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An application of Mellin-Barnes’ type integrals to the mean square of L-functions

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1 Introduction

Let $q$ be a positive integer, $s$ a complex variable and $L(s, \chi)$ the Dirichlet L-function attached to a Dirichlet character $\chi \mod q$. Note that $L(s, \chi)$ reduces to the Riemann zeta-function $\zeta(s)$ if $q = 1$.

Let $\varphi(q)$ be Euler’s function. The mean square

$$\varphi(q)^{-1} \sum_{\chi \mod q} |L(s, \chi)|^2,$$

summed over all characters $\chi \mod q$, has been studied by various authors. Let $\mu(n)$ be Mobius’ function. In the special case $s = \frac{1}{2}$, D. R. Heath-Brown [He] found the formula

$$\varphi(q)^{-1} \sum_{\chi \mod q} |L(\frac{1}{2}, \chi)|^2 = q^{-1} \sum_{k \mid q} \mu(k) T(k),$$

(1.2)

where $k$ runs through all positive divisors of $q$ and $T(k)$ has the asymptotic expansion

$$T(k) = k \left( \log \frac{k}{8\pi} + \gamma \right) + 2\zeta^2(\frac{1}{2}) + \sum_{n=0}^{2N-1} c_n k^{-\frac{8}{3}} + O(k^{-N})$$

for any integer $N \geq 1$, with Euler’s constant $\gamma$ and unspecified numerical constants $c_n$. If $q = p$ is a prime, (1.2) gives an asymptotic series in terms of $p^{-\frac{1}{2}}$, since $T(1)$ can be evaluated in a closed form. On the other hand, Y. Motohashi [Mo1], in a series of his study on higher power moments for $\zeta(s)$ and $L(s, \chi)$, applied a classical idea of F. V. Atkinson [At] to (1.1) and proved for any prime $q = p$

$$(p - 1)^{-1} \sum_{\chi \mod p} |L(\frac{1}{2} + it, \chi)|^2$$

$$= \log \frac{p}{2\pi} + 2\gamma + \text{Re} \left\{ \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + it\right) \right\} + 2p^{-\frac{1}{2}}|\zeta(\frac{1}{2} + it)|^2 \cos(\log p)$$

$$- p^{-1}|\zeta(\frac{1}{2} + it)|^2 + O(p^{-\frac{1}{2}}),$$

where $\Gamma(s)$ is the gamma-function and the constant implied in the $O$-symbol depends on $t$. More general and precise formulae have been proved in [KM1], [Ka1] and [Ka2] by refining the argument of Atkinson and Motohashi.
and its meromorphic continuation.

Let $\sigma_a(n)$ denote the sum of the $a$-th powers of positive divisors of $n$. The error term $e_N(\sigma + it; k)$ in (1.6) is of the form

$$e_N(\sigma + it; k) = \text{Re}\left\{k^{\sigma+it-N}R_N(\sigma + it, \sigma - it; k)\right\},$$

where $R_N(u, v; k)$ has the following expressions (cf. [Ka2, Lemma 2.2]):

For $\Re u < N$, $\Re v > -N + 1$ and $\Re(u + v) < 2$,

$$R_N(u, v; k) = (-1)^N(2\pi)^{u+v-1}\frac{\Gamma(N + 1 - u)}{\Gamma(v)}\int_0^1 (1 - \tau)^{N-1} \sum_{l=1}^{\infty} \sigma_{u+v-1}(l) \times \left\{e^{\frac{\pi i(u+v-1)}{2}}J_-^{(u+v-1)}(\tau, l; k) + e^{-\frac{\pi i(u+v-1)}{2}}J_+^{(u+v-1)}(\tau, l; k)\right\}d\tau$$

with

$$J_{\pm}(\tau, l; k) = \int_0^\infty y^{\pm u+N-1}(1 + k^{-1}\tau y)^{u-N-1}e^{\pm 2\pi i\tau y}dy,$$

while for $\Re u < N$, $\Re v > -N + 1$ and $\Re(u + v) > 0$,

$$R_N(u, v; k) = (-1)^N\frac{\Gamma(v + N)}{\Gamma(v)}\int_0^1 (1 - \tau)^{N-1} \sum_{l=1}^{\infty} \sigma_{1-u-v}(l) \times \left\{\tilde{J}_-^{(1-u-v)}(\tau, l; k) + \tilde{J}_+^{(1-u-v)}(\tau, l; k)\right\}d\tau$$

with

$$\tilde{J}_{\pm}(\tau, l; k) = \int_0^\infty y^{-u+N}(1 + k^{-1}\tau y)^{-v-N}e^{\pm 2\pi i\tau y}dy.$$

It is in fact possible to obtain a more explicit estimate for $e_N(\sigma + it; k)$ by applying a saddle point lemma of Atkinson [At, Lemma 1] to $J_\pm(\tau, l; k)$ and $\tilde{J}_\pm(\tau, l; k)$.

**Theorem 2** ([Ka2, Theorem 1 with $h = 0$]) For any integer $N \geq 1$, the inequality

$$e_N(\sigma + it; k) = O\{k^{\sigma-N}(|t| + 1)^{2N+\frac{1}{2}-\sigma}\}$$

holds in the region

$$\{\sigma + it; -N + 1 < \sigma < N, \ t: \text{real}\},$$

where the $O$-constant depends only on $\sigma$ and $N$.

**Remark.** It is reasonable that such a bound as in (1.10) follows, since

$$\frac{(-1)^n}{n!}k^{\sigma+it-n}\frac{\Gamma(\sigma - it + n)}{\Gamma(\sigma - it)}\zeta(\sigma - it - n)\zeta(\sigma - it + n) \ll k^{\sigma-n}(|t| + 1)^{2n+\frac{1}{2}-\sigma}$$

for $-n + 1 < \sigma < n (n \geq 1)$, see (1.5). Note that (1.11) is the best-possible, since $\zeta(\sigma + it) = \Omega(1)$ for $\sigma > 1$ as $t \to \pm \infty$.

The main aim of this paper is to provide alternative simple proofs of Theorems 1 and 2. It should be remarked that the introduction of a Mellin transform (2.3) below is a key to the considerable simplification. In Sections 2 and 3, we shall prove Theorems 1 and 2, respectively. In the final section, the inner connections between different expressions for $R_N(u, v; k)$ (see (1.8), (1.9), (2.9) and (4.1)) will be examined.
Theorem 1 ([KM1, Theorem 1], [Ka1, Theorem 3], [Ka2, Theorem 3 with $h = 0$])

Let

\[ E = \{1, 2, 3, \ldots\} \cup \{\frac{n}{2} + it; \ n: \text{integer} \leq 2, \ t: \text{real}\}. \]

Then for any integer $N \geq 1$, in the region

\[ \{\sigma + it; -N + 1 < \sigma < N + 1, \ t: \text{real}\} \]

except the points of $E$, the formula

\[
\varphi(q)^{-1} \sum_{\chi(\mod q)} |L(\sigma + it, \chi)|^2 = \zeta(2\sigma) \prod_{p \nmid q} (1 - p^{2\sigma}) \sigma + 2q^{-2\sigma} \sum_{k | q} \mu(\frac{q}{k}) T(\sigma + it; k) + \mathcal{O}(k\sigma - N) \]

holds, where $p$ runs through all prime divisors of $q$ and $T(\sigma + it; k)$ has the asymptotic expansion

\[
T(\sigma + it; k) = \sum_{n=0}^{N-1} \frac{(-1)^n k^{-n}}{n!} \Re \left\{ k^{\sigma + it} \frac{\Gamma(\sigma - it + n)}{\Gamma(\sigma - it)} \zeta(\sigma + it - n) \zeta(\sigma - it + n) \right\} + e_N(\sigma + it; k).
\]

Here $e_N(\sigma + it; k)$ is the error term satisfying

\[
e_N(\sigma + it; k) = \mathcal{O}(k\sigma - N)\]

in the region (1.3), with the $O$-constant depends only on $\sigma$, $N$ and $t$. In particular, if $q = p$ is a prime, the asymptotic series

\[
(p - 1)^{-1} \sum_{\chi(\mod p)} |L(\sigma + it, \chi)|^2
\]

holds.

Remark 1. Asymptotic formulae as in (1.4) for the exceptional points $s \in E$ can be deduced as limiting cases of Theorem 1. Important cases $\Re s = \frac{1}{2}$ and $s = 1$ are treated in [KM1, Theorem 1] and [KM2, Theorems 1 and 4], respectively.

Remark 2. In this paper, the region (1.3) in which (1.4) remains valid will be slightly improved upon our earlier results [KM1, Theorem 1], [Ka1, Theorem 3] and [Ka2, Theorem 3].

Remark 3. Similar asymptotic results for (1.1) have been independently obtained by W. Zhang [Zh2]–[Zh7] and V. V. Rane [Ra]. Their proofs are based on the use of the Hurwitz zeta-function $\zeta(s, \alpha)$ defined by

\[
\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (\Re s > 1, \ \alpha > 0),
\]
2 Proof of Theorem 1

Let

\[ Q(u,v;q) = \varphi(q)^{-1} \sum_{\chi \mod q} L(u, \chi) L(v, \overline{\chi}). \]

We suppose first that Re \( u > 1 \) and Re \( v > 1 \). Then by the orthogonality and the periodicity of characters

\[ Q(u,v;q) = \sum_{h,k=1}^{\infty} h^{-u} k^{-v} = \sum_{a=1}^{q} \sum_{m,n=0}^{\infty} (qm+a)^{-u}(qn+a)^{-v}. \]

Classifying the last inner double sum according to the conditions \( m = n \), \( m < n \) and \( m > n \), we get

\[ Q(u,v;q) = L(u+v, \chi_0) + f(u,v;q) + f(v,u;q), \tag{2.1} \]

where \( \chi_0 \) is the principal character \( \mod q \) and

\[ f(u,v;q) = \sum_{a=1}^{q} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (qm+a)^{-u}(q(m+n)+a)^{-v}. \tag{2.2} \]

Atkinson succeeded in obtaining the analytic continuation of \( f(u,v;1) \) (namely in the case of \( \zeta(s) \)), which led him to the eventual application on taking \( u = \frac{1}{2} + it \) and \( v = \frac{1}{2} - it \). Several ways are known to prove the analytic continuation of \( f(u,v;q) \). T. Meurman [Me] generalizes Atkinson's original proof to treat \( f(u,v;q) \) by Poisson's summation formula, while Motohashi [Mo1] makes use of certain loop-integral expressions for \( f(u,v;q) \). In this paper we apply

\[ (qm+a)^{-u}(q(m+n)+a)^{-v} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(-s)\Gamma(v+s)}{\Gamma(v)} (qm+a)^{-\overline{s}-u-v}(qn)^{s}ds, \tag{2.3} \]

where \( c \) is a constant fixed with \( -Re v < c < -1 \) and \( (c) \) denotes the vertical straight line from \( c - i\infty \) to \( c + i\infty \). This can be obtained by taking \( -z = qn/(qm+a) \) in

\[ \Gamma(\alpha)(1-z)^{-\alpha} = \frac{1}{2\pi i} \int_{(b)} \Gamma(\alpha+s)\Gamma(-s)(-z)^{s}ds \quad (|\arg(-z)| < \pi, -Re \alpha < b < 0), \]

which is a special case of Mellin-Barnes' integral expression for Gauss' hypergeometric function \( F(\alpha,\beta;\gamma;z) \) (cf. [WW, p.289, 14.51 Corollary]). Integrals of the type (2.3) were firstly introduced by Motohashi [Mo2] to investigate the fourth power mean of \( \zeta(s) \). Recently, A. Ivić [Iv2, Chapter 2] applied Motohashi's argument to treat the mean square of \( \zeta(s) \).

We assume for brevity that all the singularities appearing in the following argument are at most simple poles, since other cases can be treated by taking limits (see Remark...
1 of Theorem 1). Substituting (2.3) into each term in the right-hand side of (2.2), we obtain

\[
f(u,v;q) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(-s)^{\mathrm{p}}(v+s)}{\Gamma(v)} q^{-u-v} \sum_{a=1}^{q} \zeta(u+v+s, \frac{a}{q}) \zeta(-s) ds,
\]

where the interchange of the order of summation and integration can be justified, since, by virtue of the choice of \( \delta \), the variables \( u + v + s \) and \( -s \) are both in the region of absolute convergence. As we shall see in the following, the formula (2.4) will provide the analytic continuation of \( f(u,v;q) \) by deforming suitably the path of integration. Note that (c) separates the poles at \( s = -1 + n \) \((n = 0, 1, 2, \ldots)\) from the poles at \( s = 1 - u - v, -v - n \) \((n = 0, 1, 2, \ldots)\) of the integrand. If we replace (c) by the contour \( C \) which is suitably indented in such a manner as to separate the poles at \( s = 1 - u - v, -1 + n \) \((n = 0, 1, 2, \ldots)\) from the poles at \( s = -v - n \) \((n = 0, 1, 2, \ldots)\), then we get by the theorem of residues

\[
f(u,v;q) = \frac{\Gamma(u+v-1)\Gamma(1-u)}{\Gamma(v)} \zeta(u+v-1)q^{1-u-v} \prod_{p|q} (1-p^{-1}) + g(u,v;q),
\]

where

\[
g(u,v;q) = q^{-u-v} \sum_{k|q} \mu(\frac{k}{q}) \left( \int_{C} \frac{\Gamma(-s)^{\mathrm{p}}(v+s)}{\Gamma(v)} \zeta(u+v+s, \frac{k}{q}) \zeta(-s) ds \right)
\]

say. Here we applied the identities

\[
\sum_{(a,q) = 1}^{q} \zeta(w, \frac{a}{q}) = \zeta(w) \sum_{k|q} \mu(\frac{k}{q}) k^{w} = \zeta(w) q^{w} \prod_{p|q} (1-p^{-w}).
\]

Hence from (2.1), (2.5) and (2.6),

\[
Q(u,v;q) = \zeta(u+v) \prod_{p|q} (1-p^{-u-v}) + q^{-u-v} \varphi(q) \zeta(u+v-1) \Gamma(u+v-1) \times
\]

\[
\left\{ \frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)} \right\} + q^{-u-v} \sum_{k|q} \mu(\frac{k}{q}) \{ S(u,v;k) + S(v,u;k) \}
\]

holds in the region \( \Re u > 1 \) and \( \Re v > 1 \), where \( S(v,u;k) \) is expressed in the same manner as \( S(u,v;k) \).

Next we shift the path of integration to the left. We suppose at this stage that \( \Re u < 1 \) and \( \Re v > 1 \), where \( C \) can be taken as a straight line \((c_0)\) with \( -\Re v < c_0 < \min(-1, 1 - \Re(u+v))\). Let \( N \) be a positive integer and \( c_N \) a constant fixed with \( -\Re v - N < c_N < -\Re v - N + 1 \). Since the order of the integrand in (2.6) is \( O(|\Im s|^C e^{-\pi |\Im s|}) \) as \( \Im s \to \pm\infty \) \((C \) is a positive constant depending only on \( \Re s \),
Re \( u \) and Re \( v \), we can shift the path from \((c_0)\) to \((c_N)\). Collecting the residues at the poles \( s = -v - n \) \((n = 0, 1, \ldots, N - 1)\), we obtain

\[
S(u, v; k) = \sum_{n=0}^{N-1} \frac{(-1)^n \Gamma(v + n)}{n! \Gamma(v)} \zeta(u - n)\zeta(v + n)k^{u-n} + r_N(u, v; k),
\]

where

\[
r_N(u, v; k) = \frac{1}{2\pi i} \int_{(c_N)} \frac{\Gamma(-s)\Gamma(v + s)}{\Gamma(v)} \zeta(u + v + s)\zeta(-s)k^{u+v+s} ds.
\]

Here the condition on \( u \) and \( v \) can be relaxed as

\[
\text{Re } u < N + 1 \quad \text{and} \quad \text{Re } v > -N + 1.
\]

Under (2.10) we can choose \( c_N \) satisfying the condition

\[-\text{Re } v - N < c_N < \min(-1, -\text{Re } v - N + 1, 1 - \text{Re}(u + v)),\]

by which \((c_N)\) separates the poles at \( s = -v - n \) \((n = N, N + 1, N + 2, \ldots)\) from the poles at \( s = 1 - u - v, -1 + n \) \((n = 0, 1, 2, \ldots)\), \(-v - n \) \((n = 0, 1, \ldots, N - 1)\).

Now we proceed to prove Theorem 1. Taking \( u = \sigma + it \) and \( v = \sigma - it \) in (2.7), (2.8) and (2.9), we obtain (1.4) and (1.5), by noticing (2.10) and putting

\[
T(\sigma + it; k) = \text{Re}\{S(\sigma + it, \sigma - it; k)\} \quad \text{and} \quad e_N(\sigma + it; k) = \text{Re}\{r_N(\sigma + it, \sigma - it; k)\}.
\]

The error estimate (1.6) follows from

\[
r_N(u, v; k) = \frac{(-1)^N \Gamma(v + N)}{N!} \frac{\Gamma(v + N)}{\Gamma(v)} \zeta(u - N)\zeta(v + N)k^{u-N}
\]

\[
+ \frac{1}{2\pi i} \int_{(c_N+1)} \frac{\Gamma(-s)\Gamma(v + s)}{\Gamma(v)} \zeta(u + v + s)\zeta(-s)k^{u+v+s} ds.
\]

by \(-\text{Re } v - N - 1 < c_{N+1} < -\text{Re } v - N\). Furthermore (1.7) can be deduced from (2.7) by noting

\[
S(u, v; 1) + S(v, u; 1) = \zeta(u)\zeta(v) - \zeta(u + v)
\]

\[
- \zeta(u + v - 1)\Gamma(u + v - 1)\left\{ \frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)} \right\},
\]

which is the special case \( q = 1 \) of (2.7). The proof of Theorem 1 is now complete.

### 3 Proof of Theorem 2

Throughout this section, let \(-N + 1 < \sigma < N\) and \( \delta \) a constant fixed with \( 0 < \delta < \frac{1}{2} \min(N - \sigma, N - 1 + \sigma, 1)\). We write \( s = -\sigma - N + \xi + i\tau \) in (2.9). For the proof of Theorem 2, we need
Lemma. For any real $\tau$, $t$ and $\xi$ with $|\xi| \leq \delta$, we have

\begin{align}
\Gamma(\sigma + N - \xi - i\tau) &\ll (|\tau| + 1)^{\sigma + N - \xi - \frac{1}{2} e^{\frac{\pi}{2} |\tau|}}, \\
\Gamma(-N + \xi + i(\tau - t)) &\ll \begin{cases} |\tau - t|^{-N + \xi - \frac{1}{2} e^{\frac{\pi}{2} |\tau - t|}} & \text{for } |\tau - t| \geq 1, \\
|\xi + i(\tau - t)|^{-1} & \text{for } |\tau - t| \leq 1,
\end{cases} \\
\Gamma(\sigma - it)^{-1} &\ll (|t| + 1)^{\frac{1}{2} - \sigma} e^{\frac{\pi}{2} |t|}, \\
\zeta(\sigma - N + \xi + i\tau) &\ll (|\tau| + 1)^{1 - \sigma + N - \xi}, \\
\zeta(\sigma + N - \xi - i\tau) &\ll 1.
\end{align}

Here and in what follows the implied constants depend at most on $\sigma$ and $N$.

Proof. (3.1)–(3.3) follow from Stirling’s formula (cf. [1v1, p.492, (A.34)]) and the trivial bounds for $\Gamma(w)$ near the real axis. By virtue of the choice of $\delta$, (3.5) is an immediate consequence of the inequality $\zeta(w) \ll 1$ for $\Re w > 1$, while (3.4) can be proved by applying the functional equation of $\zeta(w)$.

For the proof of Theorem 2 we may restrict ourselves to the case $t \geq 2$, since the case $t \leq -2$ follows from this case by the reflection principle, and the case $|t| \leq 2$ is a simple consequence of Theorem 1.

Let $\sigma_N = \sigma + N$ and $L$ the infinite broken line joining the points $-\sigma_N - \infty$, $-\sigma_N + i(t - \delta)$, $-\sigma_N + \delta + i(t - \delta)$, $-\sigma_N + \delta + i(t + \delta)$, $-\sigma_N + i(t + \delta)$ and $-\sigma_N + \infty$. Taking $u = \sigma - it$ and $v = \sigma + it$ in (2.9), and then replacing the path $(c_N)$ by $L$, we have

$$ r_N(\sigma + it, \sigma - it; k) = \frac{1}{2\pi i} \int_L \frac{\Gamma(-s)\Gamma(\sigma - it + s)}{\Gamma(\sigma - it)} \zeta(2\sigma + s) \zeta(-s) k^{2\sigma + s} ds. \tag{3.6} $$

We shall estimate the right-hand integral in (3.6) by dividing

$$ r_N(\sigma + it, \sigma - it; k) = \frac{1}{2\pi i} \left\{ \sum_{\mu=1}^{7} I_{\mu} + \sum_{\nu=1}^{3} I_{\nu} \right\}, $$

where

\begin{align}
I_1 &= \int_{-\sigma_N - \infty}^{\sigma_N - i}, \\
I_2 &= \int_{-\sigma_N + \infty}^{-\sigma_N - i}, \\
I_3 &= \int_{-\sigma_N + \infty}^{\sigma_N + i(t - 1)}^{-\sigma_N - i}, \\
I_4 &= \int_{-\sigma_N + i(t - 1)}^{\sigma_N + i(t - \delta)}^{-\sigma_N - i}, \\
I_{5,1} &= \int_{-\sigma_N + i(t - \delta)}^{\sigma_N + i(t - \delta) + it}, \\
I_{5,2} &= \int_{-\sigma_N + i(t - \delta)}^{\sigma_N + i(t + \delta) + it}, \\
I_{5,3} &= \int_{-\sigma_N + i(t - \delta)}^{\sigma_N + i(t + \delta)}, \\
I_5 &= \int_{-\sigma_N + i(t + \delta)}^{-\sigma_N + i(t + \delta) + it}, \\
I_6 &= \int_{-\sigma_N + i(t + \delta) + it}^{-\sigma_N + i(\tau + \delta) + it}, \\
I_7 &= \int_{-\sigma_N + i(\tau + \delta) + it}^{-\sigma_N + i(\tau + \delta) + it + \infty}.
\end{align}

The treatment of $I_{\nu}$ ($\nu = 1, 2, 3$) is more delicate than that of other $I_{\mu}$'s. By Lemma and the assumption $t \geq 2$, we get

\begin{align}
I_1 &\ll k^{\sigma - N - \frac{3}{4} - \sigma} \int_{-\infty}^{-1} (-(\tau)^2 N(t - \tau)^{-N - \frac{1}{2} e^{\frac{\pi}{2} |\tau|}} d\tau \ll k^{\sigma - N - \frac{3}{4} - \sigma}, \tag{3.7} \\
I_2 &\ll k^{\sigma - N - \frac{3}{4} - \sigma} \int_{-1}^{1} (t - \tau)^{-N - \frac{1}{2} e^{\frac{\pi}{2} |\tau|}} d\tau \ll k^{\sigma - N - \frac{3}{4} - \sigma}. \tag{3.8}
\end{align}
Moreover

\[ I_3 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma} \int_1^{t-\delta} \tau^{2N}(t-\tau)^{-\frac{3}{2}}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}, \quad (3.9) \]

\[ I_4 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma} \int_{t-\delta}^{t+1} \tau^{2N}(t-\tau)^{-1}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \log \delta^{-1}, \quad (3.10) \]

where the last upper bounds in (3.9) and (3.10) are obtained by integrating by parts. Similarly to \( I_3 \) and \( I_4 \),

\[ I_6 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma} \int_{t-\delta}^{t+1} \tau^{2N}(t-\tau)^{-\frac{3}{2}}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \log \delta^{-1}, \quad (3.11) \]

\[ I_7 \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma} \int_{t-\delta}^{t+1} \tau^{2N}(t-\tau)^{-\frac{3}{2}}d\tau \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}. \quad (3.12) \]

For \( I_{5,\nu} (\nu = 1, 2, 3) \), we proceed as follows. By \( 0 < \delta < \frac{1}{2} \),

\[ I_{5,1} \ll k^{\sigma-N}t^{\frac{1}{2}-\sigma} \int_0^\delta (t-\delta)^{2N-2t}e^{\frac{1}{2}t\delta}(\xi - i\delta)^{-1}k\delta d\xi \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \max(1, (kt^{-2})^\delta), \quad (3.13) \]

\[ I_{5,2} \ll k^{\sigma-N+\delta}t^{\frac{1}{2}-\sigma} \int_{-\delta}^\delta \tau^{2N-2\delta}e^{\frac{1}{2}(t-\tau)}\delta + i(t-\tau)^{-1}d\tau \ll k^{\sigma-N+\delta}t^{2N+\frac{1}{2}-\sigma}(kt^{-2})^\delta, \quad (3.14) \]

\[ I_{5,3} \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \max(1, (kt^{-2})^\delta), \quad (3.15) \]

where the treatment of \( I_{5,3} \) is similar to (3.13). Combining (3.7)–(3.15), we obtain

\[ r_N(\sigma + it, \sigma - it; k) \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \{\log \delta^{-1} + \max(1, (kt^{-2})^\delta)\}. \quad (3.16) \]

Next let \( L' \) be the infinite broken line joining the points \( -\sigma_N - i\infty, -\sigma_N + i(t - \delta), -\sigma_N - \delta + i(t - \delta), -\sigma_N - \delta + i(t + \delta), -\sigma_N + i(t - \delta) \) and \( -\sigma_N + i\infty \). Then

\[ r_N(\sigma + it, \sigma - it; k) = \frac{(-1)^N \Gamma(\sigma - it + N)}{N! \Gamma(\sigma - it)} \zeta(\sigma + it - N)\zeta(\sigma - it + N)k^{\sigma+it-N} \]

\[ + \frac{1}{2\pi i} \int_{L'} \frac{\Gamma(-s)\Gamma(\sigma - it + s)}{\Gamma(\sigma - it)} \zeta(2\sigma + s)\zeta(-s)k^{2\sigma+s}ds. \quad (3.17) \]

The first term in the right-hand side of (3.17) is bounded as \( \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \) by (1.11). To estimate the second term, we divide

\[ \int_{L'} = \sum_{\mu = 1}^7 \sum_{\nu = 1}^3 I_{5,\mu} \]

where

\[ I_{5,1} = \int_{-\sigma_N - \delta + i(t-\delta)}^{\sigma_N - \delta + i(t-\delta)} \], \quad I_{5,2} = \int_{-\sigma_N - \delta + i(t+\delta)}^{\sigma_N + i(t+\delta)} \], \quad I_{5,3} = \int_{-\sigma_N - \delta + i(t+\delta)}^{\sigma_N + i(t+\delta)} \].
Similarly to the previous case,
\[ I_{5,\nu} \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma} \max(1,(k^{-1}t^2)^{\delta}) \quad (\nu = 1,3), \]
\[ I_{5,2} \ll k^{\sigma-N}t^{2N+\frac{1}{2}-\sigma}(k^{-1}t^2)^{\delta}. \]

Therefore
\[ r_N(\sigma + it, \sigma - it; k) \ll k^{\sigma-N}t^{2N\sigma}+1, -t k^{-12}t) \mathcal{M}(5). \]

Theorem 2 now follows from (3.16) if \( t \geq k^{\frac{1}{4}} \), and from (3.18) if \( t \leq k^{\frac{1}{4}} \), respectively.

### 4 Additional remarks

The first purpose of this section is to show how we can deduce (2.9) directly from (1.8) or (1.9). To do this we introduce a confluent hypergeometric function \( \Psi(\alpha, \gamma; z) \) defined by
\[
\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-w} w^{\alpha - 1}(1 + w)^{\gamma - a - 1} dw
\]
for \( \text{Re} \alpha > 0, |\phi| < \pi \) and \( |\phi + \arg z| < \frac{\pi}{2} \) (cf. [Er, p.256, 6.5(3)]). Rotating suitably the path of integrations for \( J_+(\tau, l; k) \) and \( \tilde{J}_+(\tau, l; k) \), we find
\[
J_+(\tau, l; k) = (k\tau^{-1})^{u+N} \Gamma(v + N) \Psi(v + N, u + v; 2\pi kl\tau^{-1}e^{-\frac{\pi}{2}})
\]
and
\[
\tilde{J}_+(\tau, l; k) = (k\tau^{-1})^{N+1-u} \Gamma(N + 1 - u) \Psi(N + 1 - u, 2 - u - v; 2\pi kl\tau^{-1}e^{\frac{\pi}{2}}). \]

Furthermore, \( J_+(\tau, l; k) \) and \( \tilde{J}_+(\tau, l; k) \) can be expressed in terms of Mellin-Barnes’ type integrals by using
\[
\Psi(\alpha, \gamma; z) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(\alpha + s)\Gamma(-s)\Gamma(1 - \gamma - s)}{\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)} z^s ds,
\]
where \(- \text{Re} \alpha < b < \min(0,1 - \text{Re} \gamma) \) and \( |\arg z| < \frac{3\pi}{2} \) (cf. [Er, p.256, 6.5(5)]). Substituting these integrals into each term in the right-hand infinite series in (1.8) and (1.9), respectively, and then applying the functional equation of \( \zeta(w) \), we can see that either (1.8) or (1.9) directly yields (2.9), by noting
\[
r_N(\sigma, v; k) = k^{u-N} R_N(u, v; k). \quad (4.1)
\]

On the other hand, (1.8) and (1.9) are connected by the transformation formula
\[
\Psi(\alpha, \gamma; z) = z^{1-\gamma} \Psi(\alpha - \gamma + 1, 2 - \gamma; z)
\]
(cf. [Er, p.257, 6.5(6)]), for details see [Ka1, Section 3]. In view of the consideration given above, this connection is embodied in (2.9) with the functional equation of \( \zeta(w) \).
In this occasion we point out an error in the preceding article [KM3]. In Section 2, we should mention that the same result as R. Sitaramachandrarao [Si] with a slightly different log-factor was independently obtained by Zhang [Zh1, Corollary], whose main theorem is proved in a more general setting.

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