two-sided sequence of the Jones projections;

$$\begin{array}{ccc}
\langle e_{-n}, \ldots, e_{-1}, e_1, \ldots, e_n \rangle & \subset & \langle e_{-n-1}, \ldots, e_{-1}, e_1, \ldots, e_{n+1} \rangle \\
\cap & & \cap \\
\langle e_{-n}, \ldots, e_{-1}, e_0, e_1, \ldots, e_n \rangle & \subset & \langle e_{-n-1}, \ldots, e_{-1}, e_0, e_1, \ldots, e_{n+1} \rangle .
\end{array}$$

Here Jones projections $\{e_i\}_{i \in \mathbf{Z}}$ satisfy the following relations;

$$\begin{array}{rcl}
e_i e_{i\pm 1} e_i = \beta^{-2} e_i, & \text{for} & i \in \mathbf{Z}, \\
e_i e_j = e_j e_i, & \text{whenever} & |i-j| \neq 2,
\end{array}$$

where $\beta = 2\cos(\pi/N)$. We remark that the above commuting squares have periodicity 2 in the sense of Wenzl (see [28, page 357] for the definition of the periodicity of commuting squares). The indices of the subfactors were first computed by M. Choda in [2]. The above subfactor is easily shown to be isomorphic to the asymptotic inclusion of the Jones subfactor with principal graph A_n. A proof of this fact follows from a general fact that the commuting square

$$\begin{array}{ccc} (M'_{-k}\cap M_0)\vee(M'_0\cap M_k) & \subset & (M'_{-k-1}\cap M_0)\vee(M'_0\cap M_{k+1}) \\ \cap & & \cap \\ M'_{-k}\vee M_k & \subset & M'_{-k-1}\vee M_{k+1}, \end{array}$$

generates the asymptotic inclusion by Popa's generating property (see [26]).

Another fundamental example is a group case which was first claimed in [20, III.3] (see [16, Appendix] for a complete proof). That is, if we start with a finite group G and consider the subfactor $R \subset R \times G$, where R is the AFD II_1 factor and G acts freely on R, then the asymptotic inclusion is of the form $R^{G\times G} \subset R^G$, where $G \times G$ acts freely on R with G embedded into $G \times G$ with a map $g \mapsto (g, g)$. This example gives one of the reasons why the asymptotic inclusion is analogous to the quantum double construction. (See [17] for more details for this analogy.)

Recently J. Erlijman has made a remark in [4] that the following commuting squares of two-sided sequence of generators $\{g_i\}_{i\in\mathbf{Z}}$ of Hecke algebra of type B, C, D produce the asymptotic inclusion for the Hecke algebra of type B, C, D subfactors of Wenzl;

$$\begin{array}{ccc} \langle g_{-n},\ldots,g_{-1},g_1,\ldots,g_n\rangle & \subset & \langle g_{-n-1},\ldots,g_{-1},g_1,\ldots,g_{n+1}\rangle \\ \cap & & \cap \\ \langle g_{-n},\ldots,g_{-1},g_0,g_1,\ldots,g_n\rangle & \subset & \langle g_{-n-1},\ldots,g_{-1},g_0,g_1,\ldots,g_{n+1}\rangle. \end{array}$$

In these cases the commuting squares have the periodicity 2. But in the Hecke algebra of type A case, they have periodicity n ($n \geq 2$) in general. And the period 2 case is nothing but the above examples of the two-sided sequences of Jones projections.

In this paper we generalize her construction of subfactors to the case we have a fusion rule algebra and quantum $6j$-symbols which produce periodic commuting squares. We prove that this construction produces the same subfactor as the asymptotic inclusion for the subfactor generated by the original periodic commuting square. We also give some examples and by applying this result in the case of fusion rule algebras of $SU(n)_k$ WZW models, which is the same as Hecke algebra of type A subfactors of Wenzl, we show that the above two-sided sequence of Hecke algebra of type A generators produces the asymptotic inclusion of the Hecke algebra subfactor of type A. This result itself has been independently obtained by J. Erlijman [5].

According to the A. Ocneanu's theory as in [25] and [9], we can get a great deal of combinatorial data of RCFT and Reshetikhin-Turaev type topological invariants by our method.

Acknowledgement The author would like to thank Prof. Y. Kawahigashi for fruitful discussions and comments.

2 Quantum double construction for subfactors arising from periodic commuting squares

We start with a finite fusion rule algebra $\mathcal{A}$ with quantum $6j$-symbols satisfying unitarity, tetrahedral symmetry, and the pentagon equation (see [21], [7] for the definitions of these). We denote that the standard basis of $\mathcal{A}$ by $\{a_j\}_{j\in J}$. Fix an element $h = a_j$ for some $j \in J$ and take finite powers h^k of h and decompose them into sums of basis in $\mathcal{A}$. Thus we get a fusion rule subalgebra $\mathcal{B}$ of $\mathcal{A}$ generated by h and quantum $6j$-symbols restricted to the subalgebra $\mathcal{B}$.

We assume that $h^n \succ * = id$, that is, h^n contains an element $* = id$ in the expression of h^n as a sum of elements in the basis of $\mathcal{B}$. Here an element id represents the identity element of the fusion rule algebra $\mathcal{A}$. We take the smallest n satisfying the condition $h^n \succ * = id$ and denote it by the same n. Then a natural finite grading arise in the elements of the basis of $\mathcal{B}$ as follows. We set Ω_k for all $k \pmod n$ a subset of $\mathcal{B}$ consists of elements of the basis of $\mathcal{B}$ which appear in the expression of h^k as a sum of the elements in the basis of $\mathcal{B}$. Then the basis of $\mathcal{B}$ is decomposed into a union of finite subsets $\Omega_0, \Omega_1, \cdots, \Omega_{n-1}$ so that they satisfy the following.

1. $* = id \in \Omega_0$, $h \in \Omega_1$
2. if x is in Ω_k, then $x \cdot h$ can be decomposed into a sum of elements in Ω_{k+1} for $k, k+1 \in \mathrm{N} \pmod n$),

Definition 2. 1 We call a subsystem (i.e., fusion rule subalgebra) $\mathcal{B}$ of a fusion rule algebra $\mathcal{A}$ *periodic* when the generator h of $\mathcal{B}$ satisfies $h^n \succ id$. The smallest such n is called the *period* of the system $\mathcal{B}$.

We remark that if the system $\mathcal{B}$ is periodic with period n, then it produces a periodic commuting square with the same period n in the following way. Hence we get a subfactor with finite index. (See [28] Lemma 1.4.) We make a double sequence of string algebra which is a modified version of the original string algebra construction from a paragroup as follows. First we put $*$ in the upper left corner. Then we pass to the right by multiplying the generator h from the right and pass to the downward direction by multiplying h from the left. In this way we get periodic commuting squares with period n both in the horizontal and vertical directions.

$$\begin{array}{ccccccccccc}
 & & A_{-1,1} & \subset & A_{-1,2} & \subset & A_{-1,3} & \cdots & \to & P \\
 & & \cap & & \cap & & \cap & & & \cap \\
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & A_{0,3} & \cdots & \to & Q \\
\cap & & \cap & & \cap & & \cap & & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & A_{1,3} & \cdots & \to & Q_1 \\
\cap & & \cap & & \cap & & \cap & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots
\end{array}$$

Note that the grading of the vertices of these commuting squares are illustrated as follows:

$$
\begin{array}{ccccccc}
 & 0 & 1 & \cdots & n-2 & n-1 & 0 & \cdots \\
0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots \\
2 & 3 & 4 & \cdots & 1 & 2 & 3 & \cdots \\
3 & 4 & 5 & \cdots & 2 & 3 & 4 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots \\
n-1 & 0 & 1 & \cdots & n-2 & n-1 & 0 & \cdots \\
0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots
\end{array}
$$

The connection on the above commuting squares is *flat* in the following sense, that is, the horizontal string algebra $A_{0,k}$ (resp. $A_{1,k}$) commutes with the vertical string algebra $A_{l,0}$ (resp. $A_{l,-1}$). (See [19], [20], [13] or [14] for the definition of the flatness in the usual period 2 case.) This definition of the flatness in the case of double sequence of periodic commuting squares is equivalent to the following condition. (cf. [14, Theorem 2.1])

$$
\begin{tikzcd}
 & \arrow[r, "\cdot\otimes h"] & \arrow[r, "\cdot\otimes h"] & & \arrow[r, "\cdot\otimes \bar{h}"] & \\
 & * \arrow[r, "\xi_1"] \arrow[d, "\eta_1"'] & x_1 \arrow[r, "\xi_2"] & x_2 \cdots & x_{2n-1} \arrow[r, "\xi_{2n}"] & * \arrow[d, "\eta_1"] \\
 \arrow[d, "h\otimes\cdot"'] & y_1 \arrow[d, "\eta_2"'] & & & & y_1 \arrow[d, "\eta_2"] \\
 & y_2 & & & & y_2 \\
 & \vdots & & & & \vdots \\
 \arrow[d, "\bar{h}\otimes\cdot"'] & y_{2m-1} \arrow[d, "\eta_{2m}"'] & & & & y_{2m-1} \arrow[d, "\eta_{2m}"] \\
 & * \arrow[r, "\xi_1"'] & x_1 \arrow[r, "\xi_2"'] & x_2 \cdots & x_{2n-1} \arrow[r, "\xi_{2n}"'] & *
\end{tikzcd}
= 1
$$

In the above diagram we pass to the right by tensoring h (resp. $\bar{h}$) from the right in the left (resp. right) half of the horizontal paths and pass to the downward by tensoring h (resp. $\bar{h}$) from the left in the upper (resp. lower) half of the vertical paths. This identity is shown by a slight modification of the original proof in the canonical period two case ([7, section 4]) if we have the pentagon relation which is one of the axioms for the quantum $6j$-symbols.

Because the vertical string algebra has period n, the vertical graphs are not the (dual) principal graph of the subfactor $P \subset Q$ in spite of the flatness when the period n is greater than 2. But we can get the principal graph of the subfactor by using the following "orientation reversing" method of [6]. (cf. [6, Corollary3.4, Corollary 3.6]) We change the construction in the vertical direction from multiplying only h to multiplying h and $\bar{h}$ alternately. Note that the system $\mathcal{B}$ contains $\bar{h}$ by the definition of periodicity of $\mathcal{B}$ and the Frobenius reciprocity, i.e., we have $h^{n-1} \succ \bar{h}$. So we obtain a subsystem of $\mathcal{B}$ consisting of elements of the basis appearing in the finite alternating products of h and $\bar{h}$. We denote this subsystem by $\mathcal{C}$. In this way we get the grading of the commuting squares changed as follows:

$$\begin{array}{cccccccc}
 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots \\
n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots \\
n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We denote these modified commuting squares as follows.

$$\begin{array}{ccccccccccc}
 & & B_{-1,1} & \subset & B_{-1,2} & \subset & B_{-1,3} & \cdots & \rightarrow & N \\
 & & \cap & & \cap & & \cap & & & \cap \\
B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} & \cdots & \rightarrow & M \\
\cap & & \cap & & \cap & & \cap & & & \cap \\
B_{1,0} & \subset & B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} & \cdots & \rightarrow & M_1 \\
\cap & & \cap & & \cap & & \cap & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots
\end{array}$$

Here we remark that the subfactor $N \subset M$ is identical to $P \subset Q$.

Because of the flatness of the connection and Wenzl's dimension estimate [28, Theorem 1.6], we conclude that the vertical graphs are the (dual) principal graph of this subfactor. (see [6, Corollary 3.4, Corollary 3.6].) So we obtain the canonical double sequence of higher relative commutants by applying the "orientation reversing" method to both horizontal and vertical directions. (See [6, Theorem 3.5].) The grading of the vertices again changes as follows:

$$\begin{array}{ccccccccc}
 & 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots \\
n-1 & 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots \\
n-1 & 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We denote these canonical commuting squares as follows:

$$\begin{array}{ccccccccccc} & & C_{-1,1} & \subset & C_{-1,2} & \subset & C_{-1,3} & \cdots & \to & N \\ & & \cap & & \cap & & \cap & & & \cap \\ C_{0,0} & \subset & C_{0,1} & \subset & C_{0,2} & \subset & C_{0,3} & \cdots & \to & M \\ \cap & & \cap & & \cap & & \cap & & & \cap \\ C_{1,0} & \subset & C_{1,1} & \subset & C_{1,2} & \subset & C_{1,3} & \cdots & \to & M_1 \\ \cap & & \cap & & \cap & & \cap & & & \cap \\ \vdots & & \vdots & & \vdots & & \vdots & & & \vdots \end{array}$$

By using this canonical Jones tower $N \subset M \subset M_1 \subset M_2 \cdots \subset M_\infty$, we can construct the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. We also get another subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ from the previous tower $P \subset Q \subset Q_1 \subset Q_2 \cdots \subset Q_\infty$.

Here we remark that the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ is generated by the following commuting squares:

$$\begin{array}{ccc} C_{n,0} \vee C_{0,n} & \subset & C_{n+1,0} \vee C_{0,n+1} \\ \cap & & \cap \\ C_{n,n} & \subset & C_{n+1,n+1}, \end{array}$$

and the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ is also generated by the commuting squares:

$$\begin{array}{ccc} A_{n,0} \vee A_{0,n} & \subset & A_{n+1,0} \vee A_{0,n+1} \\ \cap & & \cap \\ A_{n,n} & \subset & A_{n+1,n+1}. \end{array}$$

Now we give a graphical expression of bimodules arising from these subfactors $M \vee (M' \cap M_\infty) \subset M_\infty$ and $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$. First remark that the generator h can be identified with M as an M-N bimodule by using canonical commuting squares as above and graphical expression as in Figure 2.1 (see [24] and [9].)

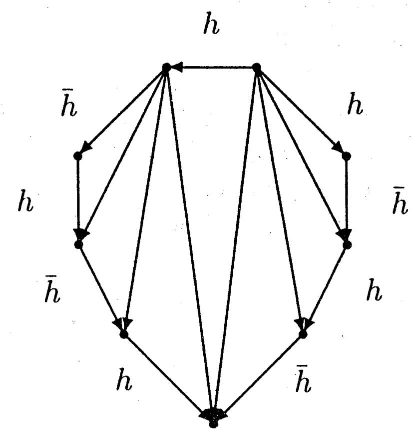

Figure 2.1

So the graphical expressions of the bimodules arising from the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ are exactly the same as the original ones in [24], [9] by this identification of $h = {}_M M_N$ and $\bar{h} = {}_N M_M$.

Next we give graphical expressions of the bimodules arising from the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$. The algebras $Q = M$, $Q \vee (Q' \cap Q_\infty) = M \vee (M' \cap M_\infty)$ and Q_∞ are expressed as in Figures 2.2, 2.3 and 2.4 respectively. Here the point is that we change all the labels of edges on the boundary from ${}_M M_N$ and ${}_N M_M$ to the generator h. In particular Figure 2.2 is exactly the 2-dimensional expression of the string algebras with period n.

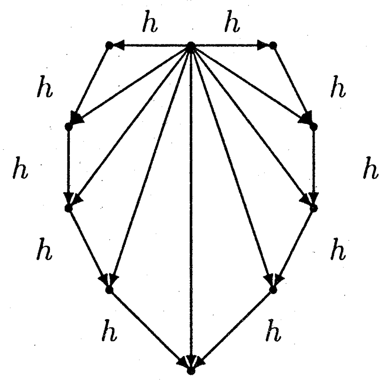

Figure 2.2

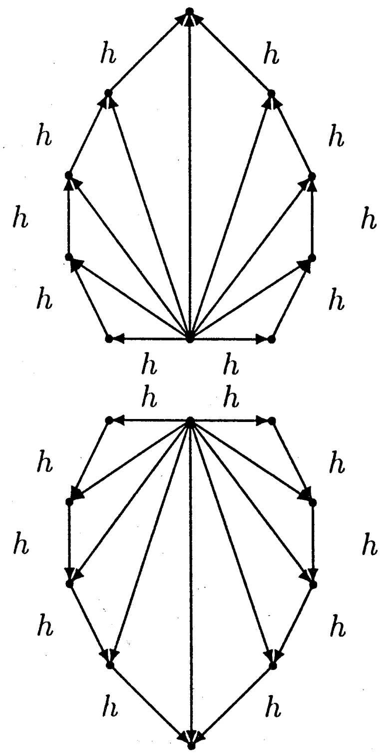

Figure 2.3

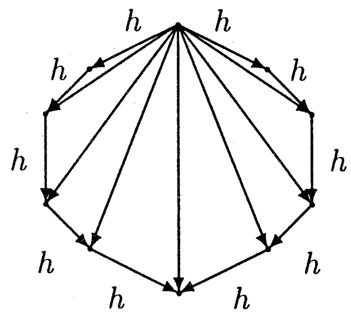

Figure 2.4

Here Figure 2.2 actually represents $Q^{\mathrm{op}} = M^{\mathrm{op}}$ but we use this expression instead of the upside-down picture for simplicity.

Similarly if we change the labels of the edges on the boundary from ${}_M M_N$ and ${}_N M_M$ to h, we get a graphical expression of Q-Q bimodule as in Figure 2.5.

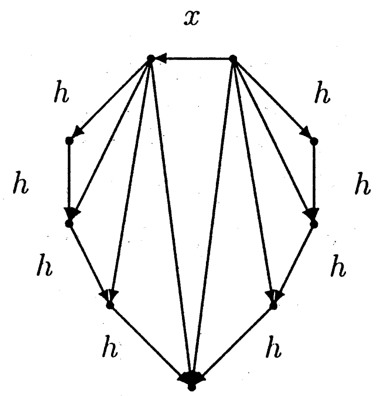

Figure 2.5

Here x is one of the elements in the basis of $\mathcal{B}$. We denote this Q-Q bimodule by K_x and call it a *surface bimodule*. In the following we use the notation $\mathcal{I} \equiv \Omega_0 \cap \mathcal{C}$ which is a subset of the basis in the system $\mathcal{C}$.

Theorem 2. 2 *The bimodule K_x above is irreducible. And the set of Q-Q bimodules $\{K_x\}_{x\in\mathcal{I}}$ makes an isomorphic system of Q-Q bimodules arising from the subfactor $(N \subset M) \cong (P \subset Q)$, i.e., the system $\{A_x\}_{x\in\mathcal{I}}$ of M-M bimodules as above.*

Proof: From the commuting squares with the following gradings

$$\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots & \longrightarrow & M = Q \\
n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots & \longrightarrow & M_1 \\
0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots & \longrightarrow & M_2 \\
n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots & \longrightarrow & M_3 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots
\end{array}$$

and the graphical expression of $Q = M$ and Q-Q bimodule, the set of bimodules $\{K_x\}_{x\in\mathcal{I}}$ really makes a system of Q-Q bimodules. The same method as in the proof of Theorem 2.1 in [9] also works by using Wenzl's dimension estimate [28, Theorem 1.6] in order to show the irreducibility of the inclusion $Q \subset R_x$. Here R_x denotes a von Neumann algebra corresponding to Figure 2.6.

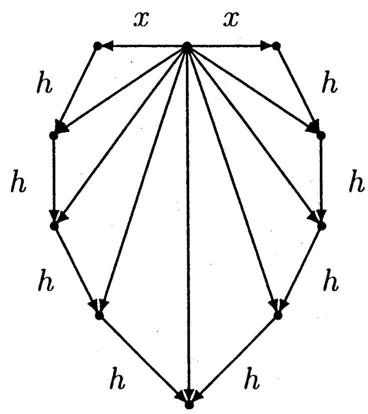

Figure 2.6

Here the edges of the boundaries are labelled by h except for the top two x's.

And a graphical inspection as in the proof of Theorem 2.1 in [9] shows that the fusion rule and quantum $6j$-symbols of the system of surface bimodules depend only on the labels on the top edges $x \in \mathcal{I}$ of the surface bimodules and do not depend on the labels of edges on the boundaries. So the above system of Q-Q bimodules $\{K_x\}_{x\in\mathcal{I}}$ have the same fusion rule and quantum $6j$-symbols as the system of Q-Q bimodule arising from the subfactor $P \subset Q$. Q.E.D.

From the above theorem we may and do use the notation $\mathcal{Q}$ for the two isomorphic system of Q-Q bimodules as in the theorem.

Similarly we get an irreducible $Q \vee (Q' \cap Q_\infty)$-$Q \vee (Q' \cap Q_\infty)$ bimodules expressed as in Figure 2.7.

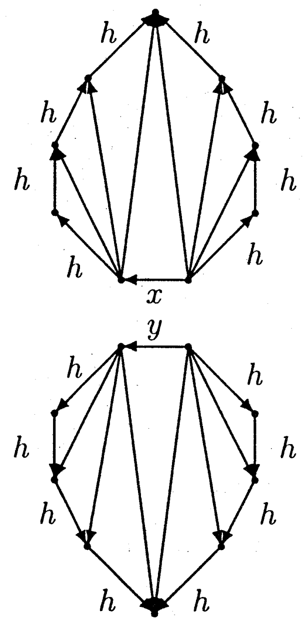

Figure 2.7

where x and y are any pair of basis in $\mathcal{I}$ and the edges on the boundaries are labelled by h. We denote this bimodule by $L_{x,y}$. The above theorem shows that this system $\{ L_{x,y} \}$ has the same fusion rule and quantum $6j$-symbols as the system $\{ B_{x,y} \}$ of $M \vee (M' \cap M_\infty)$-$M \vee (M' \cap M_\infty)$ bimodules.

We can also get the sets of *irreducible* $Q \vee (Q' \cap Q_\infty)$-$Q_\infty$ bimodules $\{ L_z \}$ and *irreducible* Q_∞-Q_∞ bimodules $\{ L_{\pi_i} \}$ as in Figure 2.8 and Figure 2.9 respectively by changing the labels of the edges on the boundaries from ${}_M M_N$ and ${}_N M_M$ to h. The irreducibility of these $Q \vee (Q' \cap Q_\infty)$-$Q_\infty$ and Q_∞-Q_∞ bimodules are shown in the same way as in [9, Theorem 4.1, Theorem 4.2].

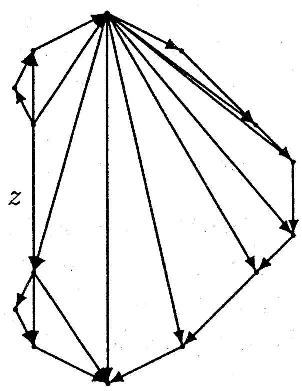

Figure 2.8

Here the labellings π_i are given by the elements of the subset of the minimal central projections of Tube $\mathcal{Q}$, the *tube algebra* of the system $\mathcal{Q}$ of Q-Q bimodules (see [25], [9] for the definition of the tube algebra) which are reachable from $*$ by the fusion graph of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ (see [25]).

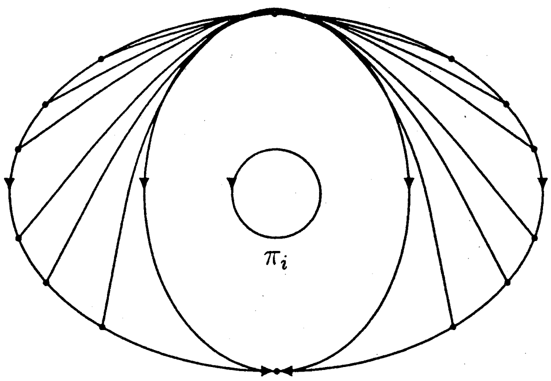

Figure 2.13

Now we can get the following theorem similarly.

Theorem 2. 3 *All the systems of bimodules $\{L_{x,y}\}$, $\{L_z\}$, $\{\overline{L}_z\}$ and $\{L_{\pi_i}\}$ consists of four kinds of bimodules arising from the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ have the same ($\mathbb{Z}_2$ graded) fusion rules and quantum $6j$-symbols as the four kinds of bimodules arising from the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$. Hence the two systems make the same paragroups.*

Proof: By the graphical inspection as in the proof of Theorem 2.1 in [9] we can easily see that the fusion rules and quantum $6j$-symbols of the system of these four kinds of bimodules depends only on the labelling of X, Y, Z and π_i. So we get the result. Q.E.D.

Corollary 2. 4 *Two subfactors $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ and $M \vee (M' \cap M_\infty) \subset M_\infty$ are isomorphic.*

Proof: Because the two subfactors have finite index and finite depth and have the same paragroups, these are isomorphic by Popa's generating theorem for strongly amenable subfactors [26]. Q.E.D.

In the following we give some applications of this result.

Example 2. 5 We start with $SU(n)$ WZW model with level k. We can get a commutative fusion rule algebra with quantum $6j$-symbols. If we take the fundamental generator h, then the resulting subfactor $(P \subset Q) \cong (N \subset M)$ is the same as Hecke algebra subfactor of type A of Wenzl. (See [6].) The above corollary shows that the following commuting squares

$$\begin{array}{ccc} \langle g_{-n}, \dots, g_{-1}, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_1, \dots, g_{n+1} \rangle \\ \cap & & \cap \\ \langle g_{-n}, \dots, g_{-1}, g_0, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_0, g_1, \dots, g_{n+1} \rangle \end{array}$$

generate the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. Here g_i's are the standard generators of Hecke algebras satisfying the following relations:

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, && \text{for} \quad i \in \mathbf{Z}, \\ g_i g_j &= g_j g_i, && \text{whenever} \quad |i-j| \geq 2, \\ g_i^2 &= (q-1) g_i + q, && \text{for} \quad i \in \mathbf{Z}, \end{aligned}$$

where $q = e^{\pm i\pi/n}$.

The isomorphism for this example has been obtained by J. Erlijman [5] independently by a different method.

Example 2. 6 Again we start with $SU(n)$ WZW model with level k and take another generator h different from the fundamental one, then the resulting subfactor is isomorphic to $pPp \subset pQ_k p$, where $P \subset Q$ denotes Hecke algebra subfactor of type A of Wenzl and p is a projection in $P' \cap Q_k$ for some k. Here Q_k is one of the von Neumann algebras in the tower $P \subset Q \subset Q_1 \subset \ldots \subset Q_k \subset \ldots$ which are generated by the double sequences of period n commuting squares as in [28]. In this case we do not have natural generators for the commuting squares as in the previous example. But our method also works in such cases and we can construct the asymptotic inclusions for many such subfactors. This is an advantage of our method.

Example 2. 7 We start with a fusion rule algebra which consists of N-N bimodules of a subfactor $N \subset M$ with principal graph D_{2m} for $m \geq 2$. If we take a bimodule corresponding to one of the two tails of the principal graph D_{2m} as a generator, then the subsystem $\mathcal{B} = \langle h \rangle$ has period 2 if m is even and period grater than 3 if m is odd. This is because the contragredient map for D_{2m} changes by mod 4, i.e., if m is even, we have $\bar{h} = h$ and if m is odd, we have $\bar{h} \neq h$.

Remark 2. 8 We can easily modify this method to the case when the subsystem $\mathcal{B}$ has more than two generators. For example if we have m generators $h_1, h_2, \cdots, h_m$ which are elements in the basis of $\mathcal{A}$, then we take $h = h_1 + h_2 + \cdots + h_m$ as a new generator. And in this case we have to change the definition (Definition 2.1) of the periodicity. We say the fusion rule subalgebra $\mathcal{B}$ generated by the above h *periodic* of with the *period* n if the Bratteli diagram for $\{h^k\}_{k=0,1,2,\cdots}$ has periodic with the periodicity n.

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