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DYNAMICAL FORMULATION OF QUANTUM LEVEL STATISTICS

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This is a framework of treating quantum mechanical perturbation theory as a classical dynamics, where the perturbation strength is regarded as the time variable. A full version is presented along the historical development in the decade of eighties. A special attention is focussed on the nature of *openness* of its statistical mechanical formulation on a rigorous basis of the present Hamiltonian level dynamics.

1. Historical outlook

A brief review is presented on the works by Pechukas[1], Yukawa[2,2a)], Nakamura and Lakshmanan[3)], Haake and his coworkers[4)], Nakamura and Mikeska[5)], and Gaspard, Rice, Mikeska and Nakamura[6)]. The first attempt to construct dynamically the joint distribution formula in the standard random matrix theory (RMT)[7)] appeared in the paper by Pechukas[1)], which influenced the rest papers a more or less and explicitly or implicitly.

We can observe that Pechukas' idea was motivated by the preceding important work of Berry-Tabor[8)] to deduce Poisson statistics from a treatment of integrable semiclassical mechanics where the Planck constant $\hbar$ was regarded as a varying parameter. He considered a Schrödinger operator $-\hbar^2 D + V(r)$, and putting $\hbar^2 = e^{-\lambda}(0 \leq \lambda < \infty)$, wrote down a set of equations of motion of the quantities $E_n(\lambda)$ and $V_{mn}(\lambda)$ i.e. the eigenvalues of the operator and the matrix elements on the eigenstate basis of the potential operator V as function of the parameter λ. Notice that the semiclassical limit $\hbar \to 0$ is to be achieved by $\lambda \to \infty$ in this parameter space. His reasoning to find a possible form of distribution function for $\{E_n(\lambda)\}$ is interesting enough to discuss, but it will be absorbed more conveniently in showing a framework of the matrix perturbation theory for hermitians devised in the subsequent paper by Yukawa[2,2a)].

The standard perturbation theory used in quantum mechanics deals with a problem to get the eigenvalues and the eigenvectors of a hermitian matrix $H_\lambda = H_0 + \lambda V$, where λ is the perturbation parameter and is real. For convenience, we confine ourselves to the real operators (H_0 and V

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and hence H_λ are real symmetric). Denote the n-th eigenvalue of H_λ and the mn element of the perturbation matrix V by $x_n(\lambda)$ and $V_{mn}(\lambda)$, respectively. To avoid ambiguity, the following assumptions are adopted:

i) The matrix space is finite dimensional, say $N \times N$.

ii) Each eigenvalue is nondegenerate.

Then, one gets without difficulty the following set of ordinary differential equations for $2N + \frac{1}{2}N(N-1)$ variables

$$\frac{dx_n}{d\lambda} = V_{nn}, \quad \frac{dV_{nn}}{d\lambda} = 2\sum_{m(\neq n)} \frac{V_{nm}V_{mn}}{x_n - x_m} \tag{1.1a, b}$$

$$\frac{dV_{mn}}{d\lambda} = \sum_{l(\neq m,n)} V_{ml}V_{ln}\left(\frac{1}{x_m - x_l} + \frac{1}{x_n - x_l}\right) - \frac{V_{mn}(V_{mm} - V_{nn})}{x_m - x_n}, \quad m \neq n. \tag{1.1c}$$

This set of equations is precisely the same as the one obtained by Pechukas (eqs.(4)$\sim$(6)[1)]) on the basis of which he argued the possible form of the distribution function $\rho(x_1, \cdots, x_n) \equiv \rho(\{x_n\})$: Let y_i, $i = 1, 2, \cdots, M$ denote a set of variables to present an incompressible flow (here $\{x_n\}$, $\{V_{mn}\}$). The distribution function $\rho(\{y_i\})$ satisfies the *equation of continuity* $\frac{\partial\rho}{\partial t} + div(\rho\dot{y}) = 0$, specifically, $div(\rho\dot{y}) = 0$ or, equivalently, $(grad\rho)\cdot\dot{y} = -\rho div\dot{y}$ for a stationary state. In the present case, the time variable is λ and in the limit $\lambda \to \infty$ (the semiclassical limit) it is assumed that a relevant distribution $\rho(\{y_i\})$ with $\{y_i\} = \{x_n\}$ is to satisfy

$$div(\rho\dot{y}) = 0, \quad \text{equivalently} \quad \frac{d\log\rho}{d\lambda} + div\dot{y} = 0. \tag{1.2}$$

Then, Pechukas answered that a possible distribution $\rho(\{x_n\})$ must be of the form, as anticipated from RMT[7)]

$$\rho(\{x_n\}) = C(\{x_n\})\prod_{m<n}|x_m - x_n|, \tag{1.3}$$

where $C(\{x_n\})$ is a normalization factor ($\int \rho dx = 1$) which may depend on $\{x_n\}$ only through some constants of motion of the flow subject to eqs.(1.1a$\sim$1b).

Proof of the above Pechukas' statement can be outlined by noting that

i) a specific function $\rho_0(\{x_n\}) \equiv \prod_{m<n}|x_m - x_n|$ satisfies eq.(1.2) (by a direct computation of $div\dot{y}$ by means of eqs.(1.1a$\sim$ 1c)),

ii) with the aid of this function ρ_0, any solution to eq.(2) must satisfy $\frac{d}{d\lambda}(\rho(\{x_n\})/\rho_0(\{x_n\})) = 0$, and hence $\rho(\{x_n\}) = C(\{x_n\}) \times \rho_0(\{x_n\})$.

As a byproduct, we can observe that the following statement is true: *For an incompressible flow subject to a set of equations of motion, a stationary distribution function $\rho(\{y_i\})$ must depend on $\{y_i\}$ only through some constants of motion of the flow i.e. $\rho(\{y_i\}) = \rho(C_1(y_i), C_2(y_i), \cdots)$, if the flow is divergenceless (i.e. $div\dot{y} = 0$).*

Wisely enough, Yukawa[2,2a)] devised a formulation of choosing another set of variables instead of $\{V_{mn}, m \neq n\}$ for which the flow meets the above condition. Namely, he proposed to change V_{mn}, $m \neq n$, into f_{mn}, where

$$f_{mn} = (x_m - x_n)V_{mn}. \tag{1.4}$$

By this change, eqs.(1.1a$\sim$ 1c)become

$$\frac{dx_n}{d\lambda} = V_{nn}, \quad \frac{dV_{nn}}{d\lambda} = 2\sum_{m(\neq n)} \frac{f_{nm}f_{mn}}{(x_n - x_m)^3} \tag{1.5a, b}$$

$$\frac{df_{mn}}{d\lambda} = \sum_{l(\neq m,n)} f_{ml}f_{ln}\left[\frac{1}{(x_m - x_n)^2} - \frac{1}{(x_n - x_l)^2}\right]. \tag{1.5c}$$

Clearly by inspection, each of the three kinds of velocity variables, $\dot{y}_i$, in the above does not contain y_i on the respective right-hand side of the equation for $\dot{y}_i$, meaning $\frac{\partial \dot{y}_i}{\partial y_i} = 0$ (in contrast to eq.(1.1c)) and $div\dot{y} = 0$. Thus, it is expected that $\rho(\{y_i\})$ is of the form $\rho(C_1\{y_i\}, C_2\{y_i\}, \cdots)$ in terms of some constants of motion $C_i(\{y_i\})$ of eqs.(1.5a$\sim$5c).

A Hamiltonian flow is typical of such divergenceless flows for which, as elementary statistical mechanics tells, the equilibrium distribution is identified with the canonical form $e^{-\beta\mathcal{H}}$ under the hypothesis that the Hamiltonian function $\mathcal{H}$ is the sole constant of motion (ergodicity hypothesis!). For a pragmatic reason, Yukawa observed already in his first paper[2)] that the first two subsets of equations of motion (1.5a) and (1.5b) are of a canonical Hamilton's form with N-particle interacting system Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \frac{1}{2}\sum_{m,n(m\neq n)} \frac{f_{mn}^2}{(x_m - x_n)^2}, \tag{1.6}$$

where the canonical momentum conjugate to x_n is defined by

$$p_n = V_{nn}. \tag{1.7}$$

Furthermore, he recognized another constant of motion

$$\mathcal{Q} = \frac{1}{2}\sum_{m,n(m\neq n)} f_{mn}^2 \tag{1.8}$$

which may play an important role for describing the fluctuation property of eigenvalues $\{x_n\}$, saying that the two constants $\mathcal{H}$ and $\mathcal{Q}$ would be sufficient for the joint distribution $\rho(\{y_i\})$. It implies that he proposed a slightly generalized canonical distribution

$$\rho_{\beta,\gamma} = \frac{1}{Z_{\beta,\gamma}}e^{-\beta\mathcal{H}-\gamma\mathcal{Q}}, \quad Z_{\beta,\gamma} = \int\int\int e^{-\beta\mathcal{H}-\gamma\mathcal{Q}}dxdpdf, \tag{1.9}$$

which he asserts to be relevant for the level statistics based on some general aspects of statistical mechanics (a more detailed account given in Sec. 3.1).

Up to this point of Yukawa's context, one would raise three basic questions as follows:

(a) Is the flow subject to eqs.(1.5a$\sim$5c) still a Hamiltonian flow, when the third set of variables $\{f_{mn}\}$ is included?

(b) How one can enumerate all the constants of motion of the flow, and on this basis is it possible to justify the form (1.9)?

(c) Is an approach to equilibrium by the flow to assure the above-mentioned statement of Pechukas?

The rest of the papers in the references we have given at the beginning, i.e. Ref.2a~Ref.6, attempted in part to clarify the above questions, and from the present viewpoint it can be said that question (a) has been answered thoroughly:

Equations of motion (1.5a~5c) represent a Hamiltonian flow to be given by a canonical Hamilton's form with properly defined Poisson bracket (P.b.)

$$\dot{y}_i = \{y_i, \mathcal{H}\}. \tag{1.10}$$

The degree of freedom of this system is equal to $N + \frac{1}{2}N(N-1)$; the number of independent elements for $N \times N$ real symmetric matrices. It is a completely integrable Hamiltonian dynamical system having a number of independent, involutive global constants of motion which is just equal to the above degree of freedom. In the literature of integrable nonlinear dynamics, this system is called *generalized Calogero-Moser system*[9] *or Euler-Calogero-Moser (ECM) system*[10]. Let us briefly discuss these two namings in view of Yukawa's second paper[2a] (based on Ref.10) and Nakamura-Lakshmanan[3] (based on Ref.9).

It is an easy observation that the first two equations (1.5a) and (1.5b) are the consequence of the canonical form (1.10) with Hamiltonian (1.6), where $\{x_n\}$ and $\{p_n\}$ play the role of the usual canonical coordinates and momenta with the standard Poisson brackets

$$\{x_n, p_n\} = \delta_{mn}, \quad \text{and} \quad \{x_m, x_n\} = \{p_m, p_n\} = 0, \tag{1.11}$$

and where f_{mn} is regarded as constant independent of λ. Suppose that f_{mn} = constant= 1 for simplicity, which leads to a simple N-body system interacting by the inverse-square pair potential called *Calogero-Moser system*[11]. What is the most natural way to enlarge the system by including the new set $\{f_{mn}\}$ compatibly with eq.(1.5c)? Physically, it is to equip each of N-particles of the CM system with an internal degree of freedom[3]. Observing that eq.(1.5c) is of the type of Euler rotation, Wojciechowski[10] discovered that the following P.b. which is characteristic of N-dimensional rotation i.e.

$$\{f_{mn}, f_{rs}\} = \frac{1}{2}(\delta_{ms}f_{rn} + \delta_{mr}f_{ns} + \delta_{ns}f_{mr} + \delta_{nr}f_{sm}) \tag{1.12}$$

realize the correct form (1.5c), when applied to the canonical equation (1.10). Such a relation is known as Lie algebra structure relation. Of coures, the same Hamiltonian (1.6) and the basic rules of P.b. (Laibniz rule and Jacobi identity) are used. This is the origin of the name of ECM. On the other hand, another context adopted in Ref.3 which attempts to represent a differential operation with respect to a vector component by means of P.b. looks rather unusual, hence we shall avoid this context hereafter. However, we retain the naming "generalized Calogero-Moser (g-CM)" instead of ECM merely because it is now traditionally used.

There exists another N-particle nonlinear system with complete integrability similar to the above g-CM, which has been used to study level statistics for quantum chaos[4,5], namely generalized Calogero-Sutherland (g-CS) system[5]. From a classification viewpoint[11] it is obtainable from g-CM just by replacing the pair potential of inverse-square type by the one of inverse-square sine function so that

$$\mathcal{H}_{gCS} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \frac{1}{2}\sum_{m,n(m\neq n)} \frac{f_{mn}^2}{4\sin^2\frac{1}{2}(\phi_m - \phi_n)} \tag{1.13}$$

where the angular variables $\{\phi_n\}$ are used instead of $\{x_n\}$. P. b.'s of the same structure as (1.11) and (1.12) are used to deduce equations of motion for $\{\phi_n\}, \{p_n\}$ and $\{f_{mn}\}$ with prescription

(1.7) and a similar one for f_{mn} (but different from (1.4)), and it was recognized that the resulting set of equations of motion traces a parameter motion of the eigenphases $\{\phi_n\}$ and the related variables of the Floquet operator[4] $F_\lambda = e^{-i\lambda V}e^{-i\lambda H_0}$ (instead of the previous $H_\lambda = H_0 + \lambda V$). A scheme of treating the statistical properties of eigenphases of unitary matrices is known as the theory of circular ensembles[7].

Therefore, the complete integrable g-CS system, $\mathcal{H}_{gCS}$ (1.13) based on eqs.((1.10),(1.11),(1.12), together with the complete integrable g-CM system, $\mathcal{H}$ on eqs.(1.10),(1.11),(1.12) for the Gaussian ensembles[7], constitutes two basic frameworks of the proposed level dynamics. We note that the last paper by Gaspard et al[6], who explored a new subject of level statistics, namely, the curvature distribution for eigenvalue motions (its account not relevant here), gave a complete list of the P.b.'s necessary for deducing the equations of motion for level dynamics.

Let us now turn back to the rest of our starting questions (b) and (c), which become presently much more difficult to answer than expected before because of the complete integrability of the g-CM/S thus discovered. Specifically, the proposed Yukawa distribution (1.9) is difficult to accept, unless some strong reason of eliminating those constants of motion other than $\mathcal{H}$ and $\mathcal{Q}$ can be provided. One of the main concern in the subsequent sections pertains to this question, and our best answer will be of an information-theoretical nature.

Grossly speaking, these questions are to be treated in a framework of *statistical mechanics of open systems.* This implies that the statistical system we investigate comprises two components; a component denoted by $\mathcal{S}$ (the system) which we are directly interested in so that its coordinates are to remain in the distribution function we are seeking at the final stage, and another component denoted by $\mathcal{R}$ (reservoir) whose coordinates are to be eliminated as irrelevant in the sought distribution function. A technical word *coarse-graining* is frequently used for this procedure in statistical physics. Clearly, for the present problem

$$\mathcal{S} = \{x_n\} \quad \text{and} \quad \mathcal{R} = \{p_n\} \times \{f_{mn}\}. \tag{1.14}$$

There are two methods of coarse-graining; static coarse-graining and dynamic coarse-graining. Once a canonical distribution for the open system $\mathcal{S} \times \mathcal{R}$ is obtained, then the static coarse-graining is sufficient to get the answer: the distribution for $\mathcal{S}$ may be written just by integrating out the larger distribution with respect to the irrelevant variables. The remaining Yukawa's context[2,2a] may be understood just by this static coarse-graining.

Yukawa's procedure of the static coarse-graining of his distribution (1.9) gave an explicit answer[2,2a]. Indeed, for the special case $\gamma = 0$ (the ordinary canonical distribution) the result agreed with Pechukas' idea, namely

$$\int\!\!\int \rho_{\beta,\gamma} \prod_n dp_n \prod_{m<n} df_{mn} = const. \prod_{m<n} |x_m - x_n|. \tag{1.15}$$

Also, it is simply regarded as the Jacobian factor for the change of variables from Pechukas to Yukawa (1.4) so that

$$\prod_{m<n} df_{mn} = \rho_0(\{x_n\}) \prod_{m<n} dV_{mn}. \tag{1.16}$$

which means that the left-hand side of the above equation yields the correct Liouville measure in the phase space for g-CM/S dynamics. This has been accepted entirely in the last paper, Gaspard et al[6], who however seems to have ignored the more general proposal of $\gamma \neq 0$ in the form of canonical distribution (1.9). The question (c) whether the prescription of Yukawa for the general case $\gamma \neq 0$ really meets the statement of Pechukas (eq.(1.3) with $C(\{x_n\})$ an effective constant of motion) is a difficult one to answer, which should be rendered into a stochastic treatment such as Brownian motion model[12] (i.e. a dynamical coarse-graining).

The purpose of the rest of this article is to make all the foregoing issues transparent, and on this basis to put the proposed distribution (1.9) on a firmer basis.

2. Generalized Calogero-Moser/Sutherland system

This section is devoted to all the supplementary matters concerning the g-CM and g-CS dynamics; equations of motion on the basis of Poisson brackets, the reason why these P.b.'s are necessitated, the decomposition of the dynamics into the translational and rotational parts, and finally the complete integrability. Always, the simplest case of the real hermitians is presented for clarity, to which the complex and quaternion hermitian cases are supplemented.

2.1. Equations of motion on the basis of Poisson brackets

g-CM dynamics

$$\mathcal{H}_{gCM} = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_{m,n(m\neq n)} \frac{|f_{mn}|^2}{(x_m - x_n)^2} \tag{2.1}$$

P.b.'s

$$\{x_m, p_n\} = \delta_{mn}, \tag{2.2a}$$

$$\{x_m, p_n\} = \{p_m, p_n\} = \{x_m, f_{rs}\} = \{p_m, f_{rs}\} = 0, \tag{2.2b}$$

$$\{f_{mn}, f_{rs}\} = \frac{1}{2}(\delta_{ms} f_{rn} + \delta_{mr} f_{ns} + \delta_{ns} f_{mr} + \delta_{nr} f_{sm}) \tag{2.2c}$$

Eqs. of motion $\dot{y}_i = \{y_i, \mathcal{H}\}$

$$\frac{dx_n}{d\lambda} = p_n, \quad \frac{dp_n}{d\lambda} = 2\sum_{m(\neq n)} \frac{|f_{mn}|^2}{(x_n - x_m)^3} \tag{2.3a, b}$$

$$\frac{df_{mn}}{d\lambda} = \sum_{l(\neq m,n)} f_{ml} f_{ln} \left[\frac{1}{(x_m - x_n)^2} - \frac{1}{(x_n - x_l)^2}\right]. \tag{2.3c}$$

Initial values at $\lambda = 0$

$$x_n = x_n^0, \quad p_n = p_n^0, \quad f_{mn} = f_{mn}^0. \tag{2.4}$$

Statement: Let H_0 and V be two arbitrary $N \times N$ real symmetric matrices and define

$$H_\lambda = H_0 + \lambda V. \tag{2.5}$$

The N eigenvalues of H_λ and $\frac{1}{2}N(N+1)$ matrix elements of V on the H_λ-eigenvector basis are denoted by $\{x_n\}, \{V_{mn}\}$, respectively. Let

$$p_n \equiv V_{nn} \quad \text{and} \quad f_{mn} \equiv [H_\lambda, V]_{mn} = (x_m - x_n) f_{mn}. \tag{2.6}$$

Then, $2N + \frac{1}{2}N(N+1)$ variables $\{x_n\}, \{p_n\}, \{f_{mn}\}$ satisfy g-CM equations (2.3a~3c). Notice that

$$f_{nm} = -f_{mn}. \tag{2.6a}$$

g-CS dynamics

$$\mathcal{H}_{gCS} = \frac{1}{2}\sum_n p_n^2 + \frac{1}{2}\sum_{m,n(m\neq n)} \frac{|f_{mn}|^2}{4\sin^2\frac{1}{2}(\phi_m - \phi_n)} \tag{2.7}$$

P.b.'s

$$\{\phi_m, p_n\} = \delta_{mn}, \tag{2.8a}$$

$$\{\phi_m, \phi_n\} = \{p_m, p_n\} = \{\phi_m, f_{rs}\} = \{p_m, f_{rs}\} = 0, \tag{2.8b}$$

$$\{f_{mn}, f_{rs}\} \text{ same as } (2.2c)$$

Eqs. of motion $\dot{y}_i = \{y_i, \mathcal{H}\}$

$$\frac{d\phi_n}{d\lambda} = p_n, \quad \frac{dp_n}{d\lambda} = 2\sum_{m(\neq n)} \frac{|f_{mn}|^2 \cos\frac{1}{2}(\phi_n - \phi_m)}{4\sin^3\frac{1}{2}(\phi_m - \phi_n)} \tag{2.9a, b}$$

$$\frac{df_{mn}}{d\lambda} = \sum_{l(\neq m,n)} f_{ml}f_{ln}\left[\frac{1}{\sin^2\frac{1}{2}(\phi_m - \phi_l)} - \frac{1}{\sin^2\frac{1}{2}(\phi_n - \phi_l)}\right]. \tag{2.9c}$$

Initial values at $\lambda = 0$

$$\phi_n = \phi_n^0, \quad p_n = p_n^0, \quad f_{mn} = f_{mn}^0. \tag{2.10}$$

Statement: In place of eq.(2.5), we investigate a unitary matrix

$$U_\lambda = e^{-i\lambda V}e^{-iH_0}. \tag{2.11}$$

The N eigenvalues of U_λ and $\frac{1}{2}N(N+1)$ matrix elements of V on the U_λ-eigenvector basis are denoted by $\{e^{-i\phi_n}\}, \{V_{mn}\}$, respectively. Let

$$p_n \equiv V_{nn} \text{ and } f_{mn} \equiv i(U_\lambda^{-1}VU_\lambda - V)_{mn} = -2\sin\frac{1}{2}(\phi_m - \phi_n)V_{mn}. \tag{2.12}$$

Then, $2N + \frac{1}{2}N(N+1)$ variables $\{\phi_n\}, \{p_n\}, \{f_{mn}\}$ satisfy g-CS equations (2.9a~2.9c).

At this stage, it is worthwhile to consider the question how the above statement is related to the problem of *complete integrability*: Simply, the statement does *not* ensure by itself that the underlying equations of motion be integrable by quadrature. Take the example of g-CM equations of motion (2.3a~3c). The statement below these equations only says that the eigenvalues of H_λ, (2.5), and the matrix elements (2.6) can be *a kind of* solutions of eqs.(2.3a~3c). It does not say that every solution to eqs.(2.3a~3c) can be expressed in this way. So, an interesting and important subject pertaining to the present dynamics is to answer the question about the validity of *converse statement*:
A solution to the g-CM equations of motion (2.3a~3c) is given by the eigenvalues of a certain real symmetric matrix (2.5) and by the matrix elements of V there, where H_0 and V are related to the set of initial values (2.4).

We say, *the g -CM equations of motion is completely integrable*, when and only when this converse statement is proved as true. Similarly,
A solution to the g-CS equations of motion (2.9~ 2.9c) is given by the eigenphases of a certain

unitary matrix (2.11) with two real symmetric matrices H_0 and V and by the matrix elements of V, where H_0 and V are related to the set of initial values (2.10).

We say, *the g -CS equations of motion is completely integrable*, when and only when this converse statement is proved as true. Our affirming argument to the converse statement will be given in Sec.2.4.

2.2. Derivation of the canonical and noncanonical Poisson brackets

Gaspard et al[6] provided a detailed discussion about P.b.'s necessary for the level dynamics, but still it is on an ad hoc basis: Here we show how eqs.(2.2a$\sim$2.2c) are necessitated from the principle of mechanics; the concept of *symplectic structure*[13]. Let us first recall the standard symplectic structure for a canonical system with f-degree of freedom: Denoting f canonical coordinates and their conjugate momenta by $(q_1, \cdots, q_f)$ and $(p_1, \cdots, p_f)$, respectively, we can write the canonical 1-form

$$\omega^{(1)} = \sum_{i=1}^{f} p_i dq_i, \tag{2.13}$$

and the canonical 2-form

$$\omega^{(2)} = \sum_{i=1}^{f} dp_i \wedge dq_i. \tag{2.14}$$

The symbol $\wedge$ denotes a multiplication introduced for exterier derivatives $dx, dy, \cdots$ which satisfy

$$(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz), \quad dy \wedge dx = -dx \wedge dy \ (dx \wedge dx = 0)$$

$$df(x_1, \cdots, dx_n) = \sum \frac{\partial f}{\partial x_i} dx_i, \text{and} \quad d^2 f (= d(df)) = 0 \text{ (closedness)}$$

which can be proved because $\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j$ (where $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$) vanishes. Thus, we can observe for the two expressions $\omega^{(1)}$ and $\omega^{(2)}$ that

$$d\omega^{(1)} = \sum_i d(p_i dq_i) = \sum_i dp_i \wedge dq_i = \omega^{(2)}$$

$$= \frac{1}{2} \sum_i (dq_i \ dp_i) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \wedge \begin{pmatrix} dq_i \\ dp_i \end{pmatrix} = \frac{1}{2} (dq \ dp) J \wedge \begin{pmatrix} dq \\ dp \end{pmatrix} \tag{2.15}$$

where $2f$ dimensional vector $(dq_1, \cdots, dq_f, dp_1, \cdots, dp_f)$ is abbreviated by $(dq \ dp)$ and the $2f \times 2f$ antisymmetric matrix is denoted by

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.16}$$

The relation $\omega^{(2)} = d\omega^{(1)}$ in the above assures the closedness of $\omega^{(2)}$ i.e.

$$d\omega^{(2)} = 0, \tag{2.17}$$

but the converse is generally not true ($d\omega^{(2)} = 0$ does not necessarily lead to $\omega^{(2)} = d\omega^{(1)}$ which is called that $\omega^{(2)}$ is exact). The Poisson bracket between two smooth functions $F(q,p)$ and $G(q,p)$ is then defined:

$$\{F, G\} = \begin{pmatrix} \frac{\partial F}{\partial q} & \frac{\partial F}{\partial p} \end{pmatrix} J^{-1} \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}$$

$$= \sum_{i=1}^{f} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right). \tag{2.18}$$

The requirements for the P.b.s' are summarized by

$$\text{i) linearity } \{c_1F_1 + c_2F_2, G\} = c_1\{F_1, G\} + c_2\{F_2, G\} \tag{2.19a}$$

$$\text{ii) antisymmetry } \{G, F\} = -\{F, G\} \ \ \textit{and} \ \ \{F, F\} = 0 \tag{2.19b}$$

$$\text{iii) Leibniz rule } \{F_1F_2, G\} = F_1\{F_2, G\} + \{F_1, G\}F_2 \tag{2.19c}$$

$$\text{iv) Jacobi identity } \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \tag{2.19d}$$

What is important about the canonical symplectic structure and the resulting standard P.b.s' discussed above is that its essential point can be extracted and extended to generally noncanonical symplectic structures: Given $2f$ dynamical variables $\{y_i\}$ and a 2-form $\omega^{(2)}$ on them written as

$$\omega^{(2)} = \sum_{i,j} \omega_{ij} dy_i \wedge dy_j. \tag{2.20}$$

Then, it is necessary and sufficient for the smooth manifold of functions of $\{y_i\}$ to allow the P.b.s' with properties (2.19a~2.19d) that the $2f \times 2f$ matrix (ω_{ij}) satisfies the following:
i') antisymmetry $\omega_{ji} = -\omega_{ij}$
ii') nonsigularity $\det|\omega_{ij}| \neq 0$ i.e. $(\omega_{ij})^{-1} \equiv (\omega^{ij})$ exists
iii') closedness of $\omega^{(2)}$ i.e. $d\omega^{(2)} = 0$, or by means of (ω^{ij})

$$\sum_{l}^{2f} \left(\frac{\partial \omega^{ik}}{\partial y_l}\omega^{lj} + \frac{\partial \omega^{kj}}{\partial y_l}\omega^{li} + \frac{\partial \omega^{ji}}{\partial y_l}\omega^{lk} \right) = 0 \ \ \text{(for Jacobi identity)},$$

and with the satisfaction of these conditions the desired P.b. is given by

$$\{F, G\} = \sum_{ij} \omega^{ij} \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial y_j}. \tag{2.21}$$

We now proceed to the application of this formula for P.b. to the g-CM/S system. The result we shall obtain can be summarized beforehand as follows: For any matrix representation of a Lie group having a set of infinitesimal generators $\{E_i\}$ which are characterized by the structure relations (i.e. the associated Lie algebra structure relations)

$$[E_i, E_j] = \sum_l c_{ij}^l E_l, \tag{2.22a}$$

there exists a smooth manifold (Poisson manifold) of the angular momenta $\{M_i\}$ on which the P.b. can be defined by

$$\{F, G\} = -\sum_{ijl} c_{ij}^l M_l \frac{\partial F}{\partial M_i}\frac{\partial G}{\partial M_j}. \tag{2.22b}$$

It is called Berezin's bracket[14)].

Before making an abstract argument to assure these facts, let us obtain the explicit form of P.b. for N-dimensional real rotation (the original group $O(N)$) by means of the above formula: The infinitesimal generator of $O(N)$ is any real antisymmetric matrix whose basis forms $\{E_{mn}\}$, where $E_{mn} = e_{mn} - e_{nm}$ (e_{mn}: matrix unit whose element vanishes only except mth and nth column)

satisfying
structure relations for Lie alg. $O(N) = A(N)$ $(dimA(N) = \frac{1}{2}N(N-1))$

$$[E_{mn}, E_{rs}] = \delta_{ms}E_{nr} + \delta_{nr}E_{ms} + \delta_{ns}E_{rm} + \delta_{mr}E_{sn} \tag{2.23a}$$

which can be obtained by using $e_{mn}e_{rs} = \delta_{nr}e_{ms}$. An inspection of eqs, (2.22a,b) shows that the P.b. between the corresponding angular momentum components is just of the opposite sign i.e.

$$\{M_{mn}, M_{rs}\} = -\delta_{ms}M_{nr} - \delta_{nr}M_{ms} - \delta_{ns}M_{rm} - \delta_{mr}M_{sn}$$

$$= \delta_{ms}M_{rn} + \delta_{nr}M_{sm} + \delta_{ns}M_{mr} + \delta_{mr}M_{ns}. \tag{2.23b}$$

Comparing this with eq.(2.2c), we see that both P.b.s' agree with each other apart from a factor 1/2 so that the consistency is recovered by setting

$$f_{mn} = \frac{1}{2}M_{mn}. \tag{2.24}$$

Both the matrix commutation relation (2.23a) and the angular momentum P.b. relation (2.23b) constitute the representation of an infinitesimal rotation in the Lie group $O(N)$, called the *adjoint* and *coadjoint representation*[13], respectively.

Here, we add a prescription how the above simplest Lie algebra matter can be extended to two other complex algebras, namely the one associated with the unitary group $U(N)$ and the symplectic group $S_p(N)$[6]. The general principle is the same as before: the only necessary thing is to establish the adjoint representation of the structure relations. The P.b.s' for f-variables can then be obtained just by taking the minus one half of the right hand side of each relation. Introducing two kinds of matrix units (antisymmetric and symmetric units),

$$E_{mn} \equiv e_{mn} - e_{nm}, \quad E^+_{mn} \equiv e_{mn} + e_{nm}, \tag{2.25}$$

we give the prescription to construct the extra structure relations for these:
structure relations for Lie alg. $U(N) = A(N) + iS(N)$ $(dim = N^2)$

$$[E_{mn}, E^+_{rs}] = -\delta_{ms}E^+_{rn} + \delta_{nr}E^+_{sm} + \delta_{ns}E^+_{mr} - \delta_{mr}E^+_{ns} \tag{2.25a}$$

$$[E^+_{mn}, E^+_{rs}] = -\delta_{ms}E_{rn} - \delta_{nr}E_{sm} + \delta_{ns}E_{mr} + \delta_{mr}E_{ns} \tag{2.25b}$$

Note. The diagonal matrix unit $E^+_{mm} \neq 0$, whereas $E_{mm} = 0$ automatically. This implies that the equations of motion to be derived by the above relation include the component such as f_{mm} which, however, is set equal to zero. (This does not occure in the simplest case $O(N)$: $f_{mm} = 0$ automatically.) The resulting dynamics is confined not in $U(N)$ but in $U(N)/T(N)$ where $T(N)$ is the N-dimensional torus of the form diag($e^{i\theta_n}$) which is a subgroup of $U(N)$ (not an invariant subgroup: thus $U(N)/T(N)$ is not a group, but still is a well-defined smooth manifold of dimension $N(N-1)$).

structure relations for Lie alg. $S_p(N) = A(N) \bigoplus (i, \tau_1, \tau_2, \tau_3) \otimes S(N)$

A quaternion algebra (generally on complex numbers)[15] is defined by a 2×2 matrix algebra generated by

$$\tau \left(\tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \tag{2.26}$$

τ_a's ($a = 1, 2, 3$) satisfy $\tau_a^2 = -1, \tau_a\tau_b = \tau_c\epsilon_{abc}$ ($\epsilon_{abc} = +$ and -1 with abc even and odd permutation of (1,2,3), respectively).

$$[E^+_{mn} \bigotimes \tau_a, E^+_{rs} \bigotimes \tau_a] = \delta_{ms}E_{rn} + \delta_{nr}E_{sm} - \delta_{ns}E_{mr} - \delta_{mr}E_{ns} \tag{2.26a}$$

$$[E^+_{mn} \bigotimes \tau_a, E^+_{rs} \bigotimes \tau_b] = \delta_{ms}E^+_{rn} \bigotimes \tau_c + \delta_{nr}E^+_{sm} \bigotimes \tau_c - \delta_{ns}E^+_{mr} \bigotimes \tau_c - \delta_{mr}E^+_{ns} \bigotimes \tau_c \tag{2.26b}$$

with (abc)=cyclic permutation of (123). Equations (2.26a,b) together with (2.23a), (2.25a,b) constitute the relations. Similar to the $U(N)$ case, the resulting dynamics is confined in $S_p(N)/T_p(N)$ where $T_p(N)$ is the N-dimensional torus of the form diag($e^{i\theta_n\tau}$) which is a (noncommutative)subgroup of $S_p(N)$.

Consequently, the Hamiltonian which covers the three cases should be modified from expressions (2.1) and (2.7): The modification may be just the replacement of the potential strengh, $|f_{mn}|^2 \to ||f_{mn}||^2$ where f_{mn} is now complex, or quaternian so that

$$||f_{mn}||^2 = \sum_{a=1}^{\nu}(f^a_{mn})^2 \quad \nu = 1\ O(N),\ 2\ \ U(N), 4\ \ S_p(N). \tag{2.27}$$

2.3. Decomposition of the dynamics into the translational and rotational parts

Let us recall the fact that the degree of freedom of our dynamics is precisely identical to the number of independent matrix elements avoiding the doubling due to symmetry. We now show how the independent coordinate and momentum variables attached to each of matrix elements into a subdynamics of the N-particles with $\{x_n \text{ (or,}\phi_n), p_n\}$ and another of the internal degree of freedom with $\{f_{mn}\}$. For simplicity, again we restrict our discussion to the symmetric hermitian (dynamics on the orthogonal group $O(N)$) and their unitary evolution matrices.

There are two representation frames of matrices; the time-independent (fixed) frame of the stating H_λ (2.5) or U_λ (2.11), and the time-dependent (moving) frame of diagonalizing these. A gross understanding of the foregoing questions about the *complete integrability* and *the canonical and noncanonical Poisson brackets* can be said that in the time-independent frame the motion becomes free, as we can see below.

Let us denote a matrix in the fixed frame by $A, B, \cdots$ etc. and the corresponding matrix in the moving frame by $\bar{A}, \bar{B}, \cdots$. First, we study the g-CM dynamics and write

$$\bar{X} = \mathcal{U}^+X\mathcal{U} = diag(x_1, x_2, \cdots, x_N) \tag{2.28a}$$

by rewriting H_λ (2.5) as X. The unitary matrix $\mathcal{U}$ can be real, orthogonal in this case, and is a complicated function of λ. We investigate an increment dX and the corresponding $d\bar{X}$ of X and $\bar{X}$, respectively, (not necessarily induced by $d\lambda$ at this moment) to see how they are related to each other:

$$d\bar{X} = \mathcal{U}^+dX\mathcal{U} + \mathcal{U}^+[\mathcal{U}d\mathcal{U}^+, X]\mathcal{U},$$

or, in the other way round, $X = \mathcal{U}\bar{X}\mathcal{U}^+$,

$$dX = \mathcal{U}d\bar{X}\mathcal{U}^+ + \mathcal{U}[\mathcal{U}^+d\mathcal{U}, \bar{X}]\mathcal{U}^+.$$

We define an infinitesimal metric associated with the matrix increment dX by

$$ds^2 = \text{Tr}(dXdX^+) = \text{Tr}(dX)^2 \ \text{ (for hermitians)}$$

and then

$$ds^2 = \mathrm{Tr}(d\bar{X})^2 + \mathrm{Tr}[\bar{\Omega}, \bar{X}]^2 + 2\mathrm{Tr}([\bar{\Omega}[\bar{X}, d\bar{X}]),$$

where

$$\bar{\Omega} \equiv \mathcal{U}^{+} d\mathcal{U} \in A(N) \tag{2.29}$$

yields an infinitesimal generator of rotation (here real antisymmetric). It can be observed that, when and only when the matrix increment $d\bar{X}$ commutes with $\bar{X}$, the metric ds^2 of dX has a decomposed form

$$ds^2 = \mathrm{Tr}(d\bar{X})^2 + \mathrm{Tr}[\bar{\Omega}, \bar{X}]^2. \tag{2.30}$$

Or, in view of the diagonalization (2.28), the decomposition (2.30) is the necessary and sufficient condition for the frame $\bar{X}$ to be the diagonalized frame of the fixed one X: thus

$$ds^2 = \sum_{n=1}^{N} (dx_n)^2 + 2 \sum_{m<n} (x_m - x_n)^2 |\omega_{mn}|^2. \tag{2.31a}$$

We can make an entirely similar argument for the g-CS dynamics, although the unitary matrix U for diagonalization is generally not orthogonal: By defining

$$\bar{U} = \mathcal{U}^{+} U \mathcal{U} = diag(e^{-i\phi_1}, e^{-i\phi_2}, \cdots, e^{-i\phi_N}) \tag{2.28b}$$

and using expression (2.29), and assuming $[dU, U] = 0$,

$$\begin{aligned} ds^2 &= \mathrm{Tr} dU dU^{+} = \mathrm{Tr} \bar{U} d\bar{U}^{+} + \mathrm{Tr}[\bar{\Omega}, \bar{U}][\bar{\Omega}, \bar{U}^{+}] \\ &= \sum_{n=1}^{N} (d\phi_n)^2 + 2 \sum_{m<n} |e^{-i\phi_m} - e^{-i\phi_n}|^2 |\bar{\Omega}_{mn}|^2. \end{aligned} \tag{2.31b}$$

Now, the kinetic energy of the g-CM/S system can be defined by

$$\begin{aligned} T &= \frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 = \frac{1}{2}\mathrm{Tr}\left(\frac{d\bar{X}}{d\lambda}\right)^2 + \frac{1}{2}\mathrm{Tr}[\bar{A}, \bar{X}]^2 \ \ (\text{or } \frac{1}{2}\mathrm{Tr}[\bar{A}, \bar{U}][\bar{A}, \bar{U}^{+}]) \\ &= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{dx_n}{d\lambda}\right)^2 + \sum_{m<n}(x_m - x_n)^2 |\bar{A}_{mn}|^2 \ \ (\text{or } \sum_{m<n} |e^{-i\phi_m} - e^{-i\phi_n}|^2 |\bar{A}_{mn}|^2) \end{aligned} \tag{2.32}$$

where

$$\bar{A} = \mathcal{U}^{+} \frac{d\mathcal{U}}{d\lambda} \tag{2.33}$$

represents the angular velocity in the moving frame. Then, the standard process of mechanics (Lagrange formulation) tells us how the conjugate momentum can be introduced by which T can be exhibited as Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\sum_{n} p_n^2 + \frac{1}{4}\sum_{m<n} \frac{|M_{mn}|^2}{|x_m - x_n|^2} \tag{2.34a}$$

$$(\text{or } \frac{1}{4}\sum_{m<n} \frac{|M_{mn}|^2}{4\sin^2\frac{1}{2}(\phi_m - \phi_n)}) \tag{2.34b}$$

where

$$p_n = \frac{dx_n}{d\lambda}, \quad \text{the momentum conjugate to the velocity,} \tag{2.35}$$

and

$$M_n = 2(x_m - x_n)^2 \bar{A}_{mn}, \quad \text{or} \quad \sin^2\frac{1}{2}(\phi_m - \phi_n)\bar{A}_{mn}, \tag{2.36}$$

the angular momentum conjugate to the angular velocity.

Thus, our remaining problem is to show that the angular momentum components defined in (2.36) satisfy the prescribed P.b.s' in the foregoing subsection i.e. Eqs.(2.22a,b). We outline this result by the explicit determination of the pertinent *symplectic structure*; the noncanonical 1 and 2 forms which can be transformed from the fixed frame to the moving one:

$$(X,V) \rightarrow (\bar{X},\bar{V}) = \mathcal{U}^{+}(X,V)\mathcal{U} \quad \text{and}$$
$$dX = \mathcal{U}(d\bar{X} + [\bar{\Omega},\bar{X}])\mathcal{U}^{+}, \quad V = \mathcal{U}\bar{V}\mathcal{U}^{+} \tag{2.37}$$

which yields

$$\omega^{(1)} = \mathrm{Tr}(VdX) = \mathrm{Tr}(\bar{V}d\bar{X} + \bar{V}[\bar{\Omega},\bar{X}]) = \mathrm{Tr}(\bar{V}d\bar{X}) + \mathrm{Tr}[\bar{X},\bar{V}]\bar{\Omega}$$
$$\omega^{(2)} = d\omega^{(1)} = \mathrm{Tr}(dV \wedge dX) = \mathrm{Tr}(d\bar{V} \wedge d\bar{X}) + \mathrm{Tr}d[\bar{X},\bar{V}] \wedge \bar{\Omega} + \mathrm{Tr}[\bar{X},\bar{V}]d\bar{\Omega}.$$

The noncanonical characteristic of the moving frame stems from the matrix 1-form $\bar{\Omega}$, eq.(2.29), for which $d\bar{\Omega} \neq 0$ but *

$$d\bar{\Omega} = -\bar{\Omega} \wedge \bar{\Omega} \quad \text{(the Maurer Cartan equation).} \tag{2.38}$$

Thus, by setting

$$[\bar{X},\bar{V}] = \frac{1}{2}\bar{M} \quad \text{(consistent to eq.(2.24)),}$$

we can write

$$\omega^{(2)} = \mathrm{Tr}(dV \wedge dX) = \mathrm{Tr}(d\bar{V} \wedge d\bar{X}) + \frac{1}{2}(\mathrm{Tr}d\bar{M} \wedge \bar{\Omega} - \mathrm{Tr}\bar{M}\bar{\Omega} \wedge \bar{\Omega})$$
$$= \sum_{n=1}^{N} dp_n \wedge dx_n + \sum_{m<n} (d\bar{M}_{mn} \wedge \bar{\Omega}_{nm} - \bar{M}_{mn}(\bar{\Omega} \wedge \bar{\Omega}_{mn})). \tag{2.39}$$

This shows that the antisymmetric matrix (ω_{ij}) in eq.(2.20) which is canonical with $f = \frac{1}{2}N(N+1)$ in the fixed frame can be decomposed into the direct sum of a canonical part ($f = N$) and a noncanonical part ($f = \frac{1}{2}N(N-1)$) in the moving frame:

$$(\omega_{ij}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \bigoplus \begin{pmatrix} 0 & I \\ -I & C \end{pmatrix}, \qquad C(\bar{\Omega} \wedge \bar{\Omega}\ part\ ) \neq 0, \tag{2.40a}$$

hence

$$(\omega_{ij})^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \bigoplus \begin{pmatrix} C & -I \\ I & 0 \end{pmatrix} \tag{2.40b}$$

showing that the fundamental P.b. relations (2.2a) and (2.2b) are indeed valid. That is to say, the translational subdynamics is canonical and separated from the rotational part (the direct sum in eq.(2.40b) shows that $\left\{ \begin{pmatrix} x_n \\ p_n \end{pmatrix}, f_{ij} \right\} = 0$). Finally, the last P.b. relation (2.2c) can be established from the following:

$$\bar{\Omega} = \sum \bar{\Omega}_i E_i \quad \text{and} \quad \bar{M} = \sum \bar{M}_i E_i, \quad \text{and}$$

* This equation can be deduced in the following manner[16]. The matrix one-form $\bar{\Omega}$ is defined by (2.29). Hence, $d\mathcal{U} = \mathcal{U}\bar{\Omega}$ and $d^2\mathcal{U} = d\mathcal{U} \wedge \bar{\Omega} + \mathcal{U}d\bar{\Omega}$. Thus, we obtain $d\bar{\Omega} = -\mathcal{U}^{+}d\mathcal{U} \wedge \bar{\Omega} = -\bar{\Omega} \wedge \bar{\Omega}$.

$$\bar{\Omega}\wedge\bar{\Omega}=\sum_{ij}\bar{\Omega}_i\wedge\bar{\Omega}_j E_i E_j=\frac{1}{2}\sum_{ij}\bar{\Omega}_i\wedge\bar{\Omega}_j[E_i,E_j]$$
$$=\frac{1}{2}\sum_{ij}\sum_{l}c_{ij}^{l}E_l\bar{\Omega}_i\wedge\bar{\Omega}_j.$$

Hence, the nonvanishing C-part in eq.(2.40a) is determined by

$$C_{ij}=-\sum_{l}c_{ij}^{l}\bar{M}_l. \tag{2.41}$$

2.4. Complete integrability

Equations of motion for the g-CM (2.3a~2.3c) and for the g-CS (2.9a~2.9c) are rederived by the matrix transformations (2.28a,2.28b) from the fixed into the moving frames: For g-CM,

$$\frac{d\bar{X}}{d\lambda}=-[\bar{A},\bar{X}]+\bar{V} \tag{2.42a}$$

$$\frac{d\bar{V}}{d\lambda}=-[\bar{A},\bar{V}] \quad \text{and} \quad \frac{d\bar{F}}{d\lambda}=-[\bar{A},\bar{F}] \tag{2.42b, c}$$

where

$$\bar{F}\equiv[\bar{X},\bar{V}]. \tag{2.42d}$$

Similarly, for g-CS, by defining $\bar{U}\equiv e^{-i\Phi}$,

$$\frac{d\Phi}{d\lambda}=-i[\bar{A},e^{-i\Phi}]e^{i\Phi}+\bar{V} \tag{2.43a}$$

$$\frac{d\bar{V}}{d\lambda}=-[\bar{A},\bar{V}] \quad \text{and} \quad \frac{d\bar{F}}{d\lambda}=-[\bar{A},\bar{F}] \tag{2.43b, c}$$

where

$$\bar{F}\equiv(e^{i\Phi}\bar{V}e^{-i\Phi}-\bar{V}). \tag{2.43d}$$

The angular velocity $\bar{A}$ is defined by eq.(2.33). Also, as we have noted in the foregoing section, the moving frame is characterized by

$$\left[\frac{d\bar{X}}{d\lambda},\bar{X}\right]=0 \ \ (or, \ \left[\frac{d\bar{\Phi}}{d\lambda},\bar{\Phi}\right]=0) \tag{2.44}$$

which yields, for g-CM by inserting eq.(2.42a) into eq.(2.44),

$$-[[\bar{A},\bar{X}]\bar{X}=[\bar{X},\bar{V}]=\bar{F}.$$

But this is a representation invariant relation so that we also have

$$-[[A,X]X]=[X,V]=F \tag{2.45}$$

in the fixed frame. Also, by recalling $X=H_0+\lambda V$ which indicates that $F=[H_0,V]$ is absolutely time-independent, we can say that *every component of the angular momentum F yields a constant of motion in the fixed frame.*

An entirely similar result is seen to hold for g-CS, if we use eqs.(2.43a), (2.43d), (2.44) and $e^{-i\Phi} = e^{-i\lambda V} e^{-iH_0}$, which shows that

$$F = i\left(e^{i\Phi} V e^{-i\Phi} - V\right) = i\left(e^{iH_0} V e^{-iH_0} - V\right). \tag{2.46}$$

We also note that the angular velocity matrix in the fixed frame is given by

$$A = \mathcal{U}\bar{A}\mathcal{U}^+ = \mathcal{U}\mathcal{U}^+ \frac{d\mathcal{U}}{d\lambda}\mathcal{U}^+ = \frac{d\mathcal{U}}{d\lambda}\mathcal{U}^+$$

which indicates that $\mathcal{U}$ can be integrated by a linear (generally, nonautonomous) evolution equation

$$\frac{d\mathcal{U}}{d\lambda} = A\mathcal{U} \qquad \mathcal{U}_{\lambda=0} = I, \tag{2.47}$$

provided that A is prescribed. The prescription is available now from the commutativity (2.44), i.e.

$$-[[A, X], X] = F \quad \text{for gCM and} \quad [[A, U]U^+] = F \quad \text{for gCS,}$$

or, in regard to the matrix elements of A,

$$A_{mn} = \frac{-f_{mn}}{(x_m - x_n)^2} \text{ for gCM or } \frac{-f_{mn}}{4\sin^2\frac{1}{2}(\phi_m - \phi_n)} \text{ for gCS.} \tag{2.48}$$

We are now able to give a full answer to the complete integrability: The g-CM/S equations of motion (2.2a∼2c) or (2.9a∼9c) can be integrated (under the respective initial condition) by quadrature, because in the fixed frame they are represented as

$$\frac{dX}{d\lambda} = V, \quad \frac{dV}{d\lambda} = 0, \quad \text{hence } X = X^0 + \lambda V^0,$$

or

$$i\frac{dU}{d\lambda} = VU, \quad \frac{dV}{d\lambda} = 0, \quad \text{hence } U = e^{-i\lambda V^0} e^{-iX^0}.$$

The solution in the moving frame can be obtained by the unitary transformation $\mathcal{U}^+(\,\cdot\,)\mathcal{U}$, where $\mathcal{U}$ is the solution of the initial-value problem (2.47) with the prescribed A-matrix (2.48): The right-hand side of eq.(2.48) is expressible in terms of the initial values of X^0 and V^0, namely, $\{x_n\}$ or $\{e^{-i\phi_n}\}$, all the eigenvalues of $X^0 + \lambda V^0$ or $e^{-i\lambda V^0} e^{-iX^0}$ and $f_{mn} = [X^0, V^0]_{mn}$ or $i(e^{iX^0} V^0 e^{-iX^0} - V^0)_{mn}$.

At the same time, we can determine the full set of the constants of motion for g-CM/S dynamics, at least, in the fixed frame:

$$diag(V_{11}, V_{22}, \cdots, V_{NN}) = diag(P_1^0, P_2^0 \cdots P_N^0) \equiv P \tag{2.49}$$

and

$$F = [X, V] \quad \text{or} \quad i(e^{iX} V e^{-iX} - V) = (f_{mn}^0). \tag{2.50}$$

The true constants of motion i.e. those ploynomials of $\{x_n\}, \{p_n\}$ and $\{f_{mn}\}$ with vanishing time-derivatives can be obtained from all the matrix invariants generated by P and F: $\bar{P} = \mathcal{U}^+ P \mathcal{U}$, $\bar{F} = \mathcal{U}^+ F \mathcal{U}$ and

$$\mathrm{Tr}(\cdots \bar{P}^{n_1} \bar{F}^{n_2} \bar{P}^{n_3}) = \mathrm{Tr}(\cdots P^{n_1} F^{n_2} P^{n_3}). \tag{2.51}$$

3. Information-theoretical basis of g-CM/S statistical mechanics

There are very few papers in the literature dealing with the aspect of the random matrix theory from a view point of *information theory.* Balian's work in 1968[18)] clarified the Gaussian property of the standard joint distribution for every independent element of a sample matrix by means of the *maximum entropy principle.* Here, we present a similar formulation which applies to possible canonical distributions of the g-CM/S Hamiltonian system in order to provide a sound basis of Yukawa's type distribution (1.9).

3.1. Ergodicity argument of Yukawa and Ishikawa

Yukawa and Ishikawa[19)] presented a supplementary discussion of justifying the proposed distribution (1.9), which has an important connection to the above context and will be outlined first. The g-CM/S dynamics, when represented in the fixed frame, is a free motion of independent particles with very high degree of freedom (= number of independent matrix elements), as we have seen in the preceding section. Without a boundary restriction on each particle, the dynamics is completely integrable, being far from ergodic: the only possible restriction would be a confinement of all the particles in a fixed-size box with periodic boundary or hard-wall boundary condition. The situation is quite similar to an ideal gas of $\sim 10^{23)}$ molecules confined in a box whose ergodic property is a basis of the statistical treatment leading to thermodynamics. Under the circumstance, almost of all the constants of motion will be destroyed, and for the g-CM/S system at hand, by choosing a hard-wall boundary condition, the only surviving one out of those $\mathrm{Tr}V^{2n}$ and $\mathrm{Tr}F^{2n}$ $(n = 1, 2, \cdots)$ will be the lowest power $n = 1$, i.e $\frac{1}{2}\mathrm{Tr}V^2$ (The Hamiltonian $\mathcal{H}$, (1.6)), $\frac{-1}{2}\mathrm{Tr}F^2$ (the square of angular momentum $\mathcal{Q}$, (1.8)).

They further added that these two specific constants of motion are *additive* quantities, namely, those whose statistical average are proportional to the system size: the only additive constants of motion of the form $\mathrm{Tr}V^{2n}$ and $\mathrm{Tr}F^{2n}$ would be the case $n = 1$.

The above additiveness statement requires a careful check before making a definite conclusion: Actually, it is not certain what is meant by the system size in the level dynamical system, because the *degree of freedom* and the *particle number* are different concepts ($f = O(N^2)$, whereas the particle number $= N$) for the present g-CM/S system. Anyway, the argument asserts the necessity to distinguish the two constants of motion $\mathcal{H}$ and $\mathcal{Q}$ from the rest.

A clearcut distinction which characterizes these two among all the constants of motion of the g-CM/S dynamics can be made from a consideration of the *ploynomial order* of the constants with respect to the perturbation matrix V such that $\mathcal{H}$ and $\mathcal{Q}$ are the only two which are quadratic with respect to (all the matrix elements of) V. The idea was first given by Hasegawa and Robnik[20)] and further refined[21)]. Here, we present the proof of this result.

3.2. Quadratic nature of the two constants of motion $\mathcal{H}$ and $\mathcal{Q}$

Let us restate our result in a precise statement: Let $\mathcal{H}$ and $\mathcal{Q}$ denote the two constant of motion of the g-CM and g-CS dynamics discussed in Sec.2, namely $\mathcal{H}_{gCM}$ in eq.(2.1) and $\mathcal{H}_{gCS}$ in eq.(2.7) respectively, and $\mathcal{Q} = \frac{1}{2}\sum_{m\neq n}|f_{mn}|^2$ with f_{mn} in eq.(2.6) and in eq.(2.12), respectively. In the invariant form, these are reexpressed as

$$\mathcal{H} = \frac{1}{2}\mathrm{Tr}V^2,$$

and

$$\mathcal{Q} = \frac{-1}{2}\mathrm{Tr}F^2 = \frac{-1}{2}\mathrm{Tr}[X, V]^2 \quad \text{for gCM}$$
$$= \frac{-1}{2}\mathrm{Tr}(e^{-iX}Ve^{iX} - V)^2 \quad \text{for gCS.}$$

They satisfy the following three conditions

i) $\{\mathcal{H}, P\} = \{\mathcal{Q}, P\} = 0$, where $P = \sum_{n=1}^{N} p_n$(total momentum)

ii) $\{\mathcal{H}, \mathcal{Q}\} = 0$

iii) $\mathcal{H}$ and $\mathcal{Q}$ are quadratic with respect to $\{V_{mn}\}$.

Conversely, among all the constants of motion of the g-CM/S system $\{\mathcal{Q}_n\}$, $n = 1, 2\cdots$; $\mathcal{Q}_0 = \mathcal{H}$, $\mathcal{Q}_1 = \mathcal{Q}\}$ no constants $\mathcal{Q}_n(n \geq 2)$ satisfy condition iii). In other words, $\mathcal{H}$ and $\mathcal{Q}$ are the only two constants of motion which are mutually independent, involutive and translationally invariant ones and, furthermore, quadratic with respect to $\{V_{mn}\}$.
Proof.[21] We state the logical process explicitly for the g-CM, and by inspection of it a completely similar process can be seen to hold for the g-CS. First, we point out that such a quadratic constant may be written as a linear combination of $\mathrm{Tr}\{a(X)Vb(X)V\}$, i.e. *trace* of a double V with two diagonal matrices $a(X)$ and $b(X)$, which can be written in the form

$$\mathcal{Q}_{g,\phi} = \frac{1}{2}\sum_n g(x_n)p_n^2 + \frac{1}{2}\sum_{m\neq n} \phi(x_m, x_n)|f_{mn}|^2$$

(cf. the relation between V_{mn} and f_{mn} $(m \neq n)$, eq.(2.50)),

and notice that the transitional invariance of $\mathcal{Q}_{g,\phi}$ (i.e. $\mathcal{Q}_{g,\phi}$ is invariant with respect to the uniform change $x_n \to x_n + a$) requires that $g(x) \equiv$ constant, and $\phi(x, y) \equiv \phi(x - y)$. Therefore,

$$\mathcal{Q}_{g,\phi} = \frac{1}{2}g_0\sum_n p_n^2 + \frac{1}{2}\sum_{m\neq n} \phi(x_m - x_n)|f_{mn}|^2$$

and

$$\frac{d}{d\lambda}\mathcal{Q}_{g_0,\phi} = \sum_m p_m \sum_{m\neq n} |f_{mn}|^2 \left[\left(\frac{d\phi}{dx}\right)_{x=x_m-x_n} + \frac{2g_0}{(x_m - x_n)^3}\right]$$
$$+\frac{1}{2}\sum_{m\neq n}\sum_{l\neq(m,n)} \mathrm{Re}(f_{mn}f_{nl}f_{lm})\phi(x_m - x_n)\left[\frac{1}{(x_m - x_l^2} - \frac{1}{(x_n - x_l^2}\right]$$

which should be set equal to 0 identically. The term-wise vanishing of the right-hand side above leads us to have two possibilities: $\frac{d\phi}{dx} = -\frac{2g_0}{x^3}$ with $\phi(x) = 1(g_0 = 0)$, or, $\phi(x) = \frac{1}{x^2}(g_0 = 1)$. The first possibility yields $\mathcal{Q}_{g,\phi} = \mathcal{Q}$, and the second $\mathcal{Q}_{g,\phi} = \mathcal{H}$, because, then, the other triple summation in the right-hand side of the expression for $\frac{d}{d\lambda}\mathcal{Q}_{g,\phi}$ above can be shown to vanish by taking the cyclic permutations of the summation indices $l \to m \to n \to l$. This also assures that $\mathcal{H}$ and $\mathcal{Q}$ are mutually involutive, i.e. $\{\mathcal{H}, \mathcal{Q}\} = 0$.

3.3. The maximum entropy principle for the g-CM/S system

The fact that the two constants of motion $\mathcal{H}$ and $\mathcal{Q}$ are quadratic with respect to $\{V_{mn}\}$ has a statistical significance that the exponential distribution (1.9) is *Gaussian* for the two sets of variables $\{p_n\}$ and $\{f_{mn}\}$ i.e. the $\mathcal{R}$-variables in eq.(1.14), where the system variables $\{x_n\}$ are fixed and the relations (1.4) and (1.7) are invoked. We can say that the present statistical g-CM/S system is an ideal open system with Gaussian reservior. The theorem we have just presented and proved now enables us to establish the most probable nature of the distribution (1.9).

Let us recall the maximum entropy principle for a one-dimensional system in the standard form[22]: Among all probability density functions $P(y)$ with a prescribed variance for y, ,i.e. $\langle (y-\langle y\rangle)^2\rangle = \sigma^2$, the Gaussian distribution

$$P_G(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\langle y\rangle)^2}; \quad \langle (y-\langle y\rangle)^2\rangle_{P_G} = \sigma^2 \tag{3.1}$$

is uniquely determined by the maximum of the entropy functional

$$h[P] \equiv -\int_{-\infty}^{\infty} P(y)\log P(y)dy; \quad \langle (y-\langle y\rangle)^2\rangle_P = \sigma^2. \tag{3.2}$$

Namely, any $P(y)$ with the variance σ^2 as above satisfies the inequality

$$h[P] \le h[P_G] = \frac{1}{2}\log(2\pi e\sigma^2), \tag{3.3}$$

where *the equality holds if and only if* $P(y) = P_G(y)$. The theorem is a consequence of the well-known Kullback divergence inequality

$$-\int_{-\infty}^{\infty} P(y)\log P(y)dy \le -\int_{-\infty}^{\infty} P(y)\log P_1(y)dy \tag{3.4}$$

$$\left(\int_{-\infty}^{\infty} P(y) = \int_{-\infty}^{\infty} P_1(y) = 1\right),$$

where the equality holds if and only if $P(y) = P_1(y)$. The result (3.3) is merely the special case $P_1(y) = P_G(y)$. We may write the above result in a simple form:

$$\max_P h[P] = h[P_G] \tag{3.5}$$

under the constraint

$$\langle y\rangle = \text{fixed, and} \tag{3.5a}$$

$$\langle (y-\langle y\rangle)^2\rangle = \text{fixed, } \sigma^2. \tag{3.5b}$$

We first apply the above to the standard RMT joint distribution for $N\times N$ hermitian matrices[18]. We denote the set of all $N\times N$ hermitian matrices (what is called *ensemble*) by $\mathcal{E}_\nu$ ($\nu = 1,2,4$ for OE, UE and SE, respectively), and the probability of a sample matrix H to lie in an interval $(H, H+dH)$ in $\mathcal{E}_\nu$ by $P_\nu(H)d^\nu H$, where

$$d^\nu H = \prod_n dH_{nn} \prod_{m<n} dH^{(1)mn} \cdots dH^{(\nu)mn}.$$

Then, the maximum entropy principle for P reads

$$\max_{P\in\varepsilon} h[P] = h[P_G] \tag{3.6}$$

under the constraint

$$\langle H_{mn}\rangle = 0 \tag{3.6a}$$

and

$$\langle H_{nn}^2\rangle = 2\langle H_{mn}^2\rangle = \sigma^2. \tag{3.6b}$$

The result yields

$$P_{G\nu}(H) = \frac{1}{Z_\nu} e^{-\frac{1}{2\sigma^2}\mathrm{Tr}H^2}. \tag{3.7}$$

The standard RMT distribution

$$P_N(x_1,\cdots,x_N)dx_1\cdots dx_N = C_{N\nu}e^{-\frac{1}{4\sigma^2}\sum x_n^2}\prod_{m<n}|x_m - x_n|^\nu dx_1\cdots dx_N \tag{3.8}$$

is known[7] to be the form $P_{G\nu}(H)d^{(\nu)}H$ after a change of the variables

$$\{H_{mn}\} \to \{x_n\} \text{ (eigenvalues of } H\text{) and all others}$$

which do not enter $P_{G\nu}(H)$ hence integrated out.

A restricted nature of the standard RMT Gaussian distribution (3.8) can be seen to stem from the uniformity of constraint (3.6b) i.e. the single constant factor σ^2, (3.6b), and our reformulation from the open-system viewpoint aims to remove this uniformity, namely, the variance function be generally x-dependent: This direction of research is in agreement with the so-called *structured random matrices*[23),24)], and the discussion to follow indeed provides such an example.

We introduce *conditional expectation* of a quantity A in the subspace of $\mathcal{R}$ out of the total space $\mathcal{S}\times\mathcal{R}$ defined in eq.(1.14):

$$\langle A\rangle_{\mathcal{R}}(= E(A|x = \text{fixed})) = \int\int A(\{x\},\{p\}\{f\})dpdf \tag{3.9}$$

which is still a function of the system variable $\{x_n\}$. This allows the expectation values generally x-dependent i.e. non-uniform. Hence our entropy principle must be by using the *conditional entropy*:

$$h_P(x) \equiv -\langle\log P\rangle_{\mathcal{R}} = -\int\int dpdf\, P(\{x\},\{p\}\{f\})\log P(\{x\},\{p\}\{f\}) \tag{3.10}$$

and

$$\max_P h_P(x) = h_G(x), \tag{3.11}$$

where the constraint is, besides

$$\langle p\rangle = \langle f\rangle = 0, \tag{3.11a}$$

$$\langle p_n^2\rangle,\ \langle f_{mn}^{(a)2}\rangle = \text{prescribed function of } \{x_n\}. \tag{3.11b}$$

We show that the maximum entropy condition (3.11) under the following variance constraint determines the distribution (1.9), i.e.

$$\rho_{\beta,\gamma} = \frac{1}{Z_{\beta,\gamma}},\quad Z_{\beta,\gamma} = \int\int\int e^{-\beta\mathcal{H}-\gamma\mathcal{Q}}dxdpdf,$$

uniquely:

$$\langle p_n^2 \rangle_{\mathcal{R}} = 1/\beta, \quad \frac{1}{(x_m - x_n)^2} \langle f_{mn}^{(a)2} \rangle_{\mathcal{R}} = \frac{1}{2\beta} \frac{1}{1 + (\gamma/\beta)(x_m - x_n)^2} \tag{3.12}$$

for the g-CM system, and

$$\langle p_n^2 \rangle_{\mathcal{R}} = 1/\beta, \quad \frac{1}{4\sin^2(\phi_m - \phi_n)/2} \langle f_{mn}^{(a)2} \rangle_{\mathcal{R}} = \frac{1}{2\beta} \frac{1}{1 + (\gamma/\beta) 4\sin^2(\phi_m - \phi_n)/2} \tag{3.13}$$

for the g-CS system. The reason for this result of the present entropy principle is as follows:

i) The special form of canonical distribution (1.9) is Gaussian with respect to the p- and f variables defined in $\mathcal{R}$, satisfying eqs.(3.11a) and (3.11b).

ii) Any canonical distribution of the form $Z^{-1} e^{-\sum_i \gamma_i Q_i}$, if it is Gaussian with respect to the p- and f variables defined in $\mathcal{R}$, with $\langle p \rangle = \langle f \rangle = 0$, must be identical to the form (1.9) on the basis of the theorem presented in Sec.3.2.

It can be observed that the latter part of expression (3.12) or (3.13) represents the mn pair potential of the g-CM or g-CS dynamics, averaged over the $\mathcal{R}$-Gaussian distribution (1.9), which is generally non-uniform although the uniform translational invariance is still retained. This is a special character of the "structuredness" in the present random matrix ensemble which generalizes the classical RMT distribution (3.8). How one can see that it actually generalizes the classical result? First, we observe that the uniform constraint in the maximum principle (3.6), i.e. (3.6b), is recovered if the special choice $\gamma = 0$ is taken in the variance expression (3.12) or (3.13). Therefore, we may expect that the resulting distribution, after coarse-graining, would emerge to be identical with the classical distribution (3.8):

$$P_N(\{x_n\}) \equiv \int\int dp df \rho_{\beta,\gamma}(\{x_n\}, \{p\}\{f\}) = C_N e^{h_G(x)}. \tag{3.14}$$

where

$$h_G(x) = \frac{1}{2} \log(2\pi e \sigma^2(x))$$

with $\sigma^2(x)$ the product of all the variance functions predicted in eq.(3.12) or (3.13). Hence, we obtain, with a properly redefined normalization factor which depends only on the ratio γ/β,

$$P_N(\{x_n, \gamma/\beta) = C_{N\nu}(\gamma/\beta) \prod_{m<n} |x_m - x_n|^\nu \left[1 + (\gamma/\beta)(x_m - x_n)^2\right]^{-\nu/2} \quad \text{for gCM}, \tag{3.15}$$

$$= C_{N\nu}(\gamma/\beta) \prod_{m<n} \left|2\sin\frac{1}{2}(\phi_m - \phi_n)\right|^\nu \left[1 + (\gamma/\beta) 4\sin^2\frac{1}{2}(\phi_m - \phi_n)\right]^{-\nu/2} \quad \text{for gCS} \tag{3.16}$$

$$\nu = 1 \text{ for OE}, \ \nu = 2 \text{ for UE}, \ \nu = 4 \text{ for SE}.$$

The question how these expressions reduce to the standard RMT distribution (3.8) will be discussed separately next.

3.4. Comments on the coarse-grained g-CM/S distribution

Case $\gamma = 0$. Result (3.15) reduces to the level repulsion factor $\rho_0(\{x_n\})$ in the beginning idea of Pechukas[1)] and that of Gaspard et al[6)], as discussed in Sec.1. Result (3.16) reduces to the same factor for the circular ensembles of Dyson[25)]. Under the circumstance, the normalization factor $C_{N\nu}$ must diverge unless the integration is restricted to a finite region of the configuration space ($|x_n| \leq L/2$ and $|\phi_n| \leq \pi$).

Case $\gamma \neq 0$ but $\gamma/\beta = O(1/N) \to 0$ for g-CM system. The consideration implies that the RMT distribution (3.8) is deducible only in an asymptotic $N \to \infty$ limit of the formula (3.15)[2,2a)]: We have, indeed,

$$P_N(\{x_n\}) = C_{N\nu}(\gamma/\beta)(\prod_{m<n} |x_m - x_n|^\nu) \exp[(-\gamma/\beta) \sum_{m<n} (x_m - x_n)^2 + O(\gamma/\beta)^2],$$

where the dominant term in the exponential is the first square term of $O(N\gamma/\beta) \times \sum_{n=1}^N x_n^2$ (the center of masses $\frac{1}{N}\sum_n x_n$ is set equal to 0), which survives in the limit $N \to \infty$ with $N\gamma/\beta$ being fixed.

Case $\gamma \neq 0, \gamma/\beta =$fixed but ϕ_n's scaled as $\phi_n = N^{-1/2}x_n$ for g-CS system. This procedure provides a possibility that the g-CS statistics becomes identical with the g-CM statistics (3.8) in the limit $N \to \infty$ [7)]. It is noted that in the standard RMT joint distribution (3.8) a further scaling $x_n \to \xi_n = N^{-1/2}x_n$ is to be made in order to get a correlation function of lower classes[7)], which coincides with the original scaling for the circular ensembles, $\phi_n = 2\pi N^{-1}x_n$, set up by Dyson[25)].

Case $\gamma/\beta \to \infty$ (with a fixed N) for both g-CM and g-CS systems This provides the uniform statistics, $P_N(\{x_n\})$ =constant, which represents no correlation between different levels (the Poisson statistics) under the assumption that no pair of levels coincides.

We may remark that the coarse-grained distribution function (3.15) or (3.16) can be regarded as the canonical one for a one-dimensional incomplete gas with repulsive pair potential

$$\sum_{m<n} \frac{1}{2}\log\left[1 + \frac{\beta/\gamma}{(x_m - x_n)^2}\right]^2, \quad \text{or} \quad \sum_{m<n} \frac{1}{2}\log\left[1 + \frac{\beta/\gamma}{4\sin^2\frac{1}{2}(\phi_m - \phi_n)}\right]^2.$$

This will enable us to have a new standpoint of further investigations.

Concluding remark In this article, I have restricted to (a basic part of) the equilibrium statistical mechanics of the g-CM and g-CS systems for level statistics. For the purpose of *dynamical formulation of level statistics* it should include aspects of stochastic processes which must be postphoned.

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