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Kyoto University
TRANSFORMATIONS APPROXIMATING A GROUP GENERATED BY THE LÉVY LAPLACIAN

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1. Introduction

Since T. Hida [6] applied the Lévy Laplacian, which was introduced by P. Lévy [25], to his theory of generalized white noise functionals, this Laplacian has been studied within the framework of white noise calculus ([8,10,11,17,23,27,30,31], etc.). On the other hand, L. Accardi et al. [1] obtained a nice relation between the Laplacian and the Yang-Mills equation. It seems an interest to consider a relation to their results [1,2].

By H.-H. Kuo [16], an infinite dimensional Fourier-Mehler transform acting on the space $(S)^*$ of generalized white noise functionals was introduced and he showed a relation between the transform and the Lévy Laplacian (see [19]). There are several Laplacian operators acting on $(S)^*$.

In this paper we discuss integral expressions of those Laplacians and groups generated by the Laplacians. In addition, we show a transform acting on $(S)^*$ approximating a group generated by the Lévy Laplacian.

The paper is organized as follows. In Section 2 we assemble some basic notations of white noise calculus. In Section 3 we explain the definitions of Laplacian operators acting on Hida distributions, and give a limiting integral expression of the Lévy Laplacian with an integral expression of the Gross Laplacian. In Section 4 we define groups generated by the Laplacian operators acting on the Hida distributions and show that Kuo's Fourier-Mehler transform is given by a composition of groups generated by the number operator and the Gross Laplacian. In addition, we give a result that the group generated by the Lévy Laplacian is approximated by groups generated by the Gross Laplacian. Finally, in the last section we introduce a transform approximating a group generated by the Lévy Laplacian. This transform includes the adjoint operator of Kuo's Fourier-Mehler transform.

2. Preliminaries

In this section, we explain some basic notations of white noise analysis following [10,15,27,29]. We begin with a Gel'fand triple $S \subset L^2(\mathbb{R}) \subset S^*$, where $S \equiv S(\mathbb{R})$ is the Schwartz space consisting of rapidly decreasing $C^\infty$-functions on $\mathbb{R}$ and $S^* \equiv S^*(\mathbb{R})$ is its dual space. An operator $A = -(d^2/du^2) + u^2 + 1$ is a densely defined self-adjoint operator on $L^2(\mathbb{R})$. There exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for
of $L^2(\mathbb{R})$ such that $Ae_\nu = 2(\nu + 1)e_\nu$. We define the norm $| \cdot |_p$ by $|f|_p = |A^p f|_0$ for $f \in \mathcal{S}$ and $p \in \mathbb{Z}$, where $| \cdot |_0$ is the $L^2(\mathbb{R})$-norm, and let $\mathcal{S}_p$ be the completion of $\mathcal{S}$ with respect to the norm $| \cdot |_p$. Then the dual space $\mathcal{S}_p'$ of $\mathcal{S}_p$ is the same as $\mathcal{S}_{-p}$ (see [13]).

The Bochner-Minlos theorem admits the existence of a probability measure $\mu$ on $\mathcal{S}^*$ such that

$$C(\xi) = \int_{\mathcal{S}^*} \exp\{i(x, \xi)\} \, d\mu(x) = \exp\{-\frac{1}{2} |\xi|^2_{0}\}, \ \xi \in \mathcal{S}.$$ 

The space $(L^2) = L^2(\mathcal{S}^*, \mu)$ of complex-valued square-integrable functionals defined on $\mathcal{S}^*$ admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where $H_n$ is the space of multiple Wiener integrals of order $n \in \mathbb{N}$ and $H_0 = \mathbb{C}$. This decomposition theorem says that each $\varphi \in (L^2)$ is uniquely represented as

$$\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n), \ f_n \in L^2_\mathbb{C}(\mathbb{R})^{\otimes n},$$

where $\mathbf{I}_n(f_n) \in H_n$ and $L^2_\mathbb{C}(\mathbb{R})^{\otimes n}$ denotes the n-th symmetric tensor product of the complexification of $L^2(\mathbb{R})$.

For each $p \in \mathbb{Z}$, we define the norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$ by

$$\|\varphi\|^2_p = \left( \sum_{n=0}^{\infty} n! |f_n|^{2}_{p,n} \right)^{1/2}, \ \varphi \in \mathcal{S}^{\otimes n}_{\mathbb{C},p} \ (S_p')$$

where $| \cdot |_{p,n}$ is the norm of $\mathcal{S}_{\mathbb{C},p}^{\otimes n}$ and $\mathcal{S}^{\otimes n}_{\mathbb{C},p}$ is the n-th symmetric tensor product of the complexification of $\mathcal{S}_p$. The norm $\| \cdot \|_0$ is nothing but the $(L^2)$-norm. We put

$$(S_p) = \{ \varphi \in (L^2); \|\varphi\|_p < \infty \}$$

for $p \in \mathbb{Z}, p \geq 0$. Let $(S_p)^*$ be the dual space of $(S_p)$. Then $(S_p)$ and $(S_p)^*$ are Hilbert spaces with the norm $\| \cdot \|_p$ and the dual norm of $\| \cdot \|_p$, respectively. We define the space $(S_p)$ for $p < 0$ by the completion of $(L^2)$ with respect to $\| \cdot \|_p$. Then $(S_p), p < 0,$ is a Hilbert space with the norm $\| \cdot \|_p$. It is easy to see that for $p > 0$, the dual space $(S_p)^*$ of $(S_p)$ is given by $(S_{-p})$. Moreover, we see that for any $p \in \mathbb{R},$

$$(S_p) = \bigoplus H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{ \mathbf{I}_n(f); f \in \mathcal{S}^{\otimes n}_{\mathbb{C}} \}$ with respect to $\| \cdot \|_p$.

Denote the projective limit space of the $(S_p), p \in \mathbb{Z}, p \geq 0$, and the inductive limit space of the $(S_p)^*, p \in \mathbb{Z}, p \geq 0$, by $(S)$ and $(S)^*$, respectively. Then $(S)$ is
a nuclear space and \((S)^*\) is nothing but the dual space of \((S)\). The space \((S)^*\) is called the space of Hida distributions or generalized white noise functionals. Since \(\exp <\cdot,\xi>\in(S)\), the S-transform is defined on \((S)^*\) by
\[
S[\Phi](\xi) = C(\xi) \ll \Phi, \exp <\cdot,\xi> \gg, \xi \in S,
\]
where \(\ll \cdot, \cdot \gg\) is the canonical pairing of \((S)^*\) and \((S)\). In [10], we can see the following fundamental properties:

i) if \(S[\Phi](\xi) = S[\Psi](\xi)\) for all \(\xi \in S\), then \(\Phi = \Psi\).

ii) if \(\Phi = \sum_{n=0}^{\infty} \Phi_n \in (S)^*\), then there exist an integer \(p\) and distributions \(f_n \in S_{C,p}^{\otimes n}\), \(n = 0, 1, 2, \ldots\), such that \(\sum_{n=0}^{\infty} n! |f_n|_{p,n}^2 < \infty\) and
\[
S[\Phi](\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle
\]
for all \(\xi \in S\).

We denote the above Hida distribution \(\Phi_n\) in ii) by the same notation \(I_n(f_n)\) for \(f_n \in S_{C,p}^{\otimes n}\).

3. Laplacian operators acting on Hida distributions

We introduce the definitions of Laplacian operators following [10] (see also [20]). Let \(F\) be a Fréchet differentiable function defined on \(S\), i.e. we assume that there exists a map \(F'\) from \(S\) to \((S)^*\) such that
\[
F(\xi + \eta) = F(\xi) + F'(\xi)(\eta) + o(\eta), \eta \in S,
\]
where \(o(\eta)\) means that there exists \(p \in \mathbb{Z}, p \geq 0,\) depending on \(\xi\) such that \(o(\eta)/|\eta|_p \rightarrow 0\) as \(|\eta|_p \rightarrow 0\). If the first variation is expressed in the form
\[
F'(\xi)(\eta) = \int_{\mathbb{R}} F'_{\xi}(\xi;u) \eta(u) \, du
\]
for every \(\eta \in S\) by using the generalized function \(F'(\xi;\cdot)\), we define the Hida derivative \(\partial_t \Phi\) of \(\Phi\) to be the generalized white noise functional whose S-transform is given by \(F'(\xi; t)\). The differentiation \(\partial_t\) is continuous from \((S)\) into itself. Its adjoint operator \(\partial_t^*\) is continuous from \((S)^*\) into itself.

Let \((\mathcal{H}, B)\) be an abstract Wiener space. Suppose \(\psi\) is a real-valued twice \(\mathcal{H}\)-differentiable function on \(B\) such that the second \(\mathcal{H}\)-derivative \(D^2 \psi(x)\) at \(x\) is a trace class operator of \(\mathcal{H}\). Then the Gross Laplacian \(\Delta_{G}\) ([4,5]) is defined by
\[
\Delta_{G} \psi(x) = \text{Trace}_{\mathcal{H}} D^2 \psi(x).
\]
The Laplacian \(\Delta_{G}\) has the expression \(\Delta_{G} \Phi = \int_{\mathbb{R}} \partial_t^2 \Phi dt\) on \((S)\) (see [17]). The Gross Laplacian is a continuous linear operator from \((S)\) into itself.
For any $\Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (S)^*$, the number operator $N$ is defined by

$$N \Phi = \sum_{n=0}^{\infty} n I_n(f_n).$$

The number operator is a continuous linear operator from $(S)^*$ into itself. The operator $N$ has the expression

$$N \Phi = \int_{\mathbb{R}} \partial_t^* \partial t \Phi \, dt$$

on $(S)^*$ (see [17]).

A Hida distribution $\Phi$ is called an $L$-functional if for each $\xi \in S$, there exist $(S[\Phi])'(\xi; \cdot) \in L_{loc}^1(\mathbb{R}) \cap S^*$ and $(S[\Phi])''(\xi; \cdot, \cdot) \in L_{loc}^1(\mathbb{R}^2) \cap S^*(\mathbb{R}^2)$ such that the first and second variations are uniquely expressed in the forms:

$$(S[\Phi])'(\xi)(\eta) = \int_{\mathbb{R}} (S[\Phi])'(\xi; u)\eta(u) \, du,$$

and

$$(S[\Phi])''(\xi)(\eta, \zeta) = \int_{\mathbb{R}} (S[\Phi])''(\xi; u)\eta(u)\zeta(u) \, du + \int_{\mathbb{R}^2} (S[\Phi])''(\xi; u, v)\eta(u)\zeta(v) \, dudv,$$

for each $\eta, \zeta \in S$, respectively and for any finite interval $T$, $\int_T (S[\Phi])''(\cdot; u) \, du$ is in $S[(S)^*]$. For any $L$-functional $\Phi \in D_L$ and any finite interval $T$ in $\mathbb{R}$, the Lévy Laplacian $\Delta_L^T$ is defined by

$$\Delta_L^T \Phi = S^{-1} \left[ \frac{1}{|T|} \int_T (S[\Phi])''(\cdot; u) \, du \right].$$

This Laplacian has the following interesting properties.

1) $\Delta_L^T = 0$ on $(L^2)$ (see [7,26]).
2) $\Delta_L^T$ is a derivation under the Wick product (see [23]).

A Hida distribution $\Phi$ is called to be normal if its $S$-transform $S[\Phi]$ is given by a finite linear combination of

$$\int_{T^k} f(u_1, \ldots, u_k)\xi(u_1)^{p_1} \cdots \xi(u_k)^{p_k} \, du_1 \cdots du_k, \quad (3.1)$$

where $T$ is a finite interval in $\mathbb{R}$, $f \in L^1(T^k)$ and $p_1, \ldots, p_k \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$. For any $p \geq 1$, the normal functional with the $S$-transform given as in (3.1) is in $D_L^T \cap (S_{-p})$, because the kernel

$$\int_{T_k} f(u_1, \ldots, u_k)\delta_{u_1}^{\otimes p_1} \otimes \cdots \otimes \delta_{u_k}^{\otimes p_k} \, du_1 \cdots du_k$$

is in $S_{-1}^{\otimes (p_1 + \cdots + p_k)}$. This functional plays the role of the polynomial in the infinite dimensional analysis. Let $\mathcal{N}_T$ denote the set of all normal functionals in $D_L^T$. For $p \geq 1$ and $\Phi \in D_L^T$, we define a $(-p)$-norm $||| \cdot |||_{-p}$ by

$$||| \Phi |||_{-p}^2 = \sum_{k=0}^{\infty} \| (\Delta_L^T)^k \Phi \|^2_{-p} \in [0, \infty).$$
and denote the completion of $\mathcal{N}_T$ with respect to the norm $|||\cdot|||_{-p}$ by $D_{-p}$. Then $D_{-p}$ is the Hilbert space with the norm $|||\cdot|||_{-p}$ and $\Delta^T L$ is a bounded linear operator from $D_{-p}$ into itself satisfying $|||\Delta^T L\Phi|||_{-p} \leq |||\Phi|||_{-p}$ for $\Phi \in D_{-p}$. We put $D = \bigcup_{p=1}^{\infty} D_{-p}$ with the inductive limit topology. Then the Laplacian $\Delta^T L$ is a continuous linear operator on $D$.

Let $D^T_L$ denote the set of all $L$-functionals $\Phi$ satisfying $S[\Phi](\eta) = 0$ for $\eta$ with $\text{supp}(\eta) \subset T^c$. In [22], Kuo obtained the following result.

**Theorem 3.1.** Suppose $\{j_\epsilon; \epsilon > 0\}$ is a family of continuous linear operators from $S^*$ into $S$ satisfying the following conditions:
(a) $j_\epsilon^* \to I$ strongly on $L^2(\mathbb{R})$ as $\epsilon \to 0$.
(b) $\lim_{\epsilon \to 0} |j_\epsilon|^{-2}_{HS} j_\epsilon^* j_\epsilon|_{HS} = 0$.
(c) There exists a uniformly bounded orthonormal basis $\{e_k; k \geq 0\}$ for $L^2(T)$ such that as $\epsilon \to 0$,

$$|j_\epsilon|^{-2}_{HS} \sum_{k=0}^{\infty} (j_\epsilon e_k)(t)^2 \to \frac{1}{|T|} \text{ in } L^2(T).$$

Then for any $\Phi$ in $D^T_L$,

$$S[\Delta^T L\Phi](\xi) = \lim_{\epsilon \to 0} |j_\epsilon|^{-2}_{HS} S[\Delta G S^{-1}(S[\Phi] \circ j_\epsilon)](\xi).$$

If $\varphi \in (S)$, the functional $S[\varphi]''(\xi)(\eta, \zeta)$, $\eta, \zeta \in S$ has an extension $S[\varphi]''(\xi)(x, y)$, $x, y \in S^*$, such that $S[\varphi]''(\xi)(x, y)$ is in $(S)$. The chaos expansions of $\Delta G \varphi$ and $S[\varphi]''(\xi)(x, x)$ for $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ in $(S)$ are given by

$$\Delta G \varphi = \sum_{n=0}^{\infty} I_n \left( (n + 2)(n + 1) \int_{\mathbb{R}} f_{n+2}(\cdot, t, t) dt \right)$$

and

$$S[\varphi]''(\xi)(x, x) = \sum_{n=0}^{\infty} n(n - 1) \int_{\mathbb{R}^n} f_n(u) \xi(u_1) \cdots \xi(u_{n-2}) x(u_{n-1}) x(u_n) du,$$

respectively. Hence the expectation of $S[\varphi]''(\xi)(\cdot, \cdot)$ is given by

$$\int_{S^*} S[\varphi]''(\xi)(x, x) d\mu(x) = \sum_{n=0}^{\infty} n(n - 1) \int_{\mathbb{R}^{n-1}} f_n(v, t, t) \xi^{(n-2)}(v) dv dt.$$

Thus we come to get Lemma 3.2.

**Lemma 3.2.** For any $\varphi \in (S)$, we have

$$S[\Delta G \varphi](\xi) = \int_{S^*} S[\varphi]''(\xi)(x, x) d\mu(x).$$

We introduce an operator $J_\epsilon$ on $(S)^*$ into $(S)$ by

$$S[J_\epsilon \Phi](\xi) = S[\Phi](j_\epsilon(\xi)), \quad \Phi \in (S)^*.$$

Using the operator $J_\epsilon$, we can obtain the following result.
Theorem 3.3. Let \( T \) be a finite interval in \( \mathbb{R} \) and \( \Phi \) an \( L \)-functional in \( D_{L}^{T} \). Then we have
\[
S[\Delta_{L}^{T}\Phi](\xi) = \lim_{\epsilon \to 0} (\theta_{\epsilon})^{2} \int_{S^{*}} S[J_{\epsilon}\Phi]''(\xi)(x, x) \, d\mu(x),
\]
where \( \theta_{\epsilon} = |j_{\epsilon}|^{-1}_{RS} \).

4. Groups generated by infinite dimensional Laplacians

We now introduce an operator \( e^{z\Delta_{G}} \), \( z \in \mathbb{C} \) by
\[
e^{z\Delta_{G}} \Phi = \sum_{n=0}^{\infty} \frac{(z\Delta_{G})^{n}}{n!} \Phi
\]
for \( \Phi \in (S) \). This operator satisfies the following properties.

Theorem 4.1 [32]. The \( e^{z\Delta_{G}} \) is a continuous linear operator from \( (S) \) into itself given by
\[
e^{z\Delta_{G}} \Phi = \sum_{n=0}^{\infty} I_{n}(\ell_{n}(\Phi; z)), \quad \ell_{n}(\Phi; z) = \sum_{m=0}^{\infty} \frac{(n + 2m)!}{n!m!} z^{m} T_{r} \otimes m * f_{n+2m}
\]
for \( \Phi = \sum_{n=0}^{\infty} I_{n}(f_{n}) \in (S) \).

Theorem 4.2 [32]. For any \( \Phi \in (S) \), we have
\[
S[e^{z\Delta_{G}} \Phi](\xi) = \int_{S^{*}} S[\Phi](\xi + \sqrt{z} x) \, d\mu(x),
\]
where the integral is defined independent of choices of the branch of \( \sqrt{z} \) since \( \mu \) is symmetric.

An infinite dimensional Fourier-Mehler transform \( F_{\theta}, \theta \in \mathbb{R} \), on \( (S)^{*} \) was defined by H.-H. Kuo [19] as follows. The transform \( F_{\theta}\Phi, \theta \in \mathbb{R} \) of \( \Phi \in (S)^{*} \) is defined by the unique Hida distribution with the \( S \)-transform
\[
S[F_{\theta}\Phi](\xi) = S[\Phi](e^{i}\xi) \exp \left[ i \frac{e^{i}\sin \theta |\xi|^{2}}{2} \right], \quad \xi \in S.
\]
Moreover, the adjoint operator \( F_{\theta}^{*} \) of \( F_{\theta} \) is given by
\[
F_{\theta}^{*} \Phi = \sum_{n=0}^{\infty} I_{n}(h_{n}(\Phi; \theta)) \text{ for } \Phi = \sum_{n=0}^{\infty} I_{n}(f_{n}) \in (S),
\]
where
\[
h_{n}(\Phi; \theta) = \sum_{m=0}^{\infty} \frac{(n + 2m)!}{n!m!} \left( \frac{i}{2} \sin \theta \right)^{m} e^{i(m+n)\theta} T_{r} \otimes m * f_{n+2m};
\]
\[ Tr = \int_{\mathbb{R}} \delta_t \otimes \delta_t \, dt. \]

This operator $F^*_\theta$ is a continuous linear operator on $(\mathcal{S})$. (For details, see [19] and also [9].) The operator $e^{i\theta N}$ is called the Fourier-Wiener transform, which is given by

\[ e^{i\theta N} \Phi = \sum_{n=0}^{\infty} e^{i\theta \ell_n} \Phi_n \]

for $\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S})$ (see [9]). The families $\{e^{i\theta \Delta_G}; \theta \in \mathbb{R}\}$ and $\{F^*_\theta; \theta \in \mathbb{R}\}$ are groups generated by $i\Delta_G, iN$ and $iN + \frac{i}{2}\Delta_G$, respectively (see [9]). Take $\Phi = \sum_{n=0}^{\infty} I_n(f_n) \in (\mathcal{S})$. From (4.1), we see that

\[ e^{\frac{i}{2}(e^{\theta \sin \theta}) \Delta_G \Phi} = \sum_{n=0}^{\infty} I_n(\ell_n(\Phi; \frac{i}{2}e^{i\theta \sin \theta})). \]

Hence,

\[ e^{i\theta N}(e^{\frac{i}{2}(e^{\theta \sin \theta}) \Delta_G \Phi}) = \sum_{n=0}^{\infty} I_n(e^{i\theta \ell_n}(\Phi; \frac{i}{2}e^{i\theta \sin \theta})). \]

Since $e^{i\theta \ell_n}(\Phi; \frac{i}{2}e^{i\theta \sin \theta}) = h_n(\Phi; \theta)$, we obtain the following relation.

**Theorem 4.3** [31].

\[ F^*_\theta = e^{i\theta N} \circ e^{\frac{i}{2}(e^{\theta \sin \theta}) \Delta_G}. \]

**Remark:** Details of Lie algebras containing $\Delta_G$ and $N$ are discussed in [28].

A $(C_0)$-group $\{G_t, t \in \mathbb{R}\}$ is given by

\[ G_t = \lim_{\epsilon \to 0} \sum_{k=0}^{n} \frac{t^k}{k!} (\Delta_L^k)^k, \]

as an operator on $D$. The group $G_t$ has naturally an analytic extension $G_z$, $z \in C$. It is easily checked that for any $\Phi \in D$ and $t \in \mathbb{R}$ there exists $p \geq 1$ such that $|||G_z \Phi|||_{-p} \leq e^{i|z|} |||\Phi|||_{-p}$.

An characterization of Hida distributions was obtained by J. Potthoff and L. Streit [29]. They say that for any $F$ in $S[(S)^*]$ and $\xi, \eta \in S$, the function $F(\xi + \lambda \eta)$, $\lambda \in \mathbb{R}$, extends to an entire function $F(\xi + z \eta)$, $z \in C$. We define an operator $g_z$, $z \in C$, acting on a Hida distribution $\Phi$ by

\[ S[g_z \Phi](\xi) = \lim_{\epsilon \to 0} S[e^{i\epsilon \Phi}]^2 \Delta_G J_{\epsilon} \Phi(\xi) \]

if the limit exists in $S[(S)^*]$. For $\Phi \in \mathcal{N}_T$ and $z \in C$, we have $g_z \Phi \in \mathcal{N}_T$. For $p \geq 1$, let $E_{-p}$ denote the collection of Hida distributions $\Phi = \sum_{n=0}^{\infty} \Phi_n$ in $(S_{-p})$ such that $\Phi_n \in \mathcal{N}_T \cap H_n^{(-p)}$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty} |||\Phi_n|||_{-p} < \infty$. Set $E = \bigcup_p E_{-p}$. It is clear that $E_{-p} \subset D_{-p}$ for $p \geq 1$ and $E \subset D$. By calculations of $g_z \Phi$ and $G_z \Phi$ for $\Phi$ whose $S$-transform $S\Phi$ is given as in (3.1), we get $g_z = G_z$ on $\mathcal{N}_T$ for $z \in C$. The continuity of $G_z$ implies the following result.

```math
\begin{align*}
\Phi = \sum_{n=0}^{\infty} \Phi_n \in (\mathcal{S}) \quad \text{(see [9]).} \end{align*}
```
Theorem 4.4. If $\Phi = \sum_{n=0}^{\infty} \Phi_n$ is in $E_{-p}$ for $p \geq 1$, then $\sum_{n=0}^{\infty} g_{z} \Phi_n \in D_{-p}$ and $G_{z} \Phi = \sum_{n=0}^{\infty} g_{z} \Phi_n$ for $z \in \mathbb{C}$. Moreover if

$$\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{S^*} |S[J_{\epsilon} \Phi_n](\xi + \sqrt{2z\theta_{\epsilon}}x)| \, d\mu(x) < \infty$$

holds for any $z \in \mathbb{C}$ and $\xi \in S$, then $g_{z} \Phi$ exists in $D_{-p}$ and $g_{z} \Phi = G_{z} \Phi$.

5. A generalization

For any $\varphi \in (S), \xi \in S$ and $z_1, z_2 \in \mathbb{C}$, the functional $S[\varphi](z_1 \xi + z_2 \eta)$, $\eta \in S$, can be extended to a functional $\widehat{S}[\varphi](z_1 \xi + z_2 y)$, $y \in S^*$, in $(S)$ (cf. [15]). We denote this functional by the same symbol $S[\varphi](z_1 \xi + z_2 y)$. Thus we can define an operator $G_{\alpha,\beta}$ from $(S)$ into itself by

$$S[G_{\alpha,\beta} \varphi](\xi) = \int_{S^*} S[\varphi](\alpha \xi + \beta x) \, d\mu(x). \quad (5.1)$$

Here we note that the right hand side of (5.1) is in $S[(S)]$. If $\alpha = 1$ or $-1$, $G_{\alpha,\beta}$ is equal to Lee's transform $L_{\alpha,\beta}$ ([24]) given by

$$L_{\alpha,\beta} \varphi(x) = \int_{S^*} \varphi(\alpha x + \beta y) \, d\mu(y), \, \varphi \in (S).$$

The transform $L_{\alpha,\beta}$ is applied to the heat equation associated with the operator $(a \Delta_{G} + bN)^k$, $k \geq 1$, $a, b \in \mathbb{C}$ with $\text{Re} b^k \leq 0$. (For details, see [3] and [14].) By the proof analogous to that of Theorem 3.2 in [32], we can obtain the following Lemma.

Lemma 5.1. If a Hida distribution $\Phi$ is in $N_T$, then

$$\lim_{\epsilon \to 0} \int_{S^*} S[J_{\epsilon} \Phi](\alpha_{\epsilon}(z) \xi + \beta_{\epsilon}(z) x) \, d\mu(x) = S[g_{z} \Phi](\xi)$$

holds for any $\xi \in S$, where $\alpha_{\epsilon}(z)$ and $\beta_{\epsilon}(z)$ are complex-valued functions of $z \in \mathbb{C}$ depending $\epsilon > 0$ such that $\alpha_{\epsilon}(z) \to 1$ and $\beta_{\epsilon}(z)/\theta_{\epsilon} \to \sqrt{2it}$ as $\epsilon \to 0$.

Proof. The proof comes from Theorem 4.4 and the following formula:

$$\int_{S^*} S[\varphi](\alpha \xi + \beta x) \, d\mu(x) = S[e^{N \log \alpha} \circ e^{\frac{\theta^2}{2} \Delta_{G}} \varphi](\xi), \, \varphi \in (S), \alpha, \beta \in \mathbb{C}.$$

$\square$

By Lemma 5.1, we have the following result which is a generalization of Theorem 4.7 in [32].
Theorem 5.2. Let $\Phi$ be a Hida distribution in $\mathcal{E}$ satisfying the condition
\[\sum_{n=0}^{\infty} \sup_{\epsilon} \int_{\mathcal{S}} |S[J_\epsilon \Phi_n](\alpha_\epsilon(z)\xi + \beta_\epsilon(z) x)| \, d\mu(x) < \infty.\]

Then
\[\lim_{\epsilon \to 0} S[G_{\alpha(z), \beta_\epsilon(z)} J_\epsilon \Phi](\xi) = S[G_z \Phi](\xi), \quad z \in \mathbb{C}, \xi \in \mathcal{S}. \quad (5.2)\]

Proof. From the assumption and the Lebesgue convergence theorem, we can calculate as follows:
\[\lim_{\epsilon \to 0} S[G_{\alpha(z), \beta_\epsilon(z)} J_\epsilon \Phi](\xi) = \lim_{\epsilon \to 0} \int_{\mathcal{S}} S[J_\epsilon \Phi](\alpha_\epsilon(z)\xi + \beta_\epsilon(z) x) \, d\mu(x)\]
\[= \sum_{n=0}^{\infty} \lim_{\epsilon \to 0} \int_{\mathcal{S}} S[J_\epsilon \Phi_n](\alpha_\epsilon(z)\xi + \beta_\epsilon(z) x) \, d\mu(x).\]

Consequently, by Lemma 5.1, we obtain (5.2). □

Theorem 4.3 admits an integral expression of the adjoint operator of Kuo’s Fourier-Mehler transform:
\[S[F^*_\theta \varphi](\xi) = \int_{\mathcal{S}} S[\varphi](e^{i\theta} \xi + \sqrt{i e^{i\theta} \sin \theta} x) \, d\mu(x), \quad \varphi \in (\mathcal{S}).\]

Hence Theorem 5.2 implies the following

Corollary 5.3. Let $\Phi$ be a Hida distribution in $\mathcal{E}$ satisfying the condition in Theorem 5.2 with
\[\alpha_\epsilon(it) = e^{2it(\theta_\epsilon)^2} \text{ and } \beta_\epsilon(it) = \sqrt{ie^{2it(\theta_\epsilon)^2} \sin(2t(\theta_\epsilon)^2)}.\]

Then
\[\lim_{\epsilon \to 0} S[G_{it} \Phi](\xi) = S[G_{it} \Phi](\xi), \quad t \in \mathbb{R}, \xi \in \mathcal{S}.\]

References


