<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>ON THE SOLUTIONS OF THUE EQUATIONS (Analytic Number Theory)</td>
</tr>
<tr>
<td>著者</td>
<td>FUJIMORI, MASAMI</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 1996年 958号 56-58</td>
</tr>
<tr>
<td>発行年</td>
<td>1996-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60466">http://hdl.handle.net/2433/60466</a></td>
</tr>
<tr>
<td>型式</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>版本</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
ON THE SOLUTIONS OF THUE EQUATIONS

MASAMI FUJIMORI (藤森雅已, 東北大理)

Let \( k/\mathbb{Q} \) be a finite extension, \( p(X, Y) \in k[X, Y] \) a homogeneous polynomial of degree \( n > 3 \) with non-zero discriminant, and \( a \in k^x = k \setminus \{0\} \). The equation

\[
p(X, Y) = aZ^n
\]
defines a regular curve \( C \) in \( \mathbb{P}^2_k \) of genus \( g = (n-1)(n-2)/2 \).

Here we obtain an estimate for the number of integral solutions and a certain information about rational points.

Remark 0.1. In [2], we required that \( p(X, Y) \) be divisible by a linear element in \( k[X, Y] \), but it is easily seen we do not have to assume that. The same results follow if we replace the map \( f^a \) there by the map \( f: C \to J = \) the Jacobian of \( C \),

\[
C(k) \ni P \to \mathcal{O}_C((2g-2)P - a \text{ canonical divisor}) \in \text{Pic}^0(C_k) \simeq J(k)
\]

which is defined over \( k \), noting that this map equals \( 2g-2 \) times \( f^a \).

1. INTEGRAL SOLUTIONS

For simplicity, we state the result only in the case of rational integral solutions. As for the algebraic \( S \)-integer version, we refer the reader to [2, Theorem 5.4].

In this section we assume that \( a \) and the coefficients of \( p(X, Y) \) are in \( \mathbb{Z} \).

In 1983, Silverman used the Jacobian variety \( J \) of \( C \) to estimate the number of integral solutions from above:

**Theorem 1.1 (Silverman [4]).** If all the exponents of prime factors of \( a \) are less than \( n \), and \( |a| \) is sufficiently large, then

\[
\#\{(x, y) \in \mathbb{Z}^2 \mid p(x, y) = a\} < n^{2n^2}(8n^3)^R,
\]

where \( R = \text{rank} \ J(\mathbb{Q}) \).

1991 Mathematics Subject Classification. Primary 11D41; Secondary 11G30.
We can think of $J(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ as a Euclidean space with some height function. He mapped $C(\mathbb{Q})$ to $J(\mathbb{Q})$ and counted the number of lattice points which lie in a ball of $J(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$.

On the other hand, Mumford had asserted in 1965 paper [3] that in general, the heights of rational points on the Jacobian which come from a curve under a certain map increase at least exponentially if the genus is greater than 1.

Putting the above two results together, we obtain

**Theorem 1.2.** If all the exponents of prime factors of $a$ are less than $n$, and $|a|$ is sufficiently large, then

$$\#\{(x, y) \in \mathbb{Z}^2 \mid p(x, y) = a\} \leq 4 \cdot 7^R,$$

where $R = \text{rank } J(\mathbb{Q})$.

**2. Rational points**

For a general curve $C$ over $k$ of genus $g > 1$, there is a result of Vojta about the distribution of the rational points of $C$ in the Jacobian variety $J$.

Let $(\cdot, \cdot)$ and $\|\cdot\|$ be respectively the inner product and the norm on $J(k) \otimes_{\mathbb{Z}} \mathbb{R}$ induced by the Néron-Tate height attached to the $\Theta$-divisor.

**Theorem 2.1 (Vojta [5], cf. [1]).** Assume $C(k) \neq \emptyset$. Regard $C \subset J$ by an appropriate map. For $\varepsilon > 0$, there exists a constant $\gamma = \gamma(C, \varepsilon)$ such that if $P, Q \in C(k)$ satisfy the inequalities $\|P\| > \gamma$ and $\|Q\| > \gamma||P||$, then

$$\langle P, Q \rangle / ||P|| ||Q|| < 1/\sqrt{g} + \varepsilon.$$

If we have a non-trivial automorphism of the curve, what can we say about the distribution of the rational points of the curve? When asking this question, we prefer to use the morphism $f : C \rightarrow J$ given by

$$C(\overline{k}) \ni P \mapsto \mathcal{O}_C((2g-2)P - \text{a canonical divisor}) \in \text{Pic}^0(C_\overline{k}) \simeq J(\overline{k}),$$

where $\overline{k}$ is an algebraic closure of $k$ and $C_\overline{k} = C \times_k \text{Spec } \overline{k}$. Automorphisms of $C$ induce norm preserving morphisms of $J$ compatible with the above map. In other words, there exists a canonically defined representation of $\text{Aut}_k C$ on the Euclidean space $(J(k) \otimes_{\mathbb{Z}} \mathbb{R}, ||\cdot||)$ which leaves the image of $C(k)$ stable.

In the case of Thue curves, we obtain the following result: Let $C$ be the Thue curve as before. For an $n$-th root $\zeta$ of unity in $k$, we have an automorphism of $C$ defined as

$$C(\overline{k}) \ni P = (x : y : z) \mapsto P_\zeta := (x : y : \zeta z) \in C(\overline{k}).$$
Proposition 2.2. For $P \in C(k)$, if $\zeta \neq 1$ and $\|fP\| \neq 0$, then
\[
\langle fP_\zeta, fP \rangle / \|fP_\zeta\|\|fP\| = -1/(n - 1).
\]

As an application of this kind of results, if $C$ is a twisted Fermat curve of degree 4, for example, we can see the rational points lie in an intersection of quadric hypersurfaces in $J(k) \otimes_{\mathbb{Z}} \mathbb{R}$. The author will explain it elsewhere.

REFERENCES


MATHEMATICAL INSTITUTE, FACULTY OF SCIENCE, TOHOKU UNIVERSITY, SENDAI 980-77, JAPAN