On the Spin-Boson Model

Asao Arai (新井朝雄)∗
Department of Mathematics, Hokkaido University, Sapporo 060, Japan
Masao Hirokawa (廣川真男)
Advanced Research Laboratory, Hitachi Ltd., Hatoyama, Saitama 350-03, Japan

The existence and uniqueness of ground states of the spin-boson Hamiltonian $H_{SB}$ are considered. The main results in the case of massive bosons include: (i)(existence) there exists a ground state without restriction for the strength of the coupling constant; (ii)(uniqueness) under a mild (nonperturbative) condition for the parameters contained in $H_{SB}$, $H_{SB}$ has only one ground state; (iii)(degeneracy) under a certain condition for the parameters of $H_{SB}$ which is weaker than that of (ii), the number of the ground states is at most two. In the case of massless bosons, the existence of a ground state of $H_{SB}$ is shown as a limit of ground states of the massive case. The methods used are nonperturbative. A generalization of the model is proposed.

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1. Introduction and the main results

The spin-boson model, which describes a two-level quantum system coupled to a quantized Bose field, has been investigated as a simplified model for atomic systems interacting

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with a quantized radiation or phonon field ([1, 2, 5, 6, 7, 9, 14] and references therein). The ground states of the model are of particular interest. Spohn [14] discussed properties of ground states defined as zero-temperature limits of positive temperature equilibrium states. Analysis related to the work of Spohn was made by Amann [1] in terms of the notion of algebraic ground states, although it treats only a discrete version of the model. Recently attention has been paid to the ground states as the eigenvectors of the Hamiltonian $H_{SB}$ of the model with eigenvalue equal to the infimum of its spectrum to analyze spectral properties of $H_{SB}$ and the process of radiative decay in the model [8, 9]. In [8] Hübner and Spohn showed that, under certain conditions for the dispersion $\omega$ for bosons, the coupling function, the coupling constant $\alpha$ and the spectral gap $\mu$ of the unperturbed two-level system, there exists a unique ground state of $H_{SB}$ and identify the spectrum of $H_{SB}$.

In this paper we focus our attention on the existence and uniqueness of ground states of the spin-boson Hamiltonian $H_{SB}$. We first consider the case where the bosons are massive (i.e., $m := \inf_k \omega(k) > 0$) and show that, as far as the existence of the ground states is concerned, no restriction is needed for the coupling constant $\alpha$, which greatly improves the result on the existence of ground states in [8] (in the massive case). The basic idea to do it is as follows: we first do a unitary transformation for $H_{SB}$ to convert it to an operator more tractable in a sense and then apply the method of constructive quantum field theory [7] to the latter operator. Moreover, by employing the min-max principle, under an additional condition for the parameters $m, \mu$ and $\alpha$, which is nonperturbative, we show that $H_{SB}$ has a unique ground state. We also suggest the possibility for $H_{SB}$ to have degenerate ground states by showing that, under a weaker condition for $m, \mu$ and $\alpha$, there exist at most two ground states of $H_{SB}$. In the case of massless bosons (i.e., $m = 0$), we construct a ground state as a weak limit of ground states in the massive case.

We now describe our main results. For mathematical generality, we consider the situation where bosons move in the $\nu$-dimensional Euclidean space $\mathbb{R}^\nu$ with $\nu \geq 1$. We take the Hilbert space of bosons to be

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^\nu)) = \bigoplus_{n=0}^{\infty} [\otimes_n^\nu L^2(\mathbb{R}^\nu)],$$  \hspace{1cm} (1.1)

the symmetric Fock space over $L^2(\mathbb{R}^\nu)$ ($\otimes_n^\nu K$ denotes the $n$-fold symmetric tensor product of a Hilbert space $K$, $\otimes_0^\nu K := \mathbb{C}$). Let $\omega$ and $\lambda$ be functions on $\mathbb{R}^\nu$ satisfying the following conditions

(A.1) For all $k \in \mathbb{R}^\nu$, $\omega(k) \geq 0$ and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma}, \quad k, k' \in \mathbb{R}^\nu.$$  \hspace{1cm} (1.2)

(A.2) The function $\lambda$ is real-valued and continuous with $\lambda, \lambda/\sqrt{\omega}, \lambda/\omega \in L^2(\mathbb{R}^\nu)$ and there exist constants $q > \nu/2$ and $K_0 > 0$ such that, for all $|k| \geq K_0$,

$$\left| \frac{\lambda(k)}{\omega(k)} \right| \leq \frac{D}{1 + |k|^q}$$
with $D$ a constant (which may depend on $q$ and $K_0$).

Throughout this paper, we assume (A.1) and (A.2).

A typical example of $\omega$ satisfying (A.1) is $\omega(k) = \sqrt{|k|^2 + m_0^2}$ with $m_0 \geq 0$ a constant.

We denote by $d\Gamma(\omega)$ the second quantization of the multiplication operator $\omega$ on $L^2(\mathbb{R}^\nu)$ and set

$$H_b = d\Gamma(\omega) = \int d^\nu k \omega(k)a(k)^*a(k),$$

where $a(k)$ is the operator-valued distribution kernel of the smeared annihilation operator $a(f) = \int a(k)f(k)^*d^\nu k$ ($f \in L^2(\mathbb{R}^\nu)$) on $\mathcal{F}$ ($f^*$ denotes the complex conjugate of $f$). The Hamiltonian of the spin-boson model is defined by

$$H_{\text{SB}} = \frac{1}{2}\mu \sigma_z \otimes I + I \otimes H_b + \alpha \sigma_x \otimes (a(\lambda)^* + a(\lambda))$$

acting in the Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F},$$

where $\sigma_x, \sigma_z$ are the standard Pauli matrices, $\mu > 0$ and $\alpha \in \mathbb{R}$ are constants denoting the spectral gap of the unperturbed two-level system and the coupling constant, respectively, and $I$ denotes identity.

For a linear operator $T$ on a Hilbert space, we denote its domain by $D(T)$. It is well known that $H_{\text{SB}}$ is self-adjoint with $D(H_{\text{SB}}) = D(I \otimes H_b)$ and

$$H_{\text{SB}} \geq -\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2,$$

where $\| \cdot \|_{L^2}$ denotes the norm of $L^2(\mathbb{R}^\nu)$.

For a self-adjoint operator $T$ bounded from below, we denote by $E(T)$ the infimum of the spectrum $\sigma(T)$ of $T$:

$$E(T) = \inf \sigma(T).$$

In this paper, an eigenvector of $T$ with eigenvalue $E(T)$ is called a ground state of $T$ (if it exists). We say that $T$ has a (resp. unique) ground state if $\dim \ker(T - E(T)) \geq 1$ (resp. $\dim \ker(T - E(T)) = 1$).

The following estimate for $E(H_{\text{SB}})$ is well known (see (2.10) below):

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H_{\text{SB}}) \leq -\frac{\mu}{2}e^{-2\alpha^2\|\lambda/\omega\|_{L^2}^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2.$$

Let

$$m := \inf_{k \in \mathbb{R}^\nu} \omega(k)$$

We have the following result on the existence of ground states of $H_{\text{SB}}$:
THEOREM 1.1. Assume (A.1), (A.2) and \( m > 0 \). Then \( H_{SB} \) has purely discrete spectrum in the interval \( [E(H_{SB}), E(H_{SB}) + m] \). In particular, \( H_{SB} \) has a ground state.

Remark: Theorem 1.1 implies that, under the same assumption, \( \inf \sigma_{ess}(H_{SB}) \geq E(H_{SB}) + m \), where \( \sigma_{ess}(\cdot) \) denotes essential spectrum, i.e., \( H_{SB} \) has a spectral gap. In a forthcoming paper, we shall show that, in fact, \( \sigma_{ess}(H_{SB}) = [E(H_{SB}) + m, \infty) \).

To state our result on the uniqueness of ground states, we introduce

\[
K_{\varepsilon}(\alpha, \mu) = \min \left\{ m(1 - \varepsilon), \frac{\mu}{2} \right\} - \frac{4\alpha^{2}\mu^{2}}{\varepsilon} \left\| \frac{\lambda}{\omega^{\sqrt{\omega}}} \right\|_{L^{2}}^{2} - 2|\alpha|\mu \left\| \frac{\lambda}{\omega} \right\|_{L^{2}}^{2},
\]

with \( \lambda \) such that \( \lambda/\omega \sqrt{\omega} \in L^{2}(\mathbb{R}^\nu) \).

Remark: If \( m > 0 \), then \( \lambda \in L^{2}(\mathbb{R}^\nu) \) implies that, for all \( s > 0 \), \( \lambda/\omega^{s} \in L^{2}(\mathbb{R}^\nu) \).

THEOREM 1.2. Assume (A.1), (A.2) and \( m > 0 \). Suppose that

\[
\sup_{0 < \varepsilon < 1} K_{\varepsilon}(\alpha, \mu) > \frac{\mu}{2} \left( 1 - e^{-2\alpha^{2}}||\lambda/\omega||_{L^{2}}^{2} \right)
\]  (1.11)

Then \( H_{SB} \) has a unique ground state.

Remark: By applying regular perturbation theory (e.g., [12, Chapt.XII]), one can easily show that there exists a constant \( \alpha_{0} > 0 \) such that, for all \( \alpha \in (-\alpha_{0}, \alpha_{0}) \), \( H_{SB} \) has a unique ground state. For arbitrarily fixed \( m > 0 \) and \( \mu > 0 \), (1.11) is satisfied if \( |\alpha| \) is sufficiently small. Thus Theorem 1.2 may be regarded as a result which improves the one obtained by regular perturbation theory. Note that (1.11) is a nonperturbative estimate in \( \alpha \), since the right hand side (RHS) of (1.11) is non-polynomial in \( \alpha \). We believe that (1.11) is a relatively good estimate to ensure the uniqueness of ground states of \( H_{SB} \) (see the proof of Theorem 1.2 in §5.2).

As is easily seen, in the case \( \mu = 0 \), \( H_{SB} \) has two-fold degenerate ground states. This fact suggests that \( H_{SB} \) with \( \mu > 0 \) also may have degenerate ground states. In this respect, we have the following result:

THEOREM 1.3. Assume (A.1), (A.2) and \( m > 0 \). Suppose that

\[
m > \frac{\mu}{2} \left( 1 - e^{-2\alpha^{2}}||\lambda/\omega||_{L^{2}}^{2} \right).
\]  (1.12)

Then the following (a) and (b) hold:

(a) There are at most two eigenvalues (counting multiplicity) of \( H_{SB} \) in the interval \( [E(H_{SB}), -\frac{\mu}{2} e^{-2\alpha^{2}}||\lambda/\omega||_{L^{2}}^{2} - \alpha^{2}||\lambda/\sqrt{\omega}||_{L^{2}}^{2}] \).

(b) The Hamiltonian \( H_{SB} \) has at most two ground states, i.e., \( \dim \ker(H_{SB} - E(H_{SB})) \leq 2 \).
Remark: Condition (1.11) implies (1.12), i.e., the latter condition is weaker than the former.

In the case of massless bosons, we have the following result on the existence of ground states of $H_{SB}$:

**Theorem 1.4.** Assume (A.1), (A.2) and $m = 0$. Suppose, in addition, that $\omega \lambda \in L^2(\mathbb{R}^\nu)$ and

$$|\alpha| < \frac{1}{||\lambda/\omega||_{L^2}}.$$  \hfill (1.13)

Then $H_{SB}$ has a ground state.

Remark: To our best knowledge, Theorem 1.4 is the first which establishes the existence of ground states of the spin-boson Hamiltonian $H_{SB}$ in the case of massless bosons.

The present paper is organized as follows. In Section 2 we review some basic facts on the spin-boson Hamiltonian $H_{SB}$. We recall a well known unitary transformation which converts $H_{SB}$ to an operator $H$ simpler in a sense. We analyze the operator $H$. To prove the existence of ground states of $H$, we introduce in Section 3 a finite volume approximation $H_V$ ($V > 0$) for $H$. In Section 4 we prove that $H_V$ converges to $H$ in the norm resolvent sense as $V \to \infty$. In Section 5 we prove Theorems 1.1–1.4. In the last section we propose a generalization of the model.

2. Some basic facts

It is well known that, for all $f \in L^2(\mathbb{R}^\nu)$, the operator

$$P(f) := i\{a(f)^* - a(f)\}$$  \hfill (2.1)

is essentially self-adjoint on the finite particle subspace

$$\mathcal{F}_0 = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} | \text{only finitely many } \Psi_n \text{'s are not zero} \}.$$  \hfill (2.2)

We denote the closure of $P(f)$ by the same symbol. Let

$$U_\pm = e^{\pm i\alpha P(\lambda/\omega)}.$$  \hfill (2.3)

Then

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} U_+ & U_- \\ U_+ & -U_- \end{pmatrix}$$  \hfill (2.4)

is unitary on $\mathcal{H}$. Moreover, we have

$$U^{-1}H_{SB}U = H - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2.$$  \hfill (2.5)
with
\[ H = I \otimes H_b + \frac{\mu}{2} (A \otimes U^2 + A^* \otimes U^2), \]  
where
\[ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

Based on (2.5), we shall consider, instead of $H_{SB}$, the operator $H$ defined by (2.6). An advantage of this approach is in that the perturbation term
\[ H_I := \frac{\mu}{2} (A \otimes U^2 + A^* \otimes U^2) \]  
(2.8)
of $H$ is a bounded self-adjoint operator. The operator norm $||H_I||$ of $H_I$ can be exactly computed:

**Lemma 2.1.** We have
\[ ||H_I|| = \frac{\mu}{2}. \]

**Proof:** We need only to use the relation $H_I = \frac{\mu}{2} U^{-1} (\sigma_z \otimes I) U$ and the fact $||\sigma_z \otimes I|| = 1$. \[ \square \]

It follows from (2.9) and the variational principle (cf. [2, 4]) that
\[ -\frac{\mu}{2} \leq E(H) \leq -\frac{\mu}{2} e^{-2\sigma^2 ||\lambda/\omega||^2} < 0. \]  
(2.10)

**Lemma 2.2.** Assume, in addition to (A.1) and (A.2), that $\omega \lambda \in L^2(\mathbb{R}^\nu)$. Let $\Psi$ be any eigenvector of $H_{SB}$. Then $\Psi \in D((I \otimes H_b)^{3/2})$.

**Proof:** By the assumption, we have $H_{SB} \Psi = E \Psi, \Psi \in D(H_{SB}) = D(I \otimes H_b)$ with $E$ an eigenvalue of $H_{SB}$. Hence
\[ (I \otimes H_b) \Psi = E \Psi - \frac{\mu}{2} (\sigma_z \otimes I) \Psi - \alpha \sigma_x \otimes [a(\lambda)^* + a(\lambda)] \Psi. \]
The vectors on the RHS except for the last one is in $D(I \otimes H_b)$. We denote by $a(\cdot)^#$ either $a(\cdot)^*$ or $a(\cdot)$. It is known that, if $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$, then $a^#(f)$ maps $D(H_b)$ into $D(H_b^{1/2})$ [3, Lemma 2.4]. Hence $\sigma_x \otimes [a(\lambda)^* + a(\lambda)] \Psi \in D((I \otimes H_b)^{1/2})$. Thus we conclude that $(I \otimes H_b) \Psi \in D((I \otimes H_b)^{1/2})$, which implies the desired result. \[ \square \]

Let
\[ N = d\Gamma(I) = \int d^\nu k a(k)^* a(k), \]  
(2.11)
the number operator on $\mathcal{F}$.

In general we denote by $(\cdot, \cdot)_{\mathcal{K}}$ and $||\cdot||_{\mathcal{K}}$ the inner product (complex linear in the second variable) and the norm of a Hilbert space $\mathcal{K}$, respectively, but, we sometimes omit the subscript $\mathcal{K}$ if there is no danger of confusion.
LEMMA 2.3. Assume, in addition to (A.1) and (A.2), that $\omega \lambda \in L^2(\mathbb{R}^\nu)$. Then, for every normalized ground state $\Omega$ of $H_{SB}$,

\[
(\Omega, I \otimes N \Omega)_H \leq \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2.
\]  

(2.12)

PROOF: Let $f$ be a function such that $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ (then $f \in L^2(\mathbb{R}^\nu)$). It follows from Lemma 2.2 and a mapping property of $a(f)^\#$ [3, Lemma 2.3] that $a(f) \Omega \in D(I \otimes H_{b}) = D(H_{SB})$. Since $H_{SB} - E(H_{SB}) \geq 0$, we have

\[
0 \leq (I \otimes a(f) \Omega, [H_{SB} - E(H_{SB})] I \otimes a(f) \Omega) = (I \otimes a(f) \Omega, [H_{SB}, I \otimes a(f)] \Omega) = (I \otimes a(f) \Omega, (-I \otimes a(\omega f) - \alpha(\sigma_x \otimes I)(f, \lambda)_{L^2}) \Omega).
\]

Hence

\[
(\Omega, I \otimes a(f)^* a(\omega f) \Omega) + \alpha(\sigma_x \otimes a(\frac{\lambda}{\omega}) \Omega, \Omega) \leq 0.
\]  

(2.13)

There exists a sequence $\{f_n\}_{n=1}^\infty$ of functions such that $\omega f_n, f_n/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ for all $n \geq 1$ and $\{\sqrt{\omega} f_n\}_{n=1}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R}^\nu)$. By (2.13), we have for all $N = 1, 2, 3, \cdots$

\[
\sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n) \Omega) + \alpha(\sigma_x \otimes a(F_N) \Omega, \Omega) \leq 0,
\]

where $F_N = \sum_{n=1}^N (f_n, \lambda)_{L^2} f_n$. It is not so difficult to show that

\[
\lim_{N \to \infty} \sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n) \Omega) = (\Omega, I \otimes N \Omega),
\]

\[
\lim_{N \to \infty} (\sigma_x \otimes a(F_N) \Omega, \Omega) = (\sigma_x \otimes a(\frac{\lambda}{\omega}) \Omega, \Omega).
\]

Hence $(\Omega, I \otimes N \Omega) + \alpha (\sigma_x \otimes a(\frac{\lambda}{\omega}) \Omega, \Omega) \leq 0$. Since $(\Omega, I \otimes N \Omega) \geq 0$, it follows that

\[
(\Omega, I \otimes N \Omega) \leq -\alpha \left( \sigma_x \otimes a(\frac{\lambda}{\omega}) \Omega, \Omega \right).
\]  

(2.14)

Applying the well known estimate

\[
||a(f) \Psi||_\mathcal{F} \leq ||f||_{L^2} ||N^{1/2} \Psi||_\mathcal{F}, \quad f \in L^2(\mathbb{R}^\nu), \Psi \in D(N^{1/2}),
\]

(2.15)

to the RHS of (2.14), we obtain

\[
(\Omega, I \otimes N \Omega) \leq |\alpha| \left\| \frac{\lambda}{\omega} \right\|_{L^2} ||(I \otimes N)^{1/2} \Omega||,
\]
which implies (2.12). \[ \square \]

Inequality (2.12) gives an upper bound for the mean of boson numbers in any normalized ground state of $H_{SB}$. Note that inequality (2.12) is independent of whether bosons are massive or massless.

3. A finite volume approximation

Let $V > 0$ be a parameter and

$$\Gamma_V = \frac{2\pi \mathbb{Z}^\nu}{V} \left\{ k = (k_1, \ldots, k_\nu) \mid k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \ldots, \nu \right\}. \tag{3.1}$$

Let

$$\mathcal{F}_V = \mathcal{F}(\ell^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} [\otimes_{s}^{n}\ell^2(\Gamma_V)] \tag{3.2}$$

the symmetric Fock space over $\ell^2(\Gamma_V)$, which describes state vectors of bosons in the finite box $[-V/2, V/2]^\nu$. Each element $\Psi$ in $\otimes_{s}^{n}\ell^2(\Gamma_V)$ can be identified with a piecewise constant function in $\otimes_{s}^{n}L^2(\mathbb{R}^\nu)$ which is a constant on each cube of volume $(2\pi/V)^{n\nu}$ centered about a lattice point

$$(k_1, \ldots, k_\nu) \in \Gamma_V \times \cdots \times \Gamma_V = \Gamma^n_V.$$ 

With this identification, $\mathcal{F}_V$ is regarded as a closed subspace of $\mathcal{F}$.

For each $k = (k_1, \cdots, k_\nu) \in \Gamma_V$, we define a function $\chi_{k,V}$ on $\mathbb{R}^\nu$ by

$$\chi_{k,V}(\ell) = \chi_{[k_1-\frac{\pi}{V}, k_1+\frac{\pi}{V}]\ell_1} \cdots \chi_{[k_\nu-\frac{\pi}{V}, k_\nu+\frac{\pi}{V}]\ell_\nu}, \quad \ell = (\ell_1, \cdots, \ell_\nu) \in \mathbb{R}^\nu, \tag{3.3}$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. We introduce

$$a_V(k) := \left(\frac{V}{2\pi}\right)^{\nu/2} a(\chi_{k,V}) = \left(\frac{V}{2\pi}\right)^{\nu/2} \int_{[-\pi/V, \pi/V]^\nu} a(k + \ell)d\ell. \tag{3.4}$$

It is easy to see that, for all $k, \ell \in \Gamma_V$,

$$[a_V(k), a_V(\ell)] = \delta_{k\ell}, \quad [a_V(k), a_V(\ell)] = 0, \tag{3.5}$$

on $\mathcal{F}_0$.

We define

$$\omega_V(k) = \omega(k_V), \quad k \in \mathbb{R}^\nu, \tag{3.6}$$

with $k_V$ a lattice point closed to $k$:

$$k_V \in \Gamma_V, \quad |k_j - (k_V)_j| \leq \frac{\pi}{V}, \quad j = 1, \cdots, \nu. \tag{3.7}$$

Let

$$H_{b,V} := d\Gamma(\omega_V) = \int d^\nu k \omega_V(k)a(k)^*a(k). \tag{3.8}$$
**Lemma 3.1.** We have

\[ D(H_{b,V}) = D(H_b) \]  \hspace{1cm} (3.9)

and there exists a constant \( c > 0 \) independent of \( V \) such that, for all \( \Psi \in D(N) \),

\[ \| (H_b - H_{b,V}) \Psi \| \leq \frac{c}{V^\gamma} \| N \Psi \|. \]  \hspace{1cm} (3.10)

**Proof:** By (1.2) and (3.7), we have for all \( k \in \mathbb{R}^\nu, |\omega(k) - \omega(kv)| \leq c/V^\gamma \) with \( c = C \pi^\nu \nu^{\gamma/2} \), from which (3.9) and (3.10) follow.

The following fact is well known:

**Lemma 3.2.** The operator \( H_{b,V} \) is reduced by \( \mathcal{F}_V \) and

\[ H_{b,V} \upharpoonright \mathcal{F}_V = \sum_{k \in \Gamma_V} \omega(k) a_V(k)^* a_V(k). \]

For notational simplicity, we set

\[ g(k) = \frac{\alpha \lambda(k)}{\omega(k)}. \]  \hspace{1cm} (3.11)

For \( K > 0 \), we define a function \( g_{K,V} \) on \( \mathbb{R}^\nu \) by

\[ g_{K,V} = \sum_{k \in \Gamma_V, |k_j| \leq K, j=1,\ldots,\nu} g(k) \chi_{k,V}. \]

**Lemma 3.3.** The function \( g_{K,V} \) converges in \( L^2(\mathbb{R}^\nu) \) as \( K \to \infty \).

**Proof:** For a constant \( K > 0 \), we put

\[ S_{K,V} = \sum_{k \in \Gamma_V, |k_j| \leq K, j=1,\ldots,\nu} \left( \frac{2\pi}{V} \right)^\nu |g(k)|^2 \]

Then, by the growth condition for \( \lambda/\omega \) in (A.2), we have

\[ S_{K,V} \leq \sum_{k \in \Gamma_V, |k| \leq K_0} \left( \frac{2\pi}{V} \right)^\nu |g(k)|^2 + \alpha^2 D^2 \sum_{k \in \Gamma_V, |k| \geq K_0} \left( \frac{2\pi}{V} \right)^\nu \frac{1}{(1 + |k|^\nu)^2} \]

\[ \leq \sum_{k \in \Gamma_V, |k| \leq K_0} \left( \frac{2\pi}{V} \right)^\nu |g(k)|^2 + \alpha^2 D^2 \int_{\mathbb{R}^\nu} \frac{1}{(1 + |k|^\nu)^2} dk < \infty. \]
Hence $S_{K,V}$ is uniformly bounded in $K$. Since $S_{K,V}$ is monotone non-decreasing in $K$, it follows that the infinite series $S_V := \sum_{k \in \Gamma_V} (\frac{2\pi}{V})^{\nu}|g(k)|^2$ converges. Let $K' \geq K$. Then we have $(g_{K,V}, g_{K',V})_{L^2} = S_{K,V} \to S_V (K \to \infty)$, which implies that $\{g_{K,V}\}_K$ is a Cauchy net.

We write

$$g_V = L^2 - \lim_{K \to \infty} g_{K,V} = \sum_{k \in \Gamma_V} g(k)\chi_{k,V}. \quad (3.12)$$

Then we have

$$P(g_V) = i\left(\frac{2\pi}{V}\right)^{\nu/2} \sum_{k \in \Gamma_V} g(k)(a_V(k)^* - a_V(k)) \quad (3.13)$$
on $\mathcal{F}_0$.

Let

$$U_\pm(V) = e^{\pm iP(g_V)}. \quad (3.14)$$

and

$$H_V = I \otimes H_{b,V} + \frac{\mu}{2}(A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2). \quad (3.15)$$

**Lemma 3.4.** The operator $H_V$ is self-adjoint with $D(H_V) = D(I \otimes H_b)$ and bounded from below with $H_V \geq -\frac{\mu}{2}. \quad (3.16)$

**Proof:** Since the operator

$$H_I(V) := \frac{\mu}{2}(A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2) \quad (3.17)$$

is bounded, the Kato-Rellich theorem gives the self-adjointness of $H_V$ with $D(H_V) = D(I \otimes H_{b,V}) = D(I \otimes H_b)$ (Lemma 3.1). Inequality (3.16) follows from the fact $||H_I(V)|| = \frac{\mu}{2}$, which can be proven in the same way as in Lemma 2.1.

In the next section, we show that $H_V$ is a finite volume approximation for $H$ in a suitable sense.

**4. Convergence of the finite volume approximation**

In this section we prove the following theorem:

**Theorem 4.1.** For all $z \in \mathbb{C}$ with $\text{Im} z \neq 0$ or $z < -\mu/2$,

$$\lim_{V \to \infty} ||(H_V - z)^{-1} - (H - z)^{-1}|| = 0. \quad (4.1)$$

To prove this theorem, we prepare some lemmas.
LEMMA 4.2.  

\[ \lim_{V \to \infty} ||g_V - g||_{L^2} = 0. \]  

(4.2)

PROOF: By the growth condition for \( \lambda/\omega \) in (A.2), one can easily show that 

\[ ||g_V||_{L^2}^2 = \sum_{k \in \Gamma_V} \left( \frac{2\pi}{V} \right)^\nu |g(k)|^2 \rightarrow \int_{\mathbb{R}^\nu} d^\nu k |g(k)|^2 = ||g||_{L^2}^2 \quad (V \to \infty). \]  

(4.3)

Let \( f \in C_0^\infty(\mathbb{R}^\nu) \) and \( \text{supp} f \subset \{ k \in \mathbb{R}^\nu \mid |k_j| \leq K_f, j = 1, \cdots, \nu \} \) with a constant \( K_f \). Then we have 

\[ (f, g_V)_{L^2} = \sum_{\ell \in \Gamma_V} \left( \frac{2\pi}{V} \right)^\nu f(\ell)^* g(\ell) + I_V, \]

where 

\[ I_V = \sum_{\ell \in \Gamma_V, |\ell_j| \leq K_f, j = 1, \cdots, \nu} g(\ell) \int_{[\ell_1 - \varphi, \ell_1 + \varphi] \times \cdots \times [\ell_\nu - \varphi, \ell_\nu + \varphi]} [f(k)^* - f(\ell)^*]d^\nu k. \]

Since \( f \) is uniformly continuous, for any \( \epsilon > 0 \), there exists a constant \( V_0 > 0 \) such that, if \( |k_j - \ell_j| \leq \pi/V_0 \), then \( |f(k) - f(\ell)| \leq \epsilon \). Hence, for all \( V \geq V_0 \), we have \( |I_V| \leq D_V \epsilon \), where 

\[ D_V = \sum_{\ell \in \Gamma_V, |\ell_j| \leq K_f, j = 1, \cdots, \nu} \left( \frac{2\pi}{V} \right)^\nu g(\ell). \]

Note that 

\[ \lim_{V \to \infty} D_V = D := \int_{[-K_f, K_f]^\nu} |g(k)|d^\nu k \leq \left( \int_{[-K_f, K_f]^\nu} |g(k)|^2 d^\nu k \right)^{1/2} (2K_f)^\nu/2 < \infty. \]

Hence \( \lim_{V \to \infty} |I_V| \leq D \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we conclude that \( \lim_{V \to \infty} I_V = 0 \). Thus we obtain 

\[ (f, g_V)_{L^2} \rightarrow (f, g)_{L^2} \quad (V \to \infty). \]  

(4.4)

By (4.3), (4.4) and a limiting argument using the denseness of \( C_0^\infty(\mathbb{R}^\nu) \) in \( L^2(\mathbb{R}^\nu) \), we obtain (4.2). \( \blacksquare \)

We say that two self-adjoint operators \( T_1 \) and \( T_2 \) on a Hilbert space strongly commute if their spectral measures commute.

LEMMA 4.3. Let \( T_1 \) and \( T_2 \) be strongly commuting self-adjoint operators on a Hilbert space. Then, for all \( \psi \in D(T_1) \cap D(T_2) \), 

\[ ||(e^{iT_1} - e^{iT_2})\psi|| \leq ||(T_1 - T_2)\psi||. \]
PROOF: Let $E_j$ be the spectral measure of $T_j$. Then there exists a unique two-dimensional spectral measure $E$ such that, for all Borel sets $B_1, B_2$ in $\mathbb{R}$, $E(B_1 \times B_2) = E_1(B_1)E_2(B_2)$. In terms of $E$, we have

$$T_j = \int \lambda_j dE(\lambda_1, \lambda_2), \quad e^{iT_j} = \int e^{i\lambda_j} dE(\lambda_1, \lambda_2), \quad j = 1, 2.$$ 

By the functional calculus and the inequality $|e^{ix} - e^{iy}| \leq |x - y|$, $x, y \in \mathbb{R}$, we have for all $\psi \in D(T_1) \cap D(T_2)$

$$||(e^{iT_1} - e^{iT_2})\psi||^2 = \int_{\mathbb{R}^2} |e^{i\lambda_1} - e^{i\lambda_2}|^2 d||E(\lambda_1, \lambda_2)\psi||^2 \leq \int_{\mathbb{R}^2} |\lambda_1 - \lambda_2|^2 d||E(\lambda_1, \lambda_2)\psi||^2 = ||(T_1 - T_2)\psi||^2.$$

Thus the desired result follows.  

Lemma 4.4.

$$||(U_\pm(V)^2 - U_\pm^2)(N + I)^{-1/2}|| \leq 4||g_V - g||. \quad (4.5)$$

**Proof:** For all real-valued functions $f_1, f_2 \in L^2(\mathbb{R}^\nu)$ and all $s, t \in \mathbb{R}$, $e^{isP(f_1)}$ commutes with $e^{itP(f_2)}$ (e.g., [11, Theorem X.43]). Hence, by a general theorem (e.g., [10, Theorem VIII.13]), $P(f_1)$ and $P(f_2)$ strongly commute. Applying this fact, we conclude that $P(g)$ and $P(g_V)$ strongly commute. Hence, by Lemma 4.3, we have for all $\Psi \in \mathcal{F}_0$,

$$||(U_\pm(V)^2 - U_\pm^2)\Psi|| \leq 2||(P(g_V) - P(g))\Psi|| \leq 2(||a(g_V - g)\Psi|| + ||a(g_V - g)^*\Psi||).$$

By (2.15) and the complementary estimate to it

$$||a(f)^*\Phi|| \leq ||f||_{L^2}||(N + I)^{1/2}\Phi||, \quad \Phi \in D(N^{1/2}), f \in L^2(\mathbb{R}^\nu),$$

we obtain

$$||(U_\pm(V)^2 - U_\pm^2)\Psi|| \leq 4||g_V - g|| \cdot ||(N + I)^{1/2}\Psi||.$$ 

Since $\mathcal{F}_0$ is a core of $N^{1/2}$, we can extend this inequality, via a simple limiting argument, to all $\Psi \in D(N^{1/2})$. Thus (4.5) follows.  

**Proof of Theorem 4.1**

We prove (4.1) in the case $\text{Im} \ z \neq 0$ (the other case can be similarly treated). Writing

$$I \otimes H_b = H - H_I$$
and using Lemma 2.1, we have

\[ ||I \otimes H_b \Psi|| \leq ||H \Psi|| + \frac{\mu}{2} ||\Psi||, \quad \Psi \in D(I \otimes H_b). \]

Let \( L = I \otimes N + I \). By the fact that \( ||N \Phi|| \leq ||H_b \Phi||/m, \Phi \in D(H_b) \), we obtain

\[ ||(L - I) \Psi|| \leq \frac{1}{m} \left( ||H \Psi|| + \frac{\mu}{2} ||\Psi|| \right), \quad \Psi \in D(I \otimes H_b), \]

which implies that, for all \( z \in \mathbb{C} \setminus \mathbb{R} \), \( L(H - z)^{-1} \) is bounded. By Lemma 3.1, \( (I \otimes H_b - I \otimes H_{b,V}) L^{-1} \) is bounded with

\[ ||(I \otimes H_b - I \otimes H_{b,V}) L^{-1}|| \leq \frac{c}{V^\gamma}. \tag{4.6} \]

We write

\[ (H_V - z)^{-1} - (H - z)^{-1} = (H_V - z)^{-1} (I \otimes H_b - I \otimes H_{b,V}) L^{-1} L(H - z)^{-1} + (H_V - z)^{-1} (H_I - H_I(V)) L^{-1/2} L^{1/2} (H - z)^{-1}. \]

Hence

\[ ||(H_V - z)^{-1} - (H - z)^{-1}|| \leq \frac{1}{|\text{Im } z|} \left( ||(H_b - H_{b,V}) L^{-1}|| \cdot ||L(H - z)^{-1}|| \right. \]

\[ \left. + ||(H_I - H_I(V)) L^{-1/2}|| \cdot ||L^{1/2} (H - z)^{-1}|| \right). \]

We have

\[ H_I - H_I(V) = \frac{\mu}{2} \{ A \otimes (U_+^2 - U_+(V)^2) + A^* \otimes (U_-^2 - U_-(V)^2) \}. \]

Hence, by Lemma 4.4, \( ||(H_I - H_I(V)) L^{-1/2}|| \leq 4 \mu \cdot ||g_V - g|| \), which, combined with Lemma 4.2, implies that \( \lim_{V \to \infty} ||(H_I - H_I(V)) L^{-1/2}|| = 0 \). By (4.6), we have \( \lim_{V \to \infty} ||(H_b - H_{b,V}) L^{-1}|| = 0 \). Thus we obtain (4.1).

5. Proof of the main results

5.1. Proof of Theorem 1.1

Let

\[ \mathcal{H}_V = \mathbb{C}^2 \otimes F_V. \]
**Lemma 5.1.** The operator \( H_V \mid \mathcal{H}_V \) has purely discrete spectrum.

**Proof:** It is well known or easy to see that \( I \otimes H_{b,V} \mid \mathcal{H}_V \) has compact resolvent. Since \( H_I(V) \) is bounded, it follows that \( H_I(V)(I \otimes H_{b,V} + i)^{-1} \mid \mathcal{H}_V \) is compact. Hence, by a general theorem [12, §XIII.4, Corollary 2], \( \sigma_{\text{ess}}(H_V \mid \mathcal{H}_V) = \sigma_{\text{ess}}(I \otimes H_{b,V} \mid \mathcal{H}_V) = \emptyset \). Thus the desired result follows.

**Lemma 5.2.**

\[
H_V \mid \mathcal{H}_V \geq E(H_V) + m.
\]

**Proof:** We decompose \( L^2(\mathbb{R}^\nu) \) as \( L^2(\mathbb{R}^\nu) = F_{1V} \oplus F_{1V}^\perp \) with \( p_{1V} = L^2(\mathbb{R}^\nu) \cap \mathcal{F}_V \).

Then \( \mathcal{F} = \mathcal{F}_V \otimes \mathcal{F}(F\perp)1V = \bigoplus \mathcal{F}^{(j)}, \) where \( \mathcal{F}^{(j)} = \mathcal{F}_V \otimes \otimes_{S}^{j}F_{1V} \).

Hence \( \mathcal{F}_V^\perp = \bigoplus_{j=1}^{\infty} \mathcal{F}^{(j)} \) and \( \mathcal{H}_V^\perp = \mathbb{C}^2 \otimes \mathcal{F}_V^\perp = \bigoplus_{j=1}^{\infty} \mathbb{C}^2 \otimes \mathcal{F}^{(j)} \).

On each \( \mathbb{C}^2 \otimes \mathcal{F}^{(j)} \), \( H_V \) has the form \( S \otimes I + I \otimes T \) with \( S = H_V \mid \mathcal{H}_V \) and \( T \) is a sum of \( j \) copies of \( H_{b,V} \), each acting on a single factor \( F_{1V} \).

Since \( T \geq jm \) on \( \otimes_{S}^{j}F_{1V} \), the assertion of the lemma follows.

**Lemma 5.3** [13, Lemma 4.6]. Let \( T_n \) and \( T \) be self-adjoint operators on a Hilbert space, which are bounded from below. Suppose that \( T_n \rightharpoonup T \) in norm resolvent sense as \( n \to \infty \) and \( T_n \) has purely discrete spectrum in \( [E(T_n), E(T_n) + c) \) with some constant \( c > 0 \). Then, \( \lim_{n \to \infty} E(T_n) = E(T) \) and \( T \) has purely discrete spectrum in \( [E(T), E(T) + c) \).

We are now ready to prove Theorem 1.1: By Lemmas 5.1 and 5.2, \( H_V \) has purely discrete spectrum in \( [E(H_V), E(H_V) + m) \). By this fact and Theorem 4.1, we can apply Lemma 5.3 to conclude that \( H \) has purely discrete spectrum in \( [E(H), E(H) + m) \), which, combined with (2.5), implies Theorem 1.1.

5.2. Proof of Theorem 1.2

The basic idea of proof is to use the min-max principle for \( H \) [12, Theorem XIII.1].

Let

\[
\mu_2(H) = \sup_{\Phi \in \mathcal{H}} U_H(\Phi)
\]

with \( U_H(\Phi) = \inf_{\Psi \in D(H), \|\Psi\|=1, \Psi \in [\Phi]^\perp} (\Psi, H\Psi) \), where \( [\Phi]^\perp = \{\Psi \in \mathcal{H} | (\Psi, \Phi) = 0\} \). We estimate \( \mu_2(H) \) from below. For this purpose, we write

\[
H = I \otimes H_b + \mu \frac{\sigma_x}{2} \otimes I + W,
\]
where

$$W = \frac{\mu}{2} \left\{ A \otimes (U^2_+ - I) + A^* \otimes (U^2_- - I) \right\}.$$  

For $\epsilon > 0$, we set

$$D_\epsilon(\alpha, \mu) = \frac{4\alpha^2 \mu^2}{\epsilon} \left\| \frac{\lambda}{\omega \sqrt{\omega}} \right\|_{L^2}^2 + 2|\alpha|\mu \left\| \frac{\lambda}{\omega} \right\|_{L^2}.$$ 

**Lemma 5.4.** For all $\epsilon > 0$ and $\Psi \in D(I \otimes H_b)$,

$$\|(\Psi, W \Psi)\| \leq \epsilon(\Psi, I \otimes H_b \Psi) + D_\epsilon(\alpha, \mu)\|\Psi\|^2. \quad (5.1)$$

**Proof:** By the fact $\|A\| = \|A^*\| = 1$ and Lemma 4.3, we have for all $\Psi \in D(I \otimes H_b)$

$$\|W \Psi\| \leq \frac{\mu}{2} \left( \|I \otimes (U^2_+ - I) \Psi\| + \|I \otimes (U^2_- - I) \Psi\| \right)$$

$$\leq 2|\alpha|\mu\|I \otimes P(\lambda/\omega) \Psi\|,$$

$$\leq 2|\alpha|\mu(\|I \otimes a(\lambda/\omega) \Psi\| + \|I \otimes a(\lambda/\omega)^* \Psi\|).$$

On the other hand, the following estimates are well known:

$$\|a(f)\psi\| \leq \|f/\sqrt{\omega}\|_{L^2} \|H_b^{1/2}\psi\|,$$

$$\|a(f)^*\psi\| \leq \|f/\sqrt{\omega}\|_{L^2} \|H_b^{1/2}\psi\| + \|f\|_{L^2} \|\psi\|, \quad f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu), \psi \in D(H_b^{1/2}).$$

Hence

$$\|W \Psi\| \leq 4|\alpha|\mu \left\| \frac{\lambda}{\omega \sqrt{\omega}} \right\|_{L^2} \|(I \otimes H_b)^{1/2}\Psi\| + 2|\alpha|\mu\|\Psi\| \left\| \frac{\lambda}{\omega} \right\|_{L^2}.$$ 

Using this estimate and the elementary inequality $xy \leq \epsilon x^2 + \frac{y^2}{4\epsilon}$ holding for all $x, y, \epsilon > 0$, we obtain (5.1).}

We now proceed to proof of Theorem 1.2. Let $\Omega_0$ be the Fock vacuum in $\mathcal{F} : \Omega_0 = \{1, 0, 0, \cdots\}$ and

$$\Phi_0 = \begin{pmatrix} \Omega_0 \\ -\Omega_0 \end{pmatrix}.$$

Then it is easy to see that

$$[\Phi_0]^\perp = \left\{ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathcal{H} \mid \Psi_1^{(0)} = \Psi_2^{(0)} \right\},$$

where we write $\Psi_j = \{\Psi_j^{(n)}\}_{n=0}^\infty \in \mathcal{F}, \Psi_j^{(n)} \in \otimes_{s}^n L^2(\mathbb{R}^\nu)$. Let $\Psi \in [\Phi_0]^\perp$. Then, by the fact $H_b \Omega_0 = 0$ and $H_b \otimes_{s}^n L^2(\mathbb{R}^\nu) \geq nm$, we have

$$\langle \Psi, I \otimes H_b \Psi \rangle \geq \sum_{j=1}^{2} \sum_{n=1}^{\infty} \langle \Psi_j^{(n)}, H_b \Psi_j^{(n)} \rangle \geq m \sum_{j=1}^{2} \sum_{n=1}^{\infty} \|\Psi_j^{(n)}\|^2.$$
Noting the fact $\Psi_1^{(0)} = \Psi_2^{(0)}$, we have

$$\frac{\mu}{2} (\Psi, \sigma_x \otimes I \Psi) = \frac{\mu}{2} \{(\Psi_1, \Psi_2) + (\Psi_2, \Psi_1)\}$$

$$\geq \frac{\mu}{2} \{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2} \|\Psi\|^2.$$ 

These estimates and Lemma 5.4 give

$$(\Psi, H \Psi) \geq m(1-\epsilon) \sum_{j=1}^{2} \sum_{n=1}^{\infty} \|\Psi_j^{(n)}\|^2 + \frac{\mu}{2} \{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2} \|\Psi\|^2 - D_\epsilon(\alpha, \mu) \|\Psi\|^2$$

$$\geq \left\{ M_\epsilon - \frac{\mu}{2} - D_\epsilon(\alpha, \mu) \right\} \|\Psi\|^2,$$

where $\epsilon$ is an arbitrary constant satisfying $0 < \epsilon < 1$ and $M_\epsilon = \min \{ m(1-\epsilon), \frac{\mu}{2} \}$. Since this inequality holds for all $\Psi \in [\Phi_0]'$, we obtain $\mu_2(H) \geq C_0$ with

$$C_0 = \sup_{0<\epsilon<1} \left\{ M_\epsilon - \frac{\mu}{2} - D_\epsilon(\alpha, \mu) \right\}.$$ 

This estimate and the min-max principle imply that $E(H)$ is a simple eigenvalue of $H$ if $E(H) < C_0$. By (2.10), if $C_0 > -\mu e^{-2\alpha^2\|\lambda/\omega\|^2}/2$ (this condition is equivalent to condition (1.11)), then $E(H) < C_0$ and hence $H$ has a unique ground state. Thus the desired result follows.

5.3. Proof of Theorem 1.3

Let

$$\mu_3(H) = \sup_{\Phi_1, \Phi_2 \in \mathcal{H}} U_H(\Phi_1, \Phi_2)$$

with $U_H(\Phi_1, \Phi_2) = \inf_{\Psi \in D(H): \|\Psi\|=1, \Psi \in [\Phi_1, \Phi_2]^\perp} (\Psi, H \Psi)$, where $[\Phi_1, \Phi_2]^\perp$ denotes the orthogonal complement of $\{\alpha \Phi_1 + \beta \Phi_2 | \alpha, \beta \in \mathbb{C}\}$. Let

$$\Phi_1 = \begin{pmatrix} \Omega_0 \\ \Omega_0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \Omega_0 \\ -\Omega_0 \end{pmatrix}.$$

Then we have

$$[\Phi_1, \Phi_2]^\perp = \mathbb{C}^2 \otimes \mathcal{G} = \mathcal{G} \bigoplus \mathcal{G}.$$

with $G = \bigoplus_{n=1}^{\infty} \otimes^{n} E^{2}(\mathbb{R}^{n})$. For all $\Psi = (\Psi_{+}, \Psi_{-}) \in [\Phi_{1}, \Phi_{2}]_{1}^{\perp} (\Psi_{\pm} \in G)$, we have

$$(\Psi, H\Psi) \geq (\Psi_{+}, H_{b}\Psi_{+}) + (\Psi_{-}, H_{b}\Psi_{-}) - \frac{\mu}{2}||\Psi||^{2}.$$  

It is easy to see that $(\Psi_{\pm}, H_{b}\Psi_{\pm}) \geq m||\Psi_{\pm}||^{2}$. Hence we obtain $(\Psi, H\Psi) \geq (m - \frac{\mu}{2})||\Psi||^{2}$, which implies that

$$\mu_{3}(H) \geq m - \frac{\mu}{2}. \quad (5.2)$$

Assume (1.12). Then, by (5.2) and (2.10), we have

$$\mu_{3}(H) > -\frac{\mu}{2} e^{-2||\lambda/\omega||^{2}} \geq E(H).$$

Hence, by the min-max principle, there are at most two eigenvalues (counting mutiplicity) of $H$ in the interval $[E(H), -\frac{\mu}{2} e^{-2||\lambda/\omega||^{2}}]$. In particular, $H$ has at most two ground states. These facts and (2.5) imply Theorem 1.3. \\n
5.4. Proof of Theorem 1.4

We apply the following fact (which may be more or less known):

**Lemma 5.5.** Let $A_{n}, n = 1, 2, \cdots$, and $A$ be self-adjoint operators on a Hilbert space $K$ having a common core $D$ such that, for all $\psi \in D$, $A_{n}\psi \to A\psi$ as $n \to \infty$. Let $\psi_{n}$ be a normalized eigenvector of $A_{n}$ with eigenvalue $E_{n}$: $A_{n}\psi_{n} = E_{n}\psi_{n}$ such that $E := \lim_{n \to \infty} E_{n}$ and $w-lim_{n \to \infty} \psi_{n} = \psi \neq 0$ exist, where $w-lim$ denotes weak limit. Then $\psi$ is an eigenvector of $A$ with eigenvalue $E$. In particular, if $\psi_{n}$ is a ground state of $A_{n}$, then $\psi$ is a ground state of $A$.

**Proof:** By the present assumption and a general theorem [10, Theorem VIII.25(a)], $A_{n}$ converges to $A$ in the strong resolvent sense as $n \to \infty$. Hence, for all $\phi \in K$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$|(\phi, (A_{n} - z)^{-1}\psi_{n}) - (\phi, (A - z)^{-1}\psi)|$$

$$= |((A_{n} - z)^{-1}\phi - (A - z)^{-1}\phi, \psi_{n})| + |((A_{n} - z)^{-1}\phi, \psi_{n} - \psi)|$$

$$\leq ||(A_{n} - z)^{-1}\phi - (A - z)^{-1}\phi|| + ||(A_{n} - z)^{-1}\phi, \psi_{n} - \psi||$$

$$\to 0 \quad (n \to \infty),$$

i.e., $\lim_{n \to \infty} (\phi, (A_{n} - z)^{-1}\psi_{n}) = (\phi, (A - z)^{-1}\psi)$. By the spectral theorem, we have $(\phi, (A_{n} - z)^{-1}\psi_{n}) = (E_{n} - z)^{-1}(\phi, \psi_{n})$. Hence we obtain $(\phi, (A - z)^{-1}\psi) = (\phi, (E - z)^{-1}\psi)$ for all $\phi \in K$, which implies that $(A - z)^{-1}\psi = (E - z)^{-1}\psi$. Thus $\psi \in D(A)$ and $A\psi = E\psi$. If $\psi_{n}$ is a ground state of $A_{n}$, then $(\phi, A_{n}\phi) \geq E_{n}||\phi||^{2}$ for all $\phi \in D$. Taking the limit $n \to \infty$ in this inequality, we obtain $(\phi, A\phi) \geq E||\phi||^{2}$. Since $D$ is a core for $A$, the last inequality extends to all $\phi \in D(A)$, which, combined with the preceding result, implies that $E = \inf \sigma(A)$. Thus $\psi$ is a ground state of $A$. \\
We now turn to the spin-boson Hamiltonian in the case \( \inf_{k \in \mathbb{R}^n} \omega(k) = 0 \). To employ the results in the case of massive bosons, we define for \( m > 0 \)
\[
\omega_m(k) = \omega(k) + m.
\]
Then (1.2) with \( \omega \) replaced by \( \omega_m \) holds for all \( m > 0 \). We introduce
\[
H_{SB}(m) = \frac{1}{2} \mu \sigma_z \otimes I + I \otimes H_b(m) + \alpha \sigma_x \otimes (a(\lambda)^* + a(\lambda))
\]
with \( H_b(m) = d \mathrm{r}(\omega_m) \).

**Lemma 5.6.** Let \( D = \mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b)] \), where \( \hat{\otimes} \) denotes algebraic tensor product. Then \( D \) is a common core for all \( H_{SB}(m) \) and \( H_{SB} \). Moreover, for all \( \Psi \in D \), \( H_{SB}(m) \Psi \to H_{SB} \Psi \) as \( m \to 0 \).

**Proof:** The first half of the lemma is well known (note that \( \mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b)] = \mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b(m))] \)). The second half follows from a direct computation. \( \blacksquare \)

We are now ready to prove Theorem 1.4. By Theorem 1.1, there exists a ground state \( \Omega(m) \) of \( H_{SB}(m) \): \( H_{SB}(m) \Omega(m) = E(H_{SB}(m)) \Omega(m) \). Without loss of generality, we can assume that \( ||\Omega(m)|| = 1 \). By (1.8), we have
\[
-\frac{\mu}{2} - \alpha^2 ||\frac{\lambda}{\sqrt{\omega_m}}||_{L^2}^2 \leq E(H_{SB}(m)) \leq -\frac{\mu}{2} e^{-2\alpha^2 ||\frac{\lambda}{\sqrt{\omega_m}}||_{L^2}^2} - \alpha^2 ||\frac{\lambda}{\sqrt{\omega_m}}||_{L^2}^2.
\]
By using the Lebesgue dominated convergence theorem, one can easily show that
\[
\lim_{m \to 0} ||\frac{\lambda}{\sqrt{\omega_m}}||_{L^2}^2 = ||\frac{\lambda}{\sqrt{\omega}}||_{L^2}^2,
\]
\[
\lim_{m \to 0} ||\frac{\lambda}{\omega_m}||_{L^2}^2 = ||\frac{\lambda}{\omega}||_{L^2}^2.
\]
\[
(5.3)
\]
Hence \( \{E(H_{SB}(m))\}_m \) is uniformly bounded in \( m \). Thus there exists a sequence \( \{m_j\}_{j=1}^{\infty} \) with \( m_1 > m_2 > \cdots > m_j \to 0 \) (\( j \to \infty \)) such that
\[
E := \lim_{j \to \infty} E(H_{SB}(m_j))
\]
and
\[
\Omega := w - \lim_{j \to \infty} \Omega(m_j)
\]
exist. We need only to show that \( \Omega \neq 0 \) (then, by Lemmas 5.6 and 5.5, \( \Omega \) is a ground state of \( H_{SB} \)).

Let \( P_0 \) be the orthogonal projection from \( \mathcal{F} \) onto the Fock vacuum state \( \{c\Omega_0|c \in \mathbb{C}\} \). It is easy to see that
\[
I \otimes P_0 \geq I - I \otimes N.
\]
If $\omega \lambda$ and $\lambda$ are in $L^2(\mathbb{R}^\nu)$, then $\omega_m \lambda \in L^2(\mathbb{R}^\nu)$. By these facts and Lemma 2.3, we have

$$\langle \Omega(m), I \otimes P_0 \Omega(m) \rangle \geq 1 - \langle \Omega(m), I \otimes N \Omega(m) \rangle \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega_m} \right\|_{L^2}^2. \quad (5.4)$$

Since the range of $I \otimes P_0$ is finite dimensional (in fact, two dimensional), we have

$$\lim_{j \to \infty} \langle \Omega(m), I \otimes P_0 \Omega(m) \rangle = \langle \Omega, I \otimes P_0 \Omega \rangle.$$

From this fact, (5.4) and the second formula in (5.3), we obtain

$$\langle \Omega, I \otimes P_0 \Omega \rangle \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2.$$

Under condition (1.13), the RHS is strictly positive. Hence $\Omega \neq 0$. ■

6. A generalization of the model

In this section we propose a generalization of the spin-boson model discussed in the preceding sections. We expect that the generalization clarify the general properties of the spin-boson model. We also have in mind applications to quantum spin systems on an infinite lattice in which spins interact with bosons too.

Let $\mathcal{H}$ be a Hilbert space and $A$ (resp. $B$) be a self-adjoint (resp. symmetric) operator on $\mathcal{H}$. The Hamiltonian of the generalized spin-boson model we propose is given by

$$H = A \otimes I + I \otimes d\Gamma(\omega) + B \otimes (a(\lambda)^* + a(\lambda))$$

acting in the Hilbert space $\mathcal{H} \otimes \mathcal{F}$.

Suppose that $A, B$ are bounded and $\lambda, \lambda/\sqrt{\omega}, \lambda/\omega$ are in $L^2(\mathbb{R}^d)$. Then

$$L_{A,B} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i||\lambda/\omega||_{L^2} B^t A e^{i||\lambda/\omega||_{L^2} B^t} e^{-t^2/2} dt - \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 B^2$$

is a bounded self-adjoint operator. We can show [4] that

$$-||A|| - ||B||^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H) \leq E(L_{A,B}). \quad (6.1)$$

In the case of the original spin-boson model (i.e., the case $H = H_{SB}$), (6.1) is just (1.8). Thus estimate (6.1) clarifies a general structure of (1.8). The results on ground states of $H_{SB}$ also can be generalized to the case of $H$. We can also develop scattering theory concerning the pair $<A \otimes I + I \otimes d\Gamma(\omega), H>$. For the details, see [4].
References