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EXTREMAL PROBLEMS IN SIEVE THEORY

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1. INTRODUCTION: THE GENERAL SIEVE AS AN EXTREMAL PROBLEM

Given a finite set $\mathcal{A}$ of integers and a set $\mathcal{P}$ of primes, the object of sieve theory is to give lower and upper bounds for the sifting function

$$S(\mathcal{A},\mathcal{P}) = \#\{a \in \mathcal{A} : p|a \Rightarrow p \notin \mathcal{P}\}.$$

Such bounds can be of great interest in prime number theory; for example, if $\mathcal{A} = \{p+2 \leq x : p \text{ prime}\}$ and

$$\mathcal{P} = \mathcal{P}(x^{1/(k+1)}) = \{p \leq x^{1/(k+1)} : \text{ prime}\}$$

with some positive integer k, then a non-trivial lower bound for $S(\mathcal{A},\mathcal{P})$ implies the existence of primes p in the interval $[\sqrt{x},x]$ such that $p+2$ has at most k prime factors. In particular, if such a bound could be proved for $k=1$ and all sufficiently large x, then the twin prime conjecture would follow. While this result seems to be beyond the reach of sieve methods, Chen ([Ch]; see also [HR]) was able to obtain non-trivial bounds in the case $k=2$ by ingeneously combining sieve methods with other techniques.

Even though sieve theory has been motivated by applications to *concrete* instances of sets $\mathcal{A}$ and $\mathcal{P}$, its success is largely due to its formulation as a *general* problem of estimating the sifting functions $S(\mathcal{A},\mathcal{P})$ under minimal assumptions on $\mathcal{A}$ and $\mathcal{P}$. Most sieve theoretic estimates are of a very general nature, giving bounds for $S(\mathcal{A},\mathcal{P})$ that depend on the specific nature of these sets only via one or more simple parameters which measure, in a certain sense, the size and density of $\mathcal{A}$ and $\mathcal{P}$, and which, in some cases, can be shown to be best-possible under a suitable set of assumptions. In other words, the object of many sieve theoretic results is to solve an extremal problem: Given a set of parameters $\lambda_i(\mathcal{A},\mathcal{P})$, find the maximum and minimum value of $S(\mathcal{A},\mathcal{P})$ among all sets $\mathcal{A}$ and $\mathcal{P}$ for which $\lambda_i(\mathcal{A},\mathcal{P}) \leq \lambda_i$ for given numbers λ_i.

The classical instance of a sieve result formulated as an extremal problem is the Rosser-Iwaniec sieve [Iw], which we now describe. This sieve involves two parameters κ and s, and the assumptions on $\mathcal{A}$ and $\mathcal{P}$ can be informally stated as follows:

(I) $\mathcal{A}$ is well distributed in residue classes $0 \bmod d$ for moduli $d \leq D$, in the sense that

$$\#\{a \in \mathcal{A} : a \equiv 0 \bmod d\} \approx \frac{\omega(d)X}{d} \quad (d \leq D, p|d \Rightarrow p \in \mathcal{P})$$

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with $X = |\mathcal{A}|$ and a multiplicative function $\omega(d)$ satisfying $\omega(p) \le \kappa$ on average over primes p.

(II) $\mathcal{P}$ is contained in $[2, D^{1/s}]$.

Under these assumptions, the "expected" value for $S(\mathcal{A},\mathcal{P})$ is

$$E(\mathcal{A},\mathcal{P}) = \prod_{p\in\mathcal{P}} \left(1 - \frac{\omega(p)}{p}\right) X.$$

The extremal problem underlying the Rosser-Iwaniec sieve method can now be formulated as follows:

Extremal Problem I (The Rosser-Iwaniec sieve). *Find best-possible functions $F_\kappa(s)$ and $f_\kappa(s)$ such that, under the assumptions (I) and (II), one has*

$$S(\mathcal{A},\mathcal{P}) \le (F_\kappa(s) + o(1))\, E(\mathcal{A},\mathcal{P}) \tag{1.1}$$

and

$$S(\mathcal{A},\mathcal{P}) \ge (f_\kappa(s) + o(1))\, E(\mathcal{A},\mathcal{P}). \tag{1.2}$$

Suitable functions F_κ and f_κ were given in the case $\kappa = 1$ by Jurkat and Richert [JR], and for arbitrary $\kappa > 0$ by Rosser and Iwaniec ([Iw2]; see also [Iw3]). Selberg [Se] has shown that the Rosser-Iwaniec functions are indeed best-possible in the cases $\kappa = 1/2$ and $\kappa = 1$ and thus completely solved the above extremal problem in these cases. In the general case, however, the problem of finding *optimal* functions F_κ and f_κ remains open.

While the Rosser-Iwaniec sieve is quite satisfactory in many respects and has proved to be a highly effective tool in analytic number theory, it is by no means the only possible way to formulate a sieve problem. For example, one might consider the following questions:

(i) Can one obtain better estimates by considering only special types of sets $\mathcal{A}$, such as $\mathcal{A} = \{n : n \le x\}$ or $\mathcal{A} = \{n : y < n \le x\}$?

(ii) Can one do better for special sets of primes $\mathcal{P}$, such as the set of all primes in a given interval $(y, x]$?

(iii) Can one obtain similar results with the parameter s, which, in effect, measures the size of the largest element of $\mathcal{P}$, replaced by another measure of the "size" of the set $\mathcal{P}$? An example of such a parameter is the sum of reciprocals of primes in $\mathcal{P}$.

The purpose of this paper is to address these and related questions, formulate extremal problems that arise out of these questions, and survey some of the results and open problems in this connection.

2. The case $\mathcal{A} = \{n : 1 \le n \le x\}$

We first consider what is, in a sense, the simplest possible type of a set $\mathcal{A}$, namely the case when $\mathcal{A}$ consists of all positive integers not exceeding a given bound x. We shall write in this case

$$S(x,\mathcal{P}) = S(\mathcal{A},\mathcal{P}).$$

For certain special sets of primes $\mathcal{P}$, the behavior of $S(x,\mathcal{P})$ is relatively easy to determine. For example, if $\mathcal{P}$ is of the form

$$\mathcal{P}(x^\alpha) = \{p \text{ prime} : p \le x^\alpha\}, \tag{2.1}$$

where $\alpha \in (0,1)$ is fixed, then an asymptotic formula for $S(x,\mathcal{P})$ is given by a classical result of Buchstab [Bu], which states that

$$S(x,\mathcal{P}(x^\alpha)) \sim \omega\left(\frac{1}{\alpha}\right)\frac{x}{\log x^\alpha} \quad (x\to\infty),$$

where $\omega(u)$ is the so-called Buchstab function. Similarly, if $\mathcal{P}$ is of the form

$$\mathcal{P}(x^\alpha, x) = \{p \text{ prime} : x^\alpha < p \le x\}, \tag{2.2}$$

then a result of Dickman and de Bruijn ([Di], [dB]) shows that

$$S(x,\mathcal{P}(x^\alpha,x)) \sim \rho\left(\frac{1}{\alpha}\right)x \quad (x\to\infty), \tag{2.3}$$

where $\rho(u)$, the Dickman function, is defined as the continuous solution to the system

$$\rho(u) = 1 \quad (0 \le u \le 1), \quad u\rho'(u) = -\rho(u-1) \quad (u > 1). \tag{2.4}$$

Similar asymptotic relations can be derived for other sufficiently "well-behaved" sets $\mathcal{P}$; see, for example, [GM], or [Iw1].

The problem of estimating $S(x,\mathcal{P})$ becomes much more difficult if one does not make any special assumptions on the nature of the set $\mathcal{P}$. In this case, the assumptions (I) and (II) of the Rosser-Iwaniec sieve hold with $D = x$, $\kappa = 1$, $\omega \equiv 1$, and any parameter s such that $p \le x^{1/s}$ for $p \in \mathcal{P}$. The bounds (1.1) and (1.2) then would indeed give non-trivial results in the case $\mathcal{P} \subset [2, x^\alpha]$ with $\alpha < 1/2$. However, if one only assumes that $\mathcal{P} \subset [2,x]$, then one is forced to take $s = 1$, in which case (1.1) and (1.2) reduce to

$$0 \le S(x,\mathcal{P}) \le (2e^\gamma + o(1))\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)x, \tag{2.5}$$

since $f_1(1) = 0$ and $F_1(1) = 2e^\gamma$, where γ denotes Euler's constant.

Thus, the question arises whether one can improve on the inequalities (2.5) without imposing stringent regularity conditions on the distribution of the primes in the set $\mathcal{P}$. The first step in this direction was made by R.R. Hall [Ha1] who showed that the constant 2 in the upper bound of (2.5) can be replaced by 1.

As to the lower bound in (2.5), Erdös and Ruzsa [ER] showed that if $\sum_{p\in\mathcal{P}} 1/p \le K$ for some constant K, then one has

$$S(x,\mathcal{P}) \ge (c(K) + o(1))x,$$

where $c(K) = e^{-e^{cK}}$ with an absolute constant c. This result suggests to obtain bounds on $S(x,\mathcal{P})$ in terms of the parameter $K(\mathcal{P}) = \sum_{p\in\mathcal{P}} 1/p$. More precisely, one is led in this way to the following extremal problem.

Extremal Problem II (The Erdös-Ruzsa sieve). *Find best-possible functions $G(K)$ and $g(K)$ such that, if $\mathcal{P} \subset [2, x]$ has sum of reciprocals at most K, then one has*

$$S(x, \mathcal{P}) \leq (G(K) + o(1)) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) x \tag{2.6}$$

and

$$S(x, \mathcal{P}) \geq (g(K) + o(1)) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) x. \tag{2.7}$$

The lower bound function $g(K)$ was determined in [Hi3], where it was shown that $g(K) = \rho(e^K)e^K$, where $\rho(u)$ is the Dickman function defined by (2.4). Note that if $\sum_{p \in \mathcal{P}} 1/p$ is approximately equal to K, then the product appearing on the right hand side of is essentially equal to e^{-K}. Thus, (2.7) may be restated as

$$S(x, \mathcal{P}) \geq (\rho(e^K) + o(1))x,$$

In view of the relation (2.3), this means that, subject to the condition $\sum_{p \in \mathcal{P}} 1/p \leq K$, $S(x, \mathcal{P})$ is asymptotically smallest when the set $\mathcal{P}$ is of the form $\mathcal{P}(x^\alpha, x)$ with $\alpha = e^{-K}$. Thus, in a sense, the most efficient way of sieving the integers $\leq x$ by a set of primes with sum of reciprocals bounded by a parameter K is by choosing the primes in this set as large as possible.

The determination of the upper bound extremal function $G(K)$ in (2.6) remains an open problem. By result of Hall quoted above, this function must satisfy $G(K) \leq e^\gamma$ for all K. In [Hi1] it was shown that $G(K) \leq \int_0^{e^K} \rho(u)du$, which is slightly better than Hall's bound since $\int_0^\infty \rho(u)du = e^\gamma$. However, it is unlikely that this bound gives the true value of $G(K)$.

We conclude this section by considering a variant of the above extremal problem that involves a weighted version of the sifting function $S(x, \mathcal{P})$, namely

$$T(x, \mathcal{P}) := \sum_{\substack{n \leq x \\ p|n \Rightarrow p \notin \mathcal{P}}} \frac{1}{n}.$$

Extremal Problem III (The weighted Erdös-Ruzsa sieve). *Find best-possible functions $H(K)$ and $h(K)$ such that, if $\mathcal{P} \subset [2, x]$ has sum of reciprocals at most K, then one has*

$$T(x, \mathcal{P}) \leq (H(K) + o(1)) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \log x \tag{2.8}$$

and

$$T(x, \mathcal{P}) \geq (h(K) + o(1)) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p}\right) \log x. \tag{2.9}$$

In this version, the problem can be completely solved: the optimal functions are $H(K) = \int_0^{e^K} \rho(u)du$ and $h(K) = 1$. The determination of $H(K)$ is given in [Hil] and based on a relatively simple variational argument. The fact that $h(K) = 1$ is essentially trivial; one simply has to observe that

$$\begin{aligned} T(x,\mathcal{P}) \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right)^{-1} &= \sum_{\substack{n\le x \\ p|n\Rightarrow p\notin\mathcal{P}}} \frac{1}{n} \cdot \sum_{\substack{m\ge 1 \\ p|m\Rightarrow p\in\mathcal{P}}} \frac{1}{m} \\ &\ge \sum_{n\le x} \frac{1}{n} = (1+o(1))\log x, \end{aligned}$$

and that this lower bound is best possible.

3. The case $\mathcal{A} = \{n : y < n \le y + x\}$

For simplicity we write $S(y,x,\mathcal{P})$ for $S(\mathcal{A},\mathcal{P})$ in this case and put

$$S^*(x,\mathcal{P}) = \max_{y\ge 0} S(y,x,\mathcal{P}), \quad S_*(x,\mathcal{P}) = \min_{y\ge 0} S(y,x,\mathcal{P}).$$

Thus, $S^*(x,\mathcal{P})$ and $S_*(x,\mathcal{P})$ represent the largest respectively smallest values of the sifting functions $S(\mathcal{A},\mathcal{P})$ when the sets $\mathcal{A}$ are intervals of length x. If x is an integer, it is easy to see via the Chinese Remainder Theorem that $S^*(x,\mathcal{P})$ and $S_*(x,\mathcal{P})$ are also equal to the maximum resp. minimum possible number of positive integers $n \le x$ that satisfy $n \not\equiv a_p \bmod p$ for each prime $p \in \mathcal{P}$, where $(a_p)_{p\in\mathcal{P}}$ is a given set of "forbidden" residue classes.

The problem of obtaining precise upper and lower bounds for the functions S^* and S_* is a difficult and largely unsolved one, even when $\mathcal{P}$ is of the special form $\mathcal{P}(x^\alpha)$ or $\mathcal{P}(x^\alpha, x)$ defined in (2.1) and (2.2). In the first case, the Rosser-Iwaniec sieve leads to the bounds

$$S^*(x,\mathcal{P}(x^\alpha)) \le (F(1/\alpha)+o(1)) \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right) x$$

and

$$S_*(x,\mathcal{P}(x^\alpha)) \ge (f(1/\alpha)+o(1)) \prod_{p\in\mathcal{P}} \left(1-\frac{1}{p}\right) x,$$

where $F = F_1$ and $f = f_1$. As remarked above, the functions F_κ and f_κ are best-possible under the general assumptions of the Rosser-Iwaniec sieve. However, it is not clear, whether these functions are still best possible if the sets $\mathcal{A}$ are restricted to intervals. Specifically, one can formulate the following extremal problem.

Extremal Problem IV (The interval sieve). *Find best-possible functions $F_0(s)$ and $f_0(s)$ such that*

$$S^*(x,\mathcal{P}(x^{1/s})) \le (F_0(s)+o(1)) \prod_{p\in\mathcal{P}(x^{1/s})} \left(1-\frac{1}{p}\right) x \tag{3.1}$$

and

$$S_*(x,\mathcal{P}(x^{1/s}) \geq (f_0(s)+o(1)) \prod_{p\in\mathcal{P}(x^{1/s})} \left(1-\frac{1}{p}\right) x. \tag{3.2}$$

This problem appears to have been first raised by Selberg [Se]. Very little is known about the functions f_0 and F_0, apart from the trivial inequalities

$$f(s) \leq f_0(s), \quad F(s) \geq F_0(s). \tag{3.3}$$

In particular, it is not known whether for any single value of s these inequalities are strict. Selberg observed that a resolution of this question in either way would have important consequences in prime number theory. For example, a strict inequality $F_0(s) < F(s)$ for some value $s \in [1,2]$ would imply the non-existence of Siegel zeros. In the other direction, if it were true that $f_0(2) = f(2)(=0)$, then it would follow that the sequence $\{p_n\}$ of primes satisfies $p_{n+1} - p_n > (\log p_n)^{2-\epsilon}$, a result which would imply a \$10,000 conjecture of Erdös.

While for bounded values of s virtually nothing is known about f_0 and F_0 beyond the inequalities (3.3) and their consequences, the asymptotic behavior of these functions is somewhat better understood. Indeed, in [HM] it was shown that as $s \to \infty$, one has

$$\log(F_0(s)-1) \sim \log(F(s)-1) \sim -s\log s \tag{3.4}$$

and an analogous relation for the differences $1-f_0(s)$ and $1-f(s)$.

It is easy to formulate an analogue of the Erdös-Ruzsa extremal problem for S^* and S_*, but this problem is wide open. In particular, obtaining non-trivial lower bounds for $S_*(x,\mathcal{P})$ under the condition that $\mathcal{P} \subset [2,x]$ and $\sum_{p\in\mathcal{P}} 1/p \leq K$ for some fixed K, appears to be quite difficult. In fact, it is conceivable (though, I believe, unlikely) that for some absolute constant K and all sufficiently large x it is possible to find a set $\mathcal{P} \subset [2,x]$ with sum of reciprocals at most K such that $S_*(x,\mathcal{P}) = 0$.

4. Extremal problems for sums of multiplicative functions.

The two extremal problems of Section 2 can be generalized to extremal problems for sums of multiplicative functions with values in [0,1]. This is not too surprising, since the function $S(x,\mathcal{P})$ may be written in the form $\sum_{n\leq x} f(n)$ where f is the (multiplicative) characteristic function of the set of integers that do not have a prime divisor belonging to $\mathcal{P}$. For example, Problem II can be generalized to multiplicative functions f with values in the interval $[0,1]$ by making the substitutions

$$S(x,\mathcal{P}) \longrightarrow S(x,f) = \sum_{n\leq x} f(n),$$

$$\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right) \longrightarrow \prod_{p\leq x}\left(1-\frac{1}{p}\right)\left(1+\sum_{m\geq 1}\frac{f(p^m)}{p^m}\right),$$

$$\sum_{p\in\mathcal{P}}\frac{1}{p} \longrightarrow \sum_{p\leq x}\frac{1-f(p)}{p}.$$

All results remain valid in this more general setting; in fact, in most cases, the results quoted earlier were first proved for multiplicative function, and then specialized to sieve estimates.

The case of multiplicative functions that take on complex or negative values appears to be much more difficult, and no results of a quality comparable to those obtained for functions with values in $[0,1]$ are known. We confine ourselves to mentioning two types of results in this direction. The first concerns an inequality of the form

$$\left|\sum_{n\le x} f(n)\right| \ll x \exp\left\{-K\sum_{p\le x}\frac{1-\operatorname{Re} f(p)}{p}\right\}, \tag{4.1}$$

where f is a multiplicative function satisfying $|f|\le 1$. Assuming that the values of f on primes fall into a given convex subset $\mathcal{D}$ of the unit disk, one can try to prove an estimate of the form (4.1) with a best-possible constant $K=K(\mathcal{D})$ in the exponent. This problem was investigated in recent papers by Hall and Tenenbaum [HT] and Hall [Ha2]. In particular, in [HT] the best-possible constant K was determined for the case when f has values in $[-1,1]$.

A second type of problem originated with a conjecture of Heath-Brown. In a slightly more general form the conjecture asserts that there exists a positive constant δ such that for all completely multiplicative functions f with values in the interval $[-1,1]$ and all $x\ge 1$ one has

$$\sum_{n\le x} f(n) \ge (-1+\delta)x.$$

This conjecture was recently proved by Hall [Ha3]. However, the best-possible value for the constant δ remains an open question. More precisely, if one defines

$$c(x)=\inf\left\{\frac{1}{x}\sum_{n\le x} f(n)\right\},$$

where the infimum is taken over all completely multiplicative functions with values in $[-1,1]$, what is the asymptotic behavior of $c(x)$? Hall's result only shows that $c(x)$ is bounded away from -1.

5. Extremal problems for solutions of integral equations

If f is a multiplicative function, then it is not hard to show that the averages

$$M(t)=e^{-t}\sum_{n\le e^t} f(n)$$

satisfy an approximate integral equation of the type

$$M(t)\approx\frac{1}{t}\int_0^t M(s)\phi(t-s)ds \quad (t\ge 0) \tag{5.1}$$

with a suitable function $\phi(s)$ that, roughly, represents the average of the values $f(p)$ when p is near e^s. Wirsing [Wi] appears to have been the first to recognize this connection between multiplicative functions and integral equations, and he made extensive use of this connection in his work on mean values of multiplicative functions. In light of the generalization described in the previous section of the extremal sieve problems of Section 2 to extremal problems for multiplicative functions, it is natural to expect that these problems have analogs for solutions of integral equations of the type (16). This is indeed the case, as we shall describe below.

Motivated by (5.1), we consider solutions to integral equations of the form

$$m(t) = \frac{1}{t}\int_0^t m(s)\phi(t-s)ds \quad (t>0). \tag{5.2}$$

We will assume that $\phi : \mathcal{R}^+ \to [0,1]$ is continuous and satisfies $\int_0^1 \frac{1-\phi(s)}{s}ds < \infty$. Under these assumptions it is not hard to see that (5.2) has a unique continuous solution $m(t) = m_\phi(t)$ normalized so that $m(0) = 1$. (For a more detailed discussion of the equation (5.2) see [Hi2].) One can then formulate the following extremal problem.

Extremal Problem V (Integral Equations). *Find best-possible functions $G(K)$ and $g(K)$ such that, if $\int_0^1 \frac{1-\phi(s)}{s} \le K$, then the solution $m_\phi(t)$ to (5.2) satisfies*

$$m_\phi(1) \le G(K)\exp\left\{-\int_0^1 \frac{1-\phi(s)}{s}ds\right\}$$

and

$$m_\phi(1) \ge g(K)\exp\left\{-\int_0^1 \frac{1-\phi(s)}{s}ds\right\}$$

The analogy to the extremal problem for multiplicative functions formulated in the previous section becomes apparent, if one notes that, after suitable rescaling, one has the correspondences

$$\frac{1}{x^t}\sum_{n\le x^t} f(n) \longrightarrow m(t),$$

$$f(p) \longrightarrow \phi(\log p/\log x),$$

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)\left(1+\sum_{m\ge1}\frac{f(p^m)}{p^m}\right) \longrightarrow \exp\left\{-\int_0^1 \frac{1-\phi(s)}{s}ds\right\},$$

$$\sum_{p\le x}\frac{1-f(p)}{p} \longrightarrow \int_0^1 \frac{1-\phi(s)}{s}ds.$$

Following Wirsing's approach, one can indeed show that this problem is equivalent to the corresponding extremal problem for multiplicative functions, and the extremal functions are the same for both cases. As in the arithmetic case, only the function $g(K)$ is known (namely, $g(K) = \rho(e^K)e^K$); the upper bound function $G(K)$ is unknown. On the surface, it would seem that the determination of $G(K)$ is a tractable problem in analysis, but I have not been able to make any substantial progress in this direction.

It should be noted that, while the problem of estimating the sifting function $S(x, \mathcal{P})$ can be transformed into an essentially equivalent, and perhaps tractable, problem in analysis, this is not the case with the more delicate problem of estimating the "interval sifting functions" $S^*(x, \mathcal{P})$ and $S_*(x, \mathcal{P})$ introduced in Section 3; it appears indeed that the estimation of these functions is an inherently arithmetic problem that is genuinely more difficult than that of estimating $S(x, \mathcal{P})$.

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