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<th>EXTENDED FORMAL POWER SERIES AND G-FUNCTIONS (Analytic Number Theory)</th>
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<td>Author(s)</td>
<td>Harase, T.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1996, 958: 90-92</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60461">http://hdl.handle.net/2433/60461</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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EXTENDED FORMAL POWER SERIES AND G-FUNCTIONS

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At first, let us consider a formal power series ring $R = k[[x]]$ where $k$ is a field. The fraction field of $R$ is $\mathbb{Q}(R) = k((x))$. Every element of $k((x))$ is expressed as a power series with finite negative exponents. But when we consider a power series ring of several indeterminates $R = k[[x_1, \ldots, x_n]]$, some elements of $\mathbb{Q}(R)$ cannot be expressed as a power series. For example, consider

$$\frac{1}{x+y} \in \mathbb{Q}(k[[x], y]).$$

Sometimes, we want to express every element of $\mathbb{Q}(R)$ as a formal power with possibly negative exponents. So I introduced extended formal power series rings. [5]

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a vector in $\mathbb{R}^m$, and let $n = (n_1, \ldots, n_m)$ be an integer vector in $\mathbb{Z}^m$. Fixing $\alpha$, $L = L(n)$ denotes the linear form

$$\alpha \cdot n = \alpha_1 n_1 + \ldots + \alpha_m n_m.$$

We abbreviate $\sum a(i) x^i$ for

$$\sum_{i_1=-\infty}^{\infty} \ldots \sum_{i_m=-\infty}^{\infty} a_{i_1 \ldots i_m} x_1^{i_1} \ldots x_m^{i_m}.$$

The following definitions are essential.

Definition 1. A subset $I \subset \mathbb{Z}^m$ is L-finite iff $\forall N \in \mathbb{Z}$

$$\#(I \cap \{n|L(n) < N\}) < \infty$$

Definition 1'. $f = \sum a(i) x^i$ is L-finite iff $I = \{i|a(i) \neq 0\}$ is L-finite.

Definition 2. $K_L = k((x))_L = k((x_1, \ldots, x_m)) = \{L - finite series\}$.

Under these definitions, we have the following:

Theorem 0. (1) $k((x))_L$ is a $k[x]$-algebra. (2) If $\alpha_1, \ldots, \alpha_m$ are linearly independent over $\mathbb{Q}$ then $K = K_L$ is a field containing $k(x)$.

Remark. When $\text{char}(k) > 0$ many results are obtained. In this note we restrict ourselves to relation to G-functions.

From now on let $k$ be a number field and $\Sigma$ be the set of all places of $k$, and $|.|_v$ be the normalized absolute value corresponding to $v \in \Sigma$. Let $f = \sum_{n=0}^{\infty} a_n x^n \in k[[x]]$. The definition of the G-function is the following.
$f$ is an $\mathrm{G}$-function iff

1. $\sigma(f) < \infty$
2. $f$ is $D$–finite.

Here $\sigma(f) = \varlimsup_{n \to \infty} \frac{1}{n} \sum_v \max_{m \leq n} (\log^+ |a_m|_v)$, and $D$-finite means that $f$ satisfies a linear differential equation over $k(x)$. It is well known that this definition is equivalent to the Siegel’s original definition. Further we may take $f$ from $k((x))$.

By using our "extended power series" we can define $\mathrm{G}$-functions of several variables naturally. That is: $f$ is an "extended" $\mathrm{G}$-function iff

1. $f \in K_L$, $\sigma(f) = \varlimsup_{N \to \infty} \frac{1}{N} \sum_v \max L(nD < N(\log^+(|a_n|_v)) < \infty$.
2. $f \in K_L$ is $D$-finite ( $f$ is contained in a $\frac{d}{dx_i}$-stable $k(\underline{x})$-vector subspace $V \subset K_L$).

Next we consider the diagonal maps. For

$$f = \sum_{n_1=0}^{\infty} \sum_{n_m=0}^{\infty} a_{n_1,...,n_m} x_1^{n_1} ... x_m^{n_m} \in k[[x_1,...,x_m]],$$

diagonal map $I$ is defined as

$$I(f) = \sum_{n=0}^{\infty} a_{n,...,n} t^n \in k[[t]].$$

It is easy to see that the diagonal map $I$ is defined for "extended formal power series rings" $K_L$.

It can be proved that if $f \in K_L$ is $D$-finite then $I(f) \in K((t))$ is also $D$-finite. So we have that

$f \in K_L : "extended" G – function \Rightarrow I(f) : G – function.$

Recall the following conjecture of Christol:

Every globally bounded $G$-function is a diagonal of some rational function. Here "globally bounded" for series $f = \sum a_n x^n$ means that coefficients $a_n \in \mathbb{O}[\frac{1}{N}]$ for every $n$, where $\mathbb{O}$ is the ring of integers in the number field $k$ and $N$ is a natural integer. In this conjecture, the rational function means an elements in $K[\underline{x}]_{(x)}$. But in our situation we can take elements from $k(\underline{x})$.

It is sometimes possible to prove an "extended" $G$-function to be a rational function. The method of Gelgond, Chudnovskys are available for elements in $k((x_1,...,x_\nu))_L$. The following is the analogy for the Chudnovskys criterion for rationality for elements in $k[[x_1,...,x_\nu]]$.

Proposition. Let $Y = (y_0,...,y_{\mu-1}) \subset K((x_1,...,x_\nu))_L$, let $\tau > 0$, and let $V \subset \Sigma$ be some subset of places of $k$. Assume that for each $v \in V$ the $y_i's$ converge on a polydisk $|x_i|_v < \kappa_{i,v}$ ($i = 1,...,\nu$). If the following inequality holds

$$ (\sigma_{notV}(Y) + \tau \sigma(Y) < \sum_{v \in V} \left[ 1 - \frac{1}{\mu} (1 + \frac{1}{\tau}) \right] \cdot \left( \sum_{i=1}^{\nu} \log \kappa_{i,v} \right),$$

then $f \in K_L : "extended" G – function \Rightarrow I(f) : G – function.$
then $y_i's$ are linearly dependent over $k(\xi)$ where $\xi = (\xi_1, \ldots, \xi_\nu), \xi_i = x_i^n$ for some $n > 0$. It is a question to prove that $y_i's$ are linearly dependent over $k(\underline{x})$.

References