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Kyoto University
Parabolic fixed points of two dimensional complex dynamical systems
(2 次元複素力学系の放物型不動点について)

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0. Introduction

Let $T$ be a holomorphic mapping of a neighborhood, $V$, of the origin, $O=(0,0)\in \mathbb{C}^2$, into $\mathbb{C}^2$ with $T(O)=O$. The germ of such a mapping is called a local analytic transformation.

Let $T$ denote the set of all local analytic transformations. Local analytic transformations $T$ and $T'$ are said to be $r$-equivalent if their power series expansion at the origin coincide up to order $r$. The equivalence class is called the $r$-jet of the local analytic transformation.

Local analytic transformations $T$ and $T'$ are said to be $r$-conjugate if there is an invertible local analytic transformation $S$ such that $S^{-1}oToS$ and $T'$ are $r$-equivalent. Let $T_I=\{T\in T \mid dT(O)=id\}$, where $dT$ denotes the differential of $T$ and $id$ denotes the identity map. The elements of $T_I$ are called parabolic local analytic transformations. Ueda[2] gave a classification of 2-jets of $T_I$.

Let $E=\{P \in \mathbb{C}^2 \mid T^n(P) \rightarrow O \text{ as } n \rightarrow \infty\}$, and $D=\{P \in \mathbb{C}^2 \mid T^n \text{ converge uniformly to } O \text{ in some neighborhood of } P \text{ as } n \rightarrow \infty\}$. If $D\neq \emptyset$, then we say $O$ has a basin of attraction.

In Ueda’s list of normal forms, the case of $N_{2,1}(\lambda)$ (case I-B in our classification):

\[
\begin{align*}
x_1 &= x + \lambda x^2 + xy + \cdots \\
y_1 &= y + (\lambda + 1)xy + y^2 + \cdots
\end{align*}
\]

has a parabolic basin if $\text{Re}(\lambda)>0$. In this note, we shall prove that the fixed point of the above type has another attractive basin of a different
type. The author does not know if they are analytically conjugate or not in the basins. Since this new type of attractive basin appears as a degenerate case of parabolic basin, we call such a basin a weakly-parabolic basin.

1. 2-jets of parabolic local analytic transformations

Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ and $g : \mathbb{C}^2 \rightarrow \mathbb{C}$ be homogeneous polynomials of degree 2, and let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a parabolic analytic transformation defined by

$$F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + f(x, y) \\ y + g(x, y) \end{array} \right).$$

Let $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ defined by

$$H \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} f(x, y) \\ g(x, y) \end{array} \right)$$
denote the homogeneous part of degree 2. We have $F = id + H$.

If an invertible local analytic transformation $S$ has a linear part $L \in GL(2, \mathbb{C})$, then the 2-jet of $S^{-1} \circ F \circ S$ is given by

$$L^{-1} \circ F \circ L = id + L^{-1} \circ H \circ L.$$ 

Hence, if parabolic local transformations $F = id + H$ and $F' = id + H'$ are 2-equivalent, then there exists a linear isomorphism $L \in GL(2, \mathbb{C})$ such that

$$L^{-1} \circ H \circ L = H'$$
and vice versa. Thus, the classification of 2-jets is reduced to the classification of homogeneous polynomial maps $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ under the conjugacy $L^{-1} \circ H \circ L$ with $L \in GL(2, \mathbb{C})$. We have several cases.

**CASE I:** $f(x, y)$ and $g(x, y)$ are mutually prime.

**CASE II:** $f(x, y)$ and $g(x, y)$ have a common factor of degree one.

**CASE III:** $f(x, y)$ or $g(x, y)$ is a scalar multiple of the other (and not both zero).

**CASE IV:** both $f(x, y)$ and $g(x, y)$ are 0.

First, let us consider the case I. Let $\pi : \mathbb{C}^2 \setminus \{O\} \rightarrow \overline{\mathbb{C}}$ denote the natural projection of $\mathbb{C}^2 \setminus \{O\}$ to the Riemann sphere $\overline{\mathbb{C}}$. Homogeneous maps $H$ and $H'$ induce rational maps of degree 2 on the Riemann
sphere. We denote the induced rational maps by $[H]: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ and $[H'] : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ respectively.

**Lemma 1.1** $H$ and $H'$ are conjugate by an element of $GL(2, \mathbb{C})$ if and only if $[H]$ and $[H']$ are conjugate by a Möbius transformation.

The classification of rational functions of degree 2 under the conjugacy of Möbius transformations is well known (e.g. see Milnor[1]). A conjugacy class of rational functions of degree two is characterized by the set of three multipliers of the fixed points. The three multipliers, say $\mu_1, \mu_2, \mu_3$, are subject to the restriction

$$\mu_1 \mu_2 \mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0.$$  

These values are invariant under the conjugacies.

If $\mu_i \neq 1$ ($i = 1, 2, 3$), then the residues at each of the fixed points

$$\lambda_i = \frac{1}{2\pi \sqrt{-1}} \int \frac{dz}{[H](z) - z} = \frac{1}{\mu_i - 1}$$

give another set of holomorphic invariants. The values $\lambda_i$ are called "translation numbers" in the normal forms studied by Ueda[2]. $\lambda_1, \lambda_2, \lambda_3$ are subject to the restriction

$$\lambda_1 + \lambda_2 + \lambda_3 = -1.$$  

Ueda[2] proved the following.

**Theorem (Ueda)** If $\text{Re} \lambda_i > 0$, then $F$ has a (parabolic) basin of attraction of the fixed point $O$ which corresponds to $\lambda_i$. If $F$ is an automorphism of a complex manifold, then the basin of attraction is isomorphic to $\mathbb{C}^2$ and the dynamics in the basin is analytically conjugate to a translation.

This theorem holds also in the cases I-B and II-A-2 below. See Ueda[2] for the proof. Our case I is divided into three sub-cases.

**Case I-A:** $[H]$ has three distinct fixed points.

**Case I-B:** $[H]$ has a double fixed point and a simple fixed point.

**Case I-C:** $[H]$ has a triple fixed point.

Normal forms as 2-jets for these cases are as follows.

(I-A) \[
\begin{align*}
x_1 &= x + \lambda_1 x^2 + (\lambda_2 + 1)xy \\
y_1 &= y + (\lambda_1 + 1)xy + \lambda_2 y^2.
\end{align*}
\]
Note that in our case I-A, we exclude the case where $\lambda_i = 0$ holds for some $i$. This case is treated as case II-A-1 and III-A-1, since in this case the components of $H$ have a common factor.

The parameter $\lambda$ in the following normal form is given by $\lambda = \frac{1}{\mu_1 - 1}$, if $\mu_1 \neq 1$ and $\mu_2 = \mu_3 = 1$, for example.

(I-B) \[
\begin{align*}
x_1 &= x + \lambda x^2 + xy \\
y_1 &= y + (\lambda + 1)xy + y^2.
\end{align*}
\]

Note that in our case I-B, we exclude the case of $\lambda = 0$, in which case the induced map $[H]$ degenerates to a Möbius transformation with an indeterminate point. This case will be treated as case II-B-1.

In case I-C, we have $\mu_1 = \mu_2 = \mu_3 = 1$.

(I-C) \[
\begin{align*}
x_1 &= x + xy \\
y_1 &= y + x^2 + y^2.
\end{align*}
\]

Next, consider the case II, where $f(x,y)$ and $g(x,y)$ have a common factor and the induced map $[H]$ defines a Möbius transformation except at the indeterminate point corresponding to the common factor. We have three possibilities for the Möbius transformation $[H]$.

Case II-A: $[H]$ has two distinct fixed points.
Case II-B: $[H]$ has a double fixed point.
Case II-C: $[H]$ is the identity.

And taking the indeterminate point, originating from the common factor, into considerations, we have sub-cases as follows.

Case II-A-1: the indeterminate point is different from the fixed points.
Case II-A-2: the indeterminate point coincides with one of the fixed points of the Möbius transformation.
Case II-B-1: the indeterminate point is different from the double fixed point.
Case II-B-2: the indeterminate point coincides with the double fixed point.

The normal form of case II-A-1 is same as the case I-A. There is a restriction on the parameters. Let $\gamma \in \mathbb{C} \setminus \{0, 1\}$ denote the multiplier at one of the fixed point of the Möbius transformation. The parameters
in the normal form are given by $\lambda_1 = \frac{-\gamma}{\gamma - 1}$, $\lambda_2 = \frac{1}{\gamma - 1}$, and $\lambda_3 = 0$.

\begin{align*}
(\text{II-A-1}) & \quad \begin{cases}
x_1 &= x + \frac{\gamma}{1 - \gamma} x^2 + \frac{\gamma}{\gamma - 1} xy + \frac{1}{1 - \gamma} xy + \frac{1}{\gamma - 1} y^2, \\
y_1 &= y + \frac{1}{\gamma - 1} y^2.
\end{cases} \\
(\text{II-A-2}) & \quad \begin{cases}
x_1 &= x + \lambda x^2, \\
y_1 &= y + (\lambda + 1) xy.
\end{cases}
\end{align*}

Here, the parameter (translation number) $\lambda$ is given by $\lambda = \frac{-\gamma}{1 - \gamma}$, for multiplier $\gamma \in \mathbb{C} \setminus \{0, 1\}$ of the Möbius transformation at the indeterminate fixed point. Note that the cases $\lambda = 0$ and $\lambda = -1$ are omitted here. These cases will be treated as cases III-A-2 and III-B-1 below.

The case II-B-1 corresponds to the exceptional case of I-B with $\lambda = 0$.

\begin{align*}
(\text{II-B-1}) & \quad \begin{cases}
x_1 &= x + xy, \\
y_1 &= y + xy + y^2.
\end{cases} \\
(\text{II-B-2}) & \quad \begin{cases}
x_1 &= x + x^2, \\
y_1 &= y + x^2 + xy.
\end{cases} \\
(\text{II-C}) & \quad \begin{cases}
x_1 &= x + x^2, \\
y_1 &= y + xy.
\end{cases}
\end{align*}

In the case III, the induced map $[H]$ yields a constant function on the Riemann sphere. We have the following sub-cases according to the common factors of the components of $H$.

CASE III-A : the components $f(x, y)$ and $g(x, y)$ have two mutually prime common factors.

CASE III-B : the components $f(x, y)$ and $g(x, y)$ have a double common factor.

The common factor defines the indeterminate points of the induced map $[H]$. The value of the constant function $[H]$ is defined except at these indeterminate points. Let $v([H])$ denote the value. Taking these points into considerations, we have following sub-cases.

CASE III-A-1 : $v([H])$ is different from the indeterminate points.
CASE III-A-2 : $v([H])$ coincides with one of the indeterminate points.
CASE III-B-1 : $v([H])$ is different from the double indeterminate point.
Case III-B-2: $v([H])$ coincides with the double indeterminate point.

The case III-A-1 falls into the normal form I-A with excepted parameters $\lambda_1 = \lambda_2 = 0$, and a simpler normal form is given by

$$\begin{cases} x_1 = x \\ y_1 = y + xy + y^2. \end{cases}$$

(III-A-1)

The normal form for case III-A-2 is obtained by setting $\lambda = 0$ in II-A-2.

$$\begin{cases} x_1 = x \\ y_1 = y + xy. \end{cases}$$

(III-A-2)

The normal form for case III-B-1 is obtained by setting $\lambda = -1$ in II-A-2.

$$\begin{cases} x_1 = x \\ y_1 = y + y^2. \end{cases}$$

(III-B-1)

$$\begin{cases} x_1 = x \\ y_1 = y + x^2. \end{cases}$$

(III-B-2)

Finally, the case IV has the 2-jet normal form

$$\begin{cases} x_1 = x \\ y_1 = y. \end{cases}$$

(IV)

Here, we note the correspondence between our classification of 2-jet normal forms of parabolic analytic transformations and that of Ueda's classification[2].

<table>
<thead>
<tr>
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<th>our classification</th>
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<td>$N_1(\lambda_1, \lambda_2, \lambda_3)$</td>
<td>I-A, II-A-1, III-A-1</td>
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<tr>
<td>$N_{2,1}(\lambda)$</td>
<td>I-B, II-B-1</td>
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<td>$N_{2,2}(\lambda)$</td>
<td>II-A-2, III-A-2, III-B-1</td>
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<tr>
<td>$N_{3,1}$</td>
<td>I-C</td>
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<tr>
<td>$N_{3,2}$</td>
<td>II-B-2</td>
</tr>
<tr>
<td>$N_{3,3}$</td>
<td></td>
</tr>
<tr>
<td>$N_4$</td>
<td>II-C</td>
</tr>
<tr>
<td>$N_0$</td>
<td>IV</td>
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2. Pseudo-parabolic fixed points

In this section, we consider the case I-B. In this case, the induced map $[H]$ has a simple fixed point and a double fixed point. The translation number $\lambda$ in the normal form I-B is related to the simple fixed point. We call a fixed point of type I-B a pseudo-parabolic fixed point. We are interested in the double fixed point of $[H]$. In order to study the behavior of the local analytic transformation in the neighborhood of the pseudo-parabolic fixed point, we consider the blow-up $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of $\mathbb{C}^2$ at $O$. we denote the exceptional curve by $\Theta = \pi^{-1}(O) \simeq \mathbb{C}$. Let $V$ be the domain of definition of the transformation $T$ and let $\widetilde{V} = \pi^{-1}(V)$. The transformation induces an analytic transformation $\overline{T} : \overline{V} \rightarrow \mathbb{C}^2$. As $dT(O) = id$, all points of the exceptional curve are fixed points of $\overline{T}$.

Let us try a blow-up in our case I-B. The $x$-axis direction, \{y = 0\}, corresponds to the simple fixed point of $[H]$, and is related to the translation number $\lambda$. To see this, we may try a blow-up with $t = \frac{y}{x}$. We obtain the following local analytic transformation.

\[
\begin{align*}
(2.3) \quad & \left\{ \begin{array}{l}
x_1 = x + (\lambda + t)x^2 + \cdots \\
t_1 = t + tx + \cdots
\end{array} \right.
\end{align*}
\]

The $y$-axis direction, \{x = 0\}, corresponds to the parabolic fixed point of $[H]$. We try a blow-up with $u = \frac{x}{y}$ and obtain the following.

\[
\begin{align*}
(2.4) \quad & \left\{ \begin{array}{l}
y_1 = y + (1 + (\lambda + 1)u)y^2 + \cdots \\
u_1 = u - u^2y + \cdots
\end{array} \right.
\end{align*}
\]

Local analytic transformations arising from such a blow-up leaves the exceptional curve invariant, and all the points in the exceptional curve are fixed points. By taking a system of local coordinates around the point in the exceptional curve, we can assume, in general, that the local analytic transformation is of the following form.

\[
\begin{align*}
(2.5) \quad & \left\{ \begin{array}{l}
x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\
y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots
\end{array} \right.
\end{align*}
\]

Local analytic transformations of the form (2.5) is called a transformation of class $S_\nu$, $\nu = 0, 1, 2, \cdots$ [resp. class $S_\infty$] if $g_1(y)$ vanishes at $y = 0$ exactly with order $\nu$ [resp. vanish identically]. For $T \in S_1$, we define the translation number $\lambda$ by

$$\lambda = \frac{f_2(0)}{g_1'(0)}.$$
The translation number $\lambda$ and the multiplier $\mu$ of the corresponding simple fixed point of $[H]$ are related by $\lambda = \frac{1}{\mu - 1}$. The translation number is also a holomorphic invariant in class $S_1$.

For $T \in S$, the order of vanishing of $g_1(y)$ at $y = 0$ is invariant under those holomorphic change of coordinates which transforms the transformation of the form (2.5) into the same form.

Let $T$ be a local analytic transformation, and $T \in S_1$. The origin has a basin of attraction if the real part of the translation number is positive. We call this basin of attraction a parabolic basin of the parabolic fixed point.

Note that (2.3) is of class $S_1$ and its translation number is $\lambda$. The transformation for the double fixed point (2.4) is of class $S_2$, which shall be discussed in the following section.

3. Weakly-parabolic basin

In this section, we consider a local analytic transformation $T \in S_2$ given by

$$x_1 = x + f_2(y)x^2 + f_3(y)x^3 + \cdots$$
$$y_1 = y + g_1(y)x + g_2(y)x^2 + \cdots,$$

where $g_1(0) = 0$, $g_1'(0) = 0$, and $g_1''(0) \neq 0$.

**Theorem 3.1** If $f_2(0) \neq 0$, local analytic transformation (3.1) has a non-empty basin of attraction.

We call this attractive basin a weakly-parabolic basin. As a preliminary, we try to simplify the transformation by local change of coordinates.

**Proposition 3.2** For any $\delta \in \mathbb{C}$, by a change of coordinates $S_\alpha : (X, Y) \mapsto (x, y)$ of the form

$$x = \alpha(Y)X$$
$$y = Y,$$

where $\alpha(Y)$ is an analytic function of $Y$, transformation (3.1) can be transformed into the form

$$X_1 = X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots$$
$$Y_1 = Y + G_1(Y)X + G_2(Y)X^2 + \cdots$$

with $F_2(Y) = 1 + \delta Y + \cdots$, $G_1(0) = G_1'(0) = 0$, and $G_1''(0) \neq 0$. 
PROOF Let $\overline{T}$ denote the transformation (3.3). As $S_\alpha$ is a local automorphism, we have $\alpha(0) \neq 0$ and

$$T \circ S_\alpha = S_\alpha \circ \overline{T}$$

holds. Expand the both sides as power series in $X$ with analytic functions in $Y$ as coefficients. By comparing the coefficients of both sides, we have

\begin{equation}
(3.4) \quad f_2(Y)(\alpha(Y))^2 = \alpha(Y)F_2(Y) + \alpha'(Y)G_1(Y)
\end{equation}

and

\begin{equation}
(3.5) \quad G_1(Y) = \alpha(Y)g_1(Y).
\end{equation}

The function $\alpha(Y)$ must satisfy the differential equation

\begin{equation}
(3.6) \quad f_2(Y)\alpha(Y) - g_1(Y)\alpha'(Y) = F_2(Y),
\end{equation}

with $\alpha(0) \neq 0$. Let

$$a_0 = \frac{1}{f_2(0)},$$

$$a_1 = \frac{1}{f_2(0)}(\delta - a_0f_2'(0)) = \frac{1}{f_2(0)}(\delta - \frac{f_2'(0)}{f_2(0)})$$

and choose the analytic function $\alpha(Y)$ as, for example,

$$\alpha(Y) = a_0 + a_1 Y.$$

We obtain the desired change of coordinates of the proposition. As $\alpha(0) = a_0 \neq 0$, the conditions for $G_1(Y)$ are satisfied.

Especially, as we have $G_1''(0) = f_2(0)g_1''(0)$, we can take $\delta = G_1''(0)/2 = f_2(0)g_1''(0)/2$ to be used in the following proposition.

**Proposition 3.3** Assume $T \in S_2$ and $f_2(y) = 1 + \delta y + O(y^2)$, with $\delta = \frac{g_1''(0)}{2}$. By a change of coordinates $S_\beta : (X, Y) \mapsto (x, y)$ of the form

\begin{equation}
(3.7) \quad \begin{cases}
x & = & X \\
y & = & \beta(Y),
\end{cases}
\end{equation}

with $\beta(0) = 0$, $\beta'(0) \neq 0$, $T$ can be transformed into $\widetilde{T} : (X, Y) \mapsto (X_1, Y_1)$,

\begin{equation}
(3.8) \quad \begin{cases}
X_1 & = & X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\
Y_1 & = & Y + G_1(Y)X + G_2(Y)X^2 + \cdots
\end{cases}
\end{equation}
with \( F_2(Y) = 1 + Y + O(Y^2) \) and \( G_1(Y) = Y^2 + O(Y^3) \).

**Proof** Compare both sides of \( T \circ S_\beta = S_\beta \circ \overline{T} \) as power series in \( X \) and obtain

\[
f_2(\beta(Y)) = F_2(Y), \quad \text{and} \quad g_1(\beta(Y)) = \beta'(Y)G_1(Y).
\]

Let \( \beta(Y) = \frac{2}{g_1(0)}Y \), for example, to get \( G_1(Y) = Y^2 + O(Y^3) \).

Note that, here, generally, a term of order 3 cannot be suppressed by an analytic change of coordinates. We have, also,

\[
F_2(Y) = f_2(\beta(Y)) = 1 + Y + O(Y^2).
\]

**Proposition 3.4** Let \( T : (x, y) \mapsto (x_1, y_1) \) be a local analytic transformation of the form

\[
\begin{align*}
x_1 &= x + f_2(y)x^2 + f_3(y)x^3 + \cdots \\
y_1 &= y + g_1(y)x + g_2(y)x^2 + \cdots,
\end{align*}
\]

and let \( S : (X, Y) \mapsto (x, y) \) be a change of local coordinates of the form

\[
\begin{align*}
x &= \alpha_1(Y)X + \alpha_2(Y)X^2 + \alpha_3(Y)X^3 + \cdots \\
y &= \beta_0(Y) + \beta_1(Y)X + \beta_2(Y)X^2 + \cdots.
\end{align*}
\]

Let \( \overline{T} : (X, Y) \mapsto (X_1, Y_1) \) be the transformation given by \( \overline{T} = S^{-1} \circ T \circ S \), with

\[
\begin{align*}
X_1 &= X + F_2(Y)X^2 + F_3(Y)X^3 + \cdots \\
Y_1 &= Y + G_1(Y)X + G_2(Y)X^2 + \cdots.
\end{align*}
\]

Then we have the followings.

\[
G_1(Y) = \frac{\alpha_1(Y)}{\beta_0'(Y)}g_1(\beta_0(Y))
\]

and

\[
F_2(Y) = \alpha_1(Y)f_2(\beta_0(Y)) - \frac{\alpha_1'(Y)}{\beta_0(Y)}g_1(\beta_0(Y)).
\]

**Proof** These are verified by an immediate computation.

**Proposition 3.5** Assume \( T \in S_2 \) is of the form (3.9) with \( f_2(y) = 1 + y + O(y^2) \) and \( g_1(y) = y^2 + O(y^3) \). By a local change of coordinates \( S \) of the form (3.10), the transformation \( T \) can be transformed into \( \overline{T} \) of the form (3.11) with \( F_2(Y) = f_2(Y) \), \( G_1(Y) = g_1(Y) \) and \( G_2(Y) = 0 \).
We set $\alpha_1(\mathrm{Y}) = 1$ and $\beta_0(\mathrm{Y}) = \mathrm{Y}$. Then proposition 3.4 guarantees that $G_1(\mathrm{Y}) = g_1(\mathrm{Y})$ and $F_2(\mathrm{Y}) = f_2(\mathrm{Y})$. Compute $S \circ \overline{T}$ and $T \circ S$ to compare the coefficients of $X^2$ in $y_1$. We get

$$G_2(\mathrm{Y}) = g_2(\mathrm{Y}) + \beta_1(\mathrm{Y}) (g_1'(\mathrm{Y}) - f_2(\mathrm{Y})) + g_1(\mathrm{Y}) (\alpha_2(\mathrm{Y}) - \beta_1'(\mathrm{Y})).$$

Hence, if we set

$$\beta_1(\mathrm{Y}) = \frac{g_2(\mathrm{Y})}{f_2(\mathrm{Y}) - g_1'(\mathrm{Y})},$$

and

$$\alpha_2(\mathrm{Y}) = \beta_1'(\mathrm{Y}),$$

we get $G_2(\mathrm{Y}) = 0$. As $f_2(\mathrm{Y}) = 1 + \mathrm{Y} + O(\mathrm{Y}^2)$ and $g_1'(\mathrm{Y}) = O(\mathrm{Y})$, $\beta_1(\mathrm{Y})$ is analytic near the origin.

4. Proof of theorem 3.1

By propositions in the previous section, we can assume

$$f_2(\mathrm{y}) = 1 + \mathrm{y} + O(\mathrm{y}^2),$$

$$g_1(\mathrm{y}) = \mathrm{y}^2 + O(\mathrm{y}^3),$$

and

$$g_2(\mathrm{y}) = 0$$

to prove theorem 3.1. Then, the transformation $T : (x, \mathrm{y}) \mapsto (x_1, y_1)$, $T \in S_2$, takes the following form

$$\begin{align*}
\{ x_1 &= x + (1 + \mathrm{y}) x^2 + O(\mathrm{y}^2 x^2) + O(\mathrm{x}^3) \\
y_1 &= \mathrm{y} + \mathrm{y}^2 x + O(\mathrm{y}^3 x) + O(\mathrm{x}^3),
\end{align*}$$

(4.1)

where $O(\varphi(\mathrm{x}, \mathrm{y}))$ implies some analytic function, say $\psi(\mathrm{x}, \mathrm{y})$, which can be written as $\psi(\mathrm{x}, \mathrm{y}) = \varphi(\mathrm{x}, \mathrm{y}) \rho(\mathrm{x}, \mathrm{y})$ for some analytic function $\rho(\mathrm{x}, \mathrm{y})$ in a neighborhood of the origin.

As we are interested in the behavior of the transformation in the $y$-axis direction near the origin, let us blow-up the origin along the $y$-axis. More precisely, we change the coordinates by

$$u = \frac{x}{y}, \quad v = \mathrm{y}$$

(4.2)

into new coordinates $(u, v)$. The origin $(0, 0)$ of $(x, \mathrm{y})$-coordinates corresponds to the exceptional curve $\overline{\mathbb{C}} \times \{0\}$ in the $(u, v)$-coordinates.
In the \((u, v)\)-coordinates, (4.1) takes the form

\[
\begin{align*}
\{ u_1 &= u + vu^2 + O(v^3u^2) + O(v^2u^3) \\
v_1 &= v + v^3u + O(v^4u) + O(v^3u^3).
\end{align*}
\]

(4.3)

Let us take a new system of coordinates defined by

\[
z = \frac{1}{u}, \quad w = \frac{1}{v},
\]

Then (4.3) is transformed into the form

\[
\begin{align*}
z_1 &= z - \frac{1}{w} h_1(z, w) \\
w_1 &= w - \frac{1}{zw} h_2(z, w),
\end{align*}
\]

(4.5)

where \( h_1(z, w) = 1 + O(\frac{1}{zw}) + O(\frac{1}{w}) \) and \( h_2(z, w) = 1 + O(\frac{1}{z^2}) + O(\frac{1}{w}) \).

We regard (4.5) as a transformation near \((\infty, \infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}\).

Take constants \( \theta_0, \theta_1, \theta_2 \) such that

\[
0 < \theta_0 < \frac{1}{8} \pi, \quad 0 < \theta_2 < \frac{1}{8} \theta_0, \quad \text{and} \quad \theta_0 + \theta_2 < \theta_1 < \frac{5}{4} \theta_0 - \theta_2.
\]

Note that \( 0 < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3} \) holds.

Choose \( r_1 \) and \( r_2 \) such that \( \frac{3}{4} < r_1 < 1 < r_2 < \frac{5}{4} \) and let

\[
\Omega = \{ z \in \mathbb{C} | |\arg_{\mathcal{Z}}| < \theta_2, r_1 < |z| < r_2 \}.
\]

For \( R_1, R_2 > 0 \), let

\[
U = \{ z \in \mathbb{C} | |\arg(-z)| < \theta_1, \text{Re} \ z < -R_1 \}
\]

and

\[
V = \{ w \in \mathbb{C} | |\arg w| < \theta_0, \text{Re} \ w > R_2 \}.
\]

Choose sufficiently large \( R_1 \) and \( R_2 \) such that

\[
h_1(z, w) \in \Omega \quad \text{and} \quad h_2(z, w) \in \Omega
\]

holds for all \((z, w) \in U \times V\), and that

\[
r_2 < R_1 R_2^2 \sin(\frac{\theta_0}{2}).
\]

Let \( \Phi: (z, w) \mapsto (z_1, w_1) \) denote the transformation (4.5) defined near \((\infty, \infty) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}\).

**Proposition 4.1** If \((z, w) \in U \times V\), then \( \Phi(z, w) = (z_1, w_1) \in U \times V\), \( \text{Re} \ z_1 < \text{Re} \ z \), and \( \text{Re} \ w_1 > \text{Re} \ w \).
**Proof** Let \((z, w) \in U \times V\). Then

\[
|\arg(\frac{1}{w}h_1(z, w))| < \theta_0 + \theta_2 < \theta_1
\]

and

\[
\text{Re}(\frac{1}{w}h_1(z, w)) > 0.
\]

Hence \(z_1 \in U\) and \(\text{Re} z_1 < \text{Re} z\) follow. Now, let \(\theta = \arg w\). Then \(-\theta_0 < \theta < \theta_0\). Note that

\[
|\arg(-\frac{1}{zw}h_2(z, w))| < \theta_0 + \theta_1 + \theta_2 < \frac{\pi}{3}
\]

and

\[
\text{Re}(-\frac{1}{zw}h_2(z, w)) > 0.
\]

First, consider the case where \(\frac{\theta_0}{2} < \theta < \theta_0\). In this case, we have

\[
\arg(-\frac{1}{zw}h_2(z, w)) < -\theta + \theta_1 + \theta_2 < \theta_0.
\]

So, we have \(\arg w_1 < \theta_0\) and \(\text{Re} w_1 > w > R_2\). On the other hand,

\[
|w_1 - w| = \left|\frac{1}{zw}h_2(z, w)\right| < \frac{r_2}{R_1R_2} < R_2 \sin \frac{\theta_0}{2}.
\]

Hence \(w_1 \in V\) in this case.

Similarly, if \(-\theta_0 < \theta < -\frac{\theta_0}{2}\), we have \(w_1 \in V\).

Next, if \(|\theta| \leq \frac{\theta_0}{2}\), we have

\[
\text{Re}(-\frac{1}{zw}h_2(z, w)) > 0 \quad \text{and} \quad |w_1 - w| < R_2 \sin \frac{\theta_0}{2},
\]

which imply \(w_1 \in V\) and \(\text{Re} w_1 > \text{Re} w\). Thus proposition 4.1 is proved.

Theorem 3.1 is a corollary of this proposition.

**References**
