On the Boundary of unbounded invariant Fatou Components of Entire Functions

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1 Definitions and Results

Let $f$ be a transcendental entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F_f$ and the Julia set $J_f$ of $f$ are defined as follows:

$$F_f := \{ z \in \mathbb{C} \mid \{ f^n \}_{n=1}^{\infty} \text{ is a normal family in a neighborhood of } z \},$$
$$J_f := \mathbb{C} \setminus F_f.$$

It is possible to consider the Julia set to be a subset of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \}$ by adding the point of infinity $\infty$ to it. This definition is mainly adopted in the case of meromorphic functions (for example, see [Ber]) and also there are some researches on convergence phenomena of Julia sets as subsets of $\hat{\mathbb{C}}$ ([Ki], [Kr], [KrK]). In this setting, $J_f$ is compact in $\hat{\mathbb{C}}$ and hence $J_f$ is rather easy to handle. But for a transcendental entire function the suitable phase space as a dynamical system is the complex plane $\mathbb{C}$, not the Riemann sphere $\hat{\mathbb{C}}$, because $\infty$ is an essential singularity of $f$ and there seems to be no reasonable way to define the value at $\infty$. So it is more natural to regard $J_f$ as a subset of $\mathbb{C}$ rather than of $\hat{\mathbb{C}}$ and hence we define $J_f$ as above and write $J_f \cup \{ \infty \}$ when we consider $J_f$ to be a subset of $\hat{\mathbb{C}}$.

A connected component $U$ of $F_f$ is called a Fatou component. A Fatou component is called a wandering domain if $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$. If there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$, $U$ is called a periodic component and it is well known that there are following four possibilities:
1. There exists a point \( z_0 \in U \) with \( f^{n_0}(z_0) = z_0 \) and \( |(f^{n_0})'(z_0)| < 1 \) and every point \( z \in U \) satisfies \( f^{n_0k}(z) \to z_0 \) as \( k \to \infty \). The point \( z_0 \) is called an attracting periodic point and the domain \( U \) is called an attracting basin.

2. There exists a point \( z_0 \in \partial U \) with \( f^{n_0}(z_0) = z_0 \) and \( (f^{n_0})'(z_0) = e^{2\pi i \theta} (\theta \in \mathbb{Q}) \) and every point \( z \in U \) satisfies \( f^{n_0k}(z) \to z_0 \) as \( k \to \infty \). The point \( z_0 \) is called a parabolic periodic point and the domain \( U \) is called a parabolic basin.

3. There exists a point \( z_0 \in U \) with \( f^{n_0}(z_0) = z_0 \) and \( (f^{n_0})'(z_0) = e^{2\pi i \theta} (\theta \in \mathbb{R} \setminus \mathbb{Q}) \) and \( f^{n_0}|U \) is conjugate to an irrational rotation of a unit disk. The domain \( U \) is called a Siegel disk.

4. For every \( z \in U \), \( f^{n_0k}(z) \to \infty \) as \( k \to \infty \). The domain \( U \) is called a Baker domain.

In particular, if \( n_0 = 1 \), \( U \) is called an invariant component. \( U \) is called completely invariant if \( U \) satisfies \( f^{-1}(U) \subseteq U \). \( U \) is called a preperiodic component if \( f^m(U) \) is a periodic component for an \( m \geq 1 \). \( U \) is called eventually periodic if \( U \) is periodic or preperiodic. It is known that eventually periodic components of a transcendental entire function are simply connected ([Ber], [EL1]) while a wandering domain can be multiply-connected ([Ba1], [Ba2], [Ba5]).

The boundary of unbounded periodic Fatou component can be extremely complicated. For example, consider the exponential family \( E_\lambda(z) := \lambda e^z \). If \( \lambda \) satisfies \( 0 < \lambda < \frac{1}{e} \), \( E_\lambda(z) \) has a unique attracting fixed point \( p_\lambda \) with an unbounded simply connected completely invariant basin \( \Omega(p_\lambda) \) and the Fatou set \( F_{E_\lambda} \) is equal to this basin ([DG]). Let \( \varphi : \mathbb{D} \to \Omega(p_\lambda) \) be a Riemann map of \( \Omega(p_\lambda) \) from a unit disk \( \mathbb{D} \), then the radial limit \( \lim_{r \nearrow 1} \varphi(re^{i\theta}) \) exists for all \( e^{i\theta} \in \partial \mathbb{D} \) and moreover the set

\[
\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}
\]

is dense in \( \partial \mathbb{D} \) ([DG]). This implies that the Riemann map is highly discontinuous and hence the boundary of \( \Omega(p_\lambda) \), which is equal to \( J_{E_\lambda} \), is extremely complicated. From this fact, it follows that \( J_{E_\lambda} \) is disconnected in \( \mathbb{C} \), since \( \varphi \) is conformal the set

\[
\varphi(\{re^{i\theta_1} \mid 0 \leq r < 1 \} \cup \{re^{i\theta_2} \mid 0 \leq r < 1 \}) \subset U \quad (\theta_1, \theta_2 \in \Theta_\infty, \ \theta_1 \neq \theta_2)
\]
is a Jordan arc in $\mathbb{C}$ and this separates $J_{E_{\lambda}}$ into two disjoint relatively open subsets.

Taking these facts into account, we shall investigate the set $\Theta_\infty$ for a genetal unbounded periodic component $U$ and also consider the following problem

**Problem**: When is the Julia set of a transcendental entire function $f$ connected or disconnected as a subset of $\mathbb{C}$?

If $f$ is a polynomial, the following criterion is well known. (For example, see [Bea] or [M]).

**Proposition A** Let $f$ be a polynomial of degree $d \geq 2$. Then the Julia set $J_f$ is connected if and only if no finite critical values of $f$ tend to $\infty$ by the iterates of $f$.

Here, a critical value is a point $p := f(c)$ for a point $c$ with $f'(c) = 0$. This is a singularity of $f^{-1}$. For polynomials we have only to consider this type of singularities but there can be another type of singularities called an asymptotic value for the transcendental case. A point $p$ is called an asymptotic value if there exists a continuous curve $L(t)$ $(0 \leq t < 1)$ called an asymptotic path with

$$
\lim_{t \to 1} L(t) = \infty \quad \text{and} \quad \lim_{t \to 1} f(L(t)) = p.
$$

A point $p$ is called a singular value if it is either a critical or an asymptotic value and we denote the set of all singular values as $\text{sing}(f^{-1})$.

If $f$ is transcendental, however, the above criterion does not hold. For example, let us consider the exponential family $E_{\lambda}(z) := \lambda e^z$ again. If $\lambda$ satisfies $0 < \lambda < \frac{1}{e}$, the unique singular value $z = 0$ (this is an asymptotic value) is attracted to the fixed point $p_{\lambda}$ and hence does not tend to $\infty$ but the Julia set $J_{E_{\lambda}}$ is disconnected as we mentioned above.

For other values of $\lambda$, for example $\lambda > \frac{1}{e}$, the singular value $z = 0$ may tend to $\infty$. If $f$ is a polynomial all of whose critical values tend to $\infty$, then $J_f$ is a Cantor set and especially disconnected. But on the other hand in this case $J_f$ is equal to the entire plain $\mathbb{C}$ ([D]) and hence connected.

Before considering the connectivity of $J_f$ in $\mathbb{C}$, we investigate the connectivity of $J_f \cup \{ \infty \}$ in $\widehat{\mathbb{C}}$. In this situation compactness of $J_f \cup \{ \infty \}$ in $\widehat{\mathbb{C}}$ makes the problem easier. Actually we can prove the following:
**Theorem 1** Let $f$ be a transcendental entire function. Then the set $J_f \cup \{\infty\}$ in $\mathbb{C}$ is connected if and only if $F_f$ has no multiply-connected wandering domains.

**Corollary 1** Under one of the following conditions, $J_f \cup \{\infty\}$ in $\mathbb{C}$ is connected.

1. $f \in B := \{f \mid \text{sing}(f^{-1}) \text{ is bounded}\}$.
2. $F_f$ has an unbounded component.
3. There exists a curve $\Gamma(t)$ ($0 \leq t < 1$) with $\lim_{t \to 1} \Gamma(t) = \infty$ such that $f|\Gamma$ is bounded. Especially $f$ has a finite asymptotic value.

Then how about $J_f$ in $\mathbb{C}$ itself? The results depend on whether $F_f$ admits an unbounded component or not. In the case when $F_f$ admits no unbounded components, we obtain the following:

**Theorem 2** Let $f$ be a transcendental entire function. If all the components of $F_f$ are bounded and simply connected, then $J_f$ is connected.

The following is an easy consequence from Theorem 1 and 2.

**Corollary 2** Let $f$ be a transcendental entire function. If all the components of $F_f$ are bounded, then $J_f$ is connected in $\mathbb{C}$ if and only if $J_f \cup \{\infty\}$ is connected in $\mathbb{C}$.

As we mentioned before, for the unbounded component $\Omega(p_{\lambda})$ of $F_{E_{\lambda}}$ the set of all angles where the Riemann map $\varphi : \mathbb{D} \to \Omega(p_{\lambda})$ admits the radial limit $\infty$ is dense in $\partial\mathbb{D}$ and this leads to the disconnectivity of $J_{E_{\lambda}}$. The Main result of this paper is the generalization of this fact. Under some conditions this result holds for various kinds of unbounded periodic Fatou components. Here, a point $p \in \partial U$ is accessible if there exists a continuous curve $L(t)$ ($0 \leq t < 1$) in $U$ with $\lim_{t \to 1} L(t) = p$.

**Main Theorem** Let $U$ be an unbounded periodic Fatou component of a transcendental entire function $f$, $\varphi : \mathbb{D} \to U$ be a Riemann map of $U$ from a unit disk $\mathbb{D}$, and

$$P_{f_{n_0}} := \bigcup_{n=0}^{\infty} (f_{n_0})^n(\text{sing}((f_{n_0})^{-1})).$$

We assume one of the following four conditions:

1. $U$ is an attracting basin of period $n_0$ and $\infty \in \partial U$ is accessible. There
exists a finite point $q \in \partial U$ with $q \notin P_{f^{m_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U \ (0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

(2) $U$ is a parabolic basin of period $n_0$ and $\infty \in \partial U$ is accessible. There exists a finite point $q \in \partial U$ with $q \notin P_{f^{m_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U \ (0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

(3) $U$ is a Siegel disk of period $n_0$ and $\infty \in \partial U$ is accessible.

(4) $U$ is a Baker domain of period $n_0$ and $f^{n_0}|U$ is not univalent. There exists a finite point $q \in \partial U$ with $q \notin P_{f^{m_0}}$, $m_0 \in \mathbb{N}$ and a continuous curve $C(t) \subset U \ (0 \leq t \leq 1)$ with $C(1) = q$ and satisfies $f^{m_0}(C) \supset C$.

Then the set

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}$$

is dense in $\partial \mathbb{D}$ in the case of (1), (2) or (3). In the case of (4), the closure $\overline{\Theta_\infty}$ contains a certain perfect set in $\partial \mathbb{D}$. In particular, $J_f$ is disconnected in all cases.

In the case of the exponential family, Devaney and Goldberg ([DG]) obtained the explicit expression

$$\varphi^{-1} \circ E_\lambda \circ \varphi(z) = \exp i \left( \frac{\mu + \bar{\mu}z}{1 + z} \right), \quad \mu \in \{z \mid \Im z > 0\}$$

for a suitable Riemann map $\varphi$ which was crucial to show the density of $\Theta_\infty$ in $\partial \mathbb{D}$. In general, of course, we cannot obtain the explicit form of $\varphi^{-1} \circ f^{m_0} \circ \varphi(z)$ so instead of it we take advantage of a property of inner functions. In general analytic function $g : \mathbb{D} \rightarrow \mathbb{D}$ is called an inner function if the radial limit $g(e^{i\theta}) := \lim_{r \nearrow 1} g(re^{i\theta})$ exists for almost every $e^{i\theta} \in \partial \mathbb{D}$ and satisfies $|g(e^{i\theta})| = 1$. It is easy to see that $\varphi^{-1} \circ f^{m_0} \circ \varphi$ is an inner function. It is known that an inner function $g$ has a unique fixed point $p \in \mathbb{D}$ called a Denjoy-Wolff point and $g^n(z)$ tends to $p$ locally uniformly on $\mathbb{D}$ ([DM]). The following is an important lemma for the proof of the Main Theorem.

**Lemma 1** Let $g : \mathbb{D} \rightarrow \mathbb{D}$ be an inner function which is not a Möbius transformation and $p$ its Denjoy-Wolff point.

(1) If $p \in \mathbb{D}$, then $\bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset \partial \mathbb{D}$ holds for every $z_0 \in \mathbb{D} \setminus E$ where $E$ is a certain exceptional set of logarithmic capacity zero.

(2) If $p \in \partial \mathbb{D}$, then $\bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset K$ holds for every $z_0 \in \mathbb{D} \setminus E$ where $E$ is a certain exceptional set of logarithmic capacity zero and $K$ is a certain perfect set in $\partial \mathbb{D}$. 
If $U$ is either an attracting basin or a parabolic basin and $g = \varphi^{-1} \circ f^{n_0} \circ \varphi$, we can say more about the set $\cup_{n=-1}^{\infty} g^{-n}(z_0)$.

**Lemma 2** Let $U$ be either an attracting basin or a parabolic basin (not necessarily unbounded) and $g = \varphi^{-1} \circ f^{n_0} \circ \varphi$. Then there exists a set $E \subset \mathbb{D}$ of logarithmic capacity zero such that

$$\frac{\sigma_n(z_0, A)}{\sigma_n(z_0, \partial \mathbb{D})} \rightarrow \frac{\text{meas} A}{2\pi} \quad (n \rightarrow \infty)$$

holds for every $z_0 \in \mathbb{D} \setminus E$ and every arc $A$ in $\partial \mathbb{D}$, where $\sigma_n(z_0, A) = \sum (1 - |\zeta|^2)$ and sum is taken over all $\zeta = |\zeta|e^{i\theta}$ with $g^n(\zeta) = z_0$ and $e^{i\theta} \in A$.

The conclusion of Lemma 2 is stronger than that of Lemma 1 (1), because it implies not only that the inverse images $g^{-n}(z_0)$ accumulate on all over $\partial \mathbb{D}$ but also that their distribution is uniform on $\partial \mathbb{D}$. We shall not give the definition of logarithmic capacity here (see [P2]). But we recall that a set of logarithmic capacity zero is extremely thin: it cannot contain a connected set with more than one point and its Hausdorff dimension is zero ([DM], [P2]).

In §2 we prove Theorem 1 and Corollary 1. §3 consists of three subsections. In §3.1 we prove Theorem 2 and make some remarks on the sufficient conditions for $f$ to admit no unbounded Fatou components. In §3.2 we prove Lemma 1 and Lemma 2 which are keys for the proof of the Main Theorem. In §3.3 we prove the Main Theorem.

## 2 Connectivity of $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$

**Proof of Theorem 1**: The following criterion is well known. (See for example [Bea], p.81, Proposition 5.1.5).

**Proposition B** Let $K$ be a compact subset in $\widehat{\mathbb{C}}$. Then $K$ is connected if and only if each component of the complement $K^c$ is simply connected.

Since $J_f \cup \{\infty\}$ is compact in $\widehat{\mathbb{C}}$, we can apply Proposition B. As we mentioned in §1, eventually periodic components are simply connected. So if a Fatou component $U$ is not simply connected, then $U$ is necessarily
a wandering domain which is not simply connected. This completes the proof.

(Proof of Corollary 1): Under the condition (1), $f^n$ cannot tend to $\infty$ through $F_f$ ([EL2]). On the other hand, $f^n$ tends to $\infty$ on any multiply-connected wandering domains ([Ba4], [EL1]). So all the Fatou components are simply connected in this case. Under the condition (2) or (3), it is known that all the Fatou components must be simply connected ([Ba4], [EL1], p.620 Corollary 1, 2).

Remark 1 (1) Let $S := \{f | \#\text{sing}(f^{-1}) < \infty\} \subset B$. Then there is even no wandering domain in $F_f$ for $f \in S$ ([GK]). For $f \in B$, $F_f$ may admit a wandering domain $U$ but $U$ must be simply connected as we mentioned above. Under an additional condition

$$J_f \cap \left(\text{derived set of } \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1}))\right) = \emptyset,$$

$f \in B$ has also no wandering domain ([BHKMT]).

(2) We can conclude that in general if $J_f \cup \{\infty\}$ is disconnected, all the Fatou components are bounded and some of which are multiply-connected wandering domains.

3 Connectivity of $J_f$ in $\mathbb{C}$

3.1 The case when all the Fatou components are bounded

Suppose that a closed connected subset $K$ in $\mathbb{C}$ is bounded. Then all the components of the complement $K^c$ other than the unique unbounded component $V$ are simply connected. (Of course, $V \cup \{\infty\} \subset \mathbb{C}$ is simply connected). If $K$ is unbounded, then all the components of $K^c$ are simply connected, but the converse is false as the example $J_{E\lambda}(0 < \lambda < \frac{1}{e})$ shows. (Compare with the Proposition B). But note that $J_{E\lambda} \cup \{\infty\}$ is connected in $\mathbb{C}$. For the connectivity of a closed subset in $\mathbb{C}$, the following criterion holds.

Proposition 1 Let $K$ be a closed subset of $\mathbb{C}$. Then $K$ is connected if and only if the boundary of each component $U$ of the complement $K^c$ is connected.
(Proof): For the 'only if' part, see [New]. Suppose that $K$ is disconnected. Then there exist two closed sets $K_1$ and $K_2$ with $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. Take a point $z_0$ with $d(z_0, K_1) = d(z_0, K_2)$ where $d$ denotes the Euclid distance in $\mathbb{C}$. Then $z_0 \in K^c$ and so let $U_0$ be the connected component of $K^c$ containing $z_0$. Since $\partial U_0$ is connected by the assumption, either $\partial U_0 \subset K_1$ or $\partial U_0 \subset K_2$. Without loss of generality we can assume $\partial U_0 \subset K_1$. On the other hand denote $r_0 := d(z_0, K_1) = d(z_0, K_2)$ and let $D_{r_0}(z_0) := \{z \mid |z - z_0| < r_0\}$. Then $\overline{D_{r_0}(z_0)} \subset U_0$ and there exists a point $w \in K_2$ with $w \in \overline{U_0}$. Since $w \in K_2 \subset K$, we have $w \in \partial U_0$ but this is a contradiction since $\partial U_0 \subset K_1$ and $K_1 \cap K_2 = \emptyset$. This completes the proof.

(Proof of Theorem 2): By Proposition 1, it is sufficient to to show that the boundary $\partial U$ is connected for each Fatou component $U$. Since $U$ is bounded, the boundary of $U$ as a subset of $\mathbb{C}$ and the one as the subset of $\mathbb{C}$ coincide. Hence $U$ is simply connected if and only if $\partial U$ is connected ([Bea], p.81, Proposition 5.1.4). This completes the proof.

Remark 2 (1) Since a non-simply connected Fatou component is necessarily a wandering domain, the assumption of Theorem 2 is equivalent to that all the components of $F_f$ are bounded and $F_f$ admits no multiply-connected wandering domains.

(2) Several sufficient conditions are known for a transcendental entire function $f$ to admit no unbounded Fatou components as follows:

(i) ([Ba3]) $\log M(r) = O((\log r)^p)$ (as $r \to \infty$) where $M(r) = \sup_{|z|=r} |f(z)|$ and $1 < p < 3$.

(ii) ([S]) There exists $\varepsilon \in (0, 1)$ such that $\log \log M(r) < \frac{(\log r)^{\frac{3}{2}}}{(\log \log r)^\varepsilon}$ for large $r$.

(iii) ([S]) The order of $f$ is less than $\frac{1}{2}$ and $\log M(2r) / \log M(r) \to c$ (finite constant) as $r \to \infty$.

Note that the condition (ii) includes the condition (i).

3.2 A property of inner functions

(Proof of Lemma 1): If $g$ is a finite Blaschke product, then $g$ is a rational function of degree $d \geq 2$. It is well known that in general the
closure of the set of all the inverse images of $z_0$ by a rational function $R$ of degree $d \geq 2$ contains its Julia set $J_R$ for any $z_0$ which is not a Fatou exceptional point ([Bea], p.79, Theorem 4.2.7). If the Denjoy-Wolff point $p$ is in $\mathbb{D}$, then $J_g = \partial \mathbb{D}$ and if the Denjoy-Wolff point $p$ is in $\partial \mathbb{D}$, then $J_g = \partial \mathbb{D}$ or at least $J_g$ is a perfect set in $\partial \mathbb{D}$. In any cases all the inverse images of $z_0$ are in $\mathbb{D}$ for every $z_0 \in \mathbb{D}$. So our assertion holds for

$$E = E(g) := \{ z \mid z \text{ is a Fatou exceptional point for } g \}$$

and we have $\#E(g) \leq 2$, which implies that $E$ is a set of logarithmic capacity zero.

If $g$ is not a finite Blaschke product, then by Frostman’s theorem ([G], p.79, Theorem 6.4) there exists a set $E_1 \subset \mathbb{D}$ of capacity zero such that $T_a \circ g$ is a Blaschke product for every $a \in \mathbb{D} \setminus E_1$, where $T_a(z) := \frac{z-a}{1-\overline{a}z}$.

Therefore $B := T_a \circ g \circ T_a^{-1}$ is also a Blaschke product. By applying Frostman’s theorem to each $g^n$, we obtain the set $\cup_{n=1}^{\infty} E_i$ of logarithmic capacity zero such that $T_a \circ g^n \circ T_a^{-1} = B^n$ holds and each $B^n$ is a Blaschke product for every $a \in \mathbb{D} \setminus (\cup_{n=1}^{\infty} E_i)$. Now it is sufficient to prove our lemma for $B$, so we concentrate on a fixed $a \in \mathbb{D} \setminus (\cup_{n=1}^{\infty} E_i)$ and corresponding Blaschke product $B = T_a \circ g \circ T_a^{-1}$. Let $A_n \subset \partial \mathbb{D}$ be the set of accumulation points of $B^{-n}(0)$, then $A_n$ is closed and $B^n$ can be analytically continued to a meromorphic function on $\mathbb{C} \setminus A_n$ by the reflection principle ([G], p.75, Theorem 6.1). In other words, $A_n$ is equal to the set of singularities of $B^n$ (that is, points at which $B^n(z)$ does not extend analytically). There exists a set $E_{B_n}$ of logarithmic capacity zero such that $A_n$ is equal to the set of accumulation points of $B^{-n}(p)$ for $p \in \mathbb{D} \setminus E_{B_n}$ ([G], Theorem 6.6). Let $E := \cup_{n=1}^{\infty} E_{B_n}$, then $E$ is a set of capacity zero and for every $z_0 \in \mathbb{D} \setminus E$ we have $\cup_{n=1}^{\infty} B^{-n}(z_0) \supset \cup_{n=1}^{\infty} A_n$.

First let us consider the case when the Denjoy-Wolff point $p$ is in $\mathbb{D}$. Suppose that $\cup_{n=1}^{\infty} B^{-n}(z_0) \supset \partial \mathbb{D}$ does not hold for a $z_0 \in \mathbb{D} \setminus E$, then there exists a open set $V$ with $V \cap \partial \mathbb{D} \neq \emptyset$ such that $B^n$ can be defined on $V$ for every $n \in \mathbb{N}$ and $V \cap (\cup_{n=1}^{\infty} B^{-n}(z_0)) = \emptyset$. We take $V$ as the maximal set satisfying this property. Let $W := V \cap \partial \mathbb{D}$. Since $B$ is not a finite Blaschke product, we have $\#\{B^{-1}(0)\} = \infty$ and so $A_1 \neq \emptyset$. Hence for a $z_0 \in \mathbb{D} \setminus E$ we have $W \neq \partial \mathbb{D}$. So there exists a point $\alpha \in \partial \mathbb{D} \setminus W$. Since $B^n$ cannot take the values $z_0, \frac{1}{z_0}$ and $\alpha$, $\{B^n[V]\}_{n=1}^{\infty}$ is a normal family. Then by the dynamics of $B$ on $\mathbb{D}$, we have $B^n|V \rightarrow p$ locally uniformly. But on the other hand $B^n|(V \cap (\mathbb{D})^c) \rightarrow \frac{1}{p}$ by the construction of the extension,
which is a contradiction. Hence $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset \partial D$ holds in this case.

Next we consider the case when the Denjoy-Wolff point $p$ is on the boundary of $D$. Let $K := \bigcup_{n=1}^{\infty} A_n$ and suppose that $K \neq \partial D$. Then $B^n$ is defined on $\mathbb{C} \setminus K$ for every $n \in \mathbb{N}$. Obviously $K$ is closed. If $K$ consists of a single point, say $\beta$, then we have $B(\partial D \setminus \{\beta\}) \subset \partial D \setminus \{\beta\}$ and $B((\partial D \setminus \{\beta\})$ is one to one since $B$ is extended by the reflection principle. It follows that $B$ is a Möbius transformation, which is a contradiction. By the similar argument, we can prove $\# K \geq 3$. Then $K$ cannot have an isolated point. If this is not the case, let $\beta \in K$ be an isolated point. Then $\beta$ is an essential singularity and hence by Picard's theorem, $B$ takes all but exceptional two values in $\mathbb{C}$ infinitely often. This contradicts the fact that $B(\mathbb{C} \setminus K) \subset \mathbb{C} \setminus K$ and $\# K \geq 3$. Therefore it follows that $K$ is a perfect set. Since $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset K$ holds for every $z_0 \in D \setminus E$, this completes the proof.

(Proof of Lemma 2): In the case when $U$ is an attracting basin, the result is a special case of Theorem 3 in [P1]. In the case when $U$ is a parabolic basin, the result follows by combining the series of theorems in [DM] (Theorem 6.1, Theorem 4.2, Corollary 4.3, Theorem 3.1) together with the Theorem 3 in [P1].

3.3 In the case when $F_f$ admits an unbounded component — On the Boundary of unbounded invariant Fatou Components

(Proof of Main Theorem): In what follows we assume that $n_0 = 1$ (that is, $U$ is an invariant component) and $m_0 = 1$ for simplicity. This causes no loss of generality, because we have only to consider $f^{m_0}$ instead of $f$ in general cases.

Case (1) Since $\infty$ is accessible, there exists a continuous curve $L(t)$ ($0 \leq t < 1$) in $U$ with $\lim_{t \to 1} L(t) = \infty$. By deforming $L(t)$ slightly, we construct a new curve $L(t)$ satisfying the following condition.

Lemma 3 There exists a curve $L(t)$ ($0 \leq t < 1$) with $\lim_{t \to 1} L(t) = \infty$ such that every branch of $f^{-n}$ can be analytically continued along it for every $n \in \mathbb{N}$.

(Proof): We may assume that $L(0) \notin P_f$, since $q \notin P_f$ we have $U \not\subset P_f$. Let $p_0 := L(0), p_1, p_2, \ldots$ be points on $L$ such that all the piecewise linear line segments connecting $p_0, p_1, p_2, \ldots$ lie in $U$. Let $F_{n}^{(1)}, F_{n}^{(2)}, \ldots, F_{n}^{(m)}, \ldots$
be all the branches of $f^{-n}$ which take values on $U$. The range of the suffix $m$ may be finite or infinite. Define

$$\Theta^{(m)}_{n}(p_{0}) := \{e^{i\theta} \mid F^{(m)}_{n} \text{can be analytically continued along the ray}$$
$$\text{from} \ p_{0} \ \text{in the direction} \ \theta \} \quad (n = 1, 2, \ldots).$$

Then by the next Gross's Star Theorem ([Nev]), it follows that $\Theta^{(m)}_{n}(p_{0})$ has full measure in $\partial \mathbb{D}$.

**Lemma C (Gross's Star Theorem)** Let $f$ be an entire function and $F$ a branch of $f^{-1}$ defined in the neighborhood of $p_{0} \in \mathbb{C}$. Then $F$ can be analytically continued along almost all rays from $p_{0}$ in the direction $\theta$.

Then the set

$$\Theta(p_{0}) := \bigcap_{n \geq 1, m \geq 1} \Theta^{(m)}_{n}(p_{0})$$

has also full measure in $\partial \mathbb{D}$. Hence by changing $p_{1}$ slightly to a point $p'_{1}$, the segments $\overline{p_{0}p'_{1}}$ and $\overline{p'_{1}p_{2}}$ lie in $U$ and all the branches $F^{(m)}_{n}$ ($n \geq 1, \ m \geq 1$) can be analytically continued along $\overline{p_{0}p'_{1}}$. By the same method, we can find a point $p'_{2}$ close to $p_{2}$ such that the segment $\overline{p'_{1}p'_{2}}$ lies in $U$ and has the same property as above. By repeating this argument, we can prove the Lemma 3. \qed

Let $l^{(m)}_{n}(t) := F^{(m)}_{n}(L(t))$ then we have $\lim_{t \to 1} l^{(m)}_{n}(t) = \infty$. For suppose this is false, then there exist an increasing sequence of parameter values $t_{1} < t_{2} < \cdots < t_{k} < \cdots$ and a finite point $\alpha$ with $\lim_{k \to \infty} l^{(m)}_{n}(t_{k}) = \alpha \neq \infty$. Then it follows that $\lim_{k \to \infty} L(t_{k}) = f^{n}(\alpha) \neq \infty$ and this contradicts the fact $\lim_{k \to \infty} L(t_{k}) = \infty$.

Let $\varphi : \mathbb{D} \to U$ be a Riemann map of $U$. Then

$$\Gamma(t) := \varphi^{-1}(L(t)) \quad \text{and} \quad \gamma^{(m)}_{n}(t) := \varphi^{-1}(l^{(m)}_{n}(t))$$

are curves in $\mathbb{D}$ landing at a point in $\partial \mathbb{D}$. This fact is not so trivial but follows from the proposition in [P2] (p.29, Proposition 2.14). We may assume that $\Gamma(t)$ lands at $z = 1 \in \partial \mathbb{D}$ for simplicity. If $\lim_{t \to 1} \gamma^{(m_{0})}_{n_{0}}(t) = e^{i\theta_{0}}$, then since $\lim_{t \to 1} \varphi(\gamma^{(m_{0})}_{n_{0}}(t)) = \lim_{t \to 1} l^{(m_{0})}_{n_{0}}(t) = \infty$, it follows that there exists the radial limit $\lim_{r \to 1} \varphi(re^{i\theta_{0}})$ and this is equal to $\infty$. This fact follows from the theorem in [P2] (p.34, Theorem 2.16). Therefore it is sufficient to show that the set of all the landing points of $\gamma^{(m)}_{n}(t)$ ($n \geq 1, m \geq 1$) is dense in $\partial \mathbb{D}$.
Let \( g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \to \mathbb{D} \). Then by Fatou’s theorem \( \varphi \) has radial limit \( \varphi(e^{i\theta}) = \lim_{r \to 1} \varphi(re^{i\theta}) \in \partial U \) and non-constant for almost every \( e^{i\theta} \in \partial \mathbb{D} \). Hence \( f \circ \varphi(re^{i\theta}) \) is a curve landing at a point in \( \partial U \setminus \{\infty\} \) for almost every \( e^{i\theta} \in \partial \mathbb{D} \). Therefore it follows that \( \lim_{r \to 1} \varphi^{-1} \circ f \circ \varphi(re^{i\theta}) \in \partial \mathbb{D} \) a.e. and thus \( g \) is an inner function. Let \( \overline{C} := \varphi^{-1}(C) \) then by the same reason for \( \Gamma(t) \), \( \overline{C} \) is a curve in \( \mathbb{D} \) with an end point \( \bar{q} \in \partial U \) satisfying \( g(\overline{C}) \supset \overline{C} \).

From the dynamics of \( g : \mathbb{D} \to \mathbb{D} \), it follows that the set \( \bigcup_{n=0}^{\infty} g^n(\overline{C}) \cup \{\bar{p}, \bar{q}\} \) is compact in \( \mathbb{D} \) where \( \bar{p} = \varphi^{-1}(p) \) and \( \bar{p} \) is an attracting fixed point of \( g \) and the distance between this set and \( z = 1 \) is positive. Hence there exists \( \varepsilon_0 > 0 \) such that

\[
U_{\varepsilon_0}(1) \cap \left\{ \bigcup_{n=0}^{\infty} g^n(\overline{C}) \cup \{\bar{p}, \bar{q}\} \right\} = \emptyset
\]  

(1)

Since \( \Gamma(t) \) lands at \( z = 1 \), there exists \( t_0 \in [0,1) \) such that \( \Gamma|[t_0,1) \subset U_{\varepsilon_0}(1) \). So by rewriting \( \Gamma|[t_0,1) \) to \( \Gamma(t) (0 \leq t < 1) \) we may assume that \( \Gamma(t) \subset U_{\varepsilon_0}(1) \) for \( 0 \leq t < 1 \). Let \( K := \{z \mid |z| \leq 1 - \varepsilon_0\} \) then since every point in \( \mathbb{D} \) tends to \( \bar{p} \) under \( g^n \) and \( K \) is compact, there exists \( n_1 \in \mathbb{N} \) such that for every \( N \geq n_1 \) we have \( g^N(K) \subset U_{\varepsilon}(\bar{p}) \). Then we have \( \gamma^{(m)}_N(t) \subset \overline{C} \) for every \( N \geq n_1 \). For suppose that \( \gamma^{(m)}_N(t) \cap K \neq \emptyset \), then by operating \( f^N \) we have \( \Gamma(t) \cap K \neq \emptyset \) which contradicts \( \Gamma(t) \subset U_{\varepsilon_0}(1) \).

Now suppose that the conclusion does not hold. Then there exists

\[
(\theta_1, \theta_2) := \{e^{i\theta} \mid \theta_1 < \theta < \theta_2\} \subset \partial \mathbb{D} \quad \text{with} \quad \Theta_{\infty} \cap (\theta_1, \theta_2) = \emptyset.
\]

By changing the starting point \( \Gamma(0) \) slightly, if necessary, we may assume that the points \( \gamma_n^{(m)}(0) \) (\( n, m = 1, 2, \cdots \)) accumulate to all over \( \partial \mathbb{D} \) by Lemma 1 (1) while the end points \( \gamma_n^{(m)}(1) := \lim_{t \to 1} \gamma_n^{(m)}(t) \) (\( n, m = 1, 2, \cdots \)) are not in \( (\theta_1, \theta_2) \). Therefore there exists \( \gamma^{(m)}_{n_1}(t) \) such that \( \gamma^{(m)}_{n_1}(t) \subset K^c \) and \( \gamma^{(m)}_{n_1}(1) \in \partial \mathbb{D} \setminus (\theta_1, \theta_2) \).

On the other hand there exist inverse images \( g^{-n}(\overline{C}) \) which have limit points on \( (\theta_1, \theta_2) \) densely. The reason is as follows: Since \( q \notin P_f \), there exists a neighborhood \( V \) of \( q \) such that all the branches \( F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)} \) can be defined. Let \( V_0 \subset V \) is a neighborhood of \( q \) with \( \overline{V_0} \subset V \). We may assume that \( C \subset V_0 \). Define

\[
c_n^{(m)}(t) := F_n^{(m)}(C(t)), \quad \overline{c}_n^{(m)}(t) := \varphi^{-1}(c_n^{(m)}(t)).
\]

Then \( c_n^{(m)}(t) \) is a curve in \( U \) landing at a point in \( \partial U \) and \( \overline{c}_n^{(m)}(t) \) is a curve in \( \mathbb{D} \) landing at a point in \( \partial \mathbb{D} \) by the same reason as before. Let
$(\theta_3, \theta_4) \subset (\theta_1, \theta_2)$ be any subarc of $(\theta_1, \theta_2)$. By changing the starting point \(\tilde{C}(0)\) slightly, if necessary, we may assume that the points \(c_{n}^{(m)}(0) \quad (n, m = 1, 2, \ldots)\) accumulate to $(\theta_3, \theta_4)$ by Lemma 1 (1). Since radial limits of \(\varphi\) exist and non-constant almost everywhere, by changing \(\theta_3\) and \(\theta_4\) slightly if necessary, we may assume that there exist the finite values \(\varphi(e^{i\theta_3})\) and \(\varphi(e^{i\theta_4})\) with \(\varphi(e^{i\theta_3}) \neq \varphi(e^{i\theta_4})\). Then \(c_{n}^{(m)}(0)\) accumulate on \(\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})\). In general the family of single-valued analytic branch of \(f^{-n} \quad (n = 1, 2, \ldots)\) on a domain \(U_0\) is normal and furthermore if \(U_0 \cap J_f \neq \emptyset\), any local uniform limit of a subsequence in the family is constant ([Bea], p.193, Theorem 9.2.1, Lemma 9.2.2). So the family \(\{F_{n}^{(m)}|V_0\} \) is normal and all its limit functions are constant and hence for a suitable subsequence the diameter of \(c_{n_k}^{(m_k)}(t)\) tends to zero, that is, \(c_{n_k}^{(m_k)}(t)\) must land at a point in \(\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})\) if the constant limit is finite. Therefore \(c_{n_k}^{(m_k)}(t)\) must land at a point in \((\theta_3, \theta_4)\). If the constant limit is \(\infty\), for large enough \(n_k\) the curves \(c_{n_k}^{(m_k)}\) cannot intersect both \(\varphi(re^{i\theta_3}) \mid 0 \leq r \leq 1\) and \(\varphi(re^{i\theta_4}) \mid 0 \leq r \leq 1\) which are bounded set, since the convergence is uniform on \(V_0\). Hence again we can conclude that \(c_{n_k}^{(m_k)}(t)\) must land at a point in \(\partial U \cap \varphi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})\) and therefore \(c_{n_k}^{(m_k)}(t)\) must land at a point in \((\theta_3, \theta_4)\). This proves the assertion.

Then there exists \(c_{N_1}^{(M_1)}\) such that \(\gamma_{n_1}^{(m_1)} \cap c_{N_1}^{(M_1)} \neq \emptyset\). We may assume that \(n_1 > N_1\). Let \(u \in \gamma_{n_1}^{(m_1)} \cap c_{N_1}^{(M_1)}\) then since \(u \in \gamma_{n_1}^{(m_1)}\), we have \(g^{n_1}(u) \in U_{\varepsilon_2}(1)\). On the other hand since \(u \in c_{N_1}^{(M_1)}\) and \(n_1 > N_1\), we have \(g^{n_1}(u) \in \bigcup_{n=0}^{\infty} g^n(\tilde{C})\) which contradicts (1). Therefore \(\Theta_{\infty}\) is dense in \(\partial \mathbb{D}\). Disconnectivity of \(J_f\) easily follows by the same argument as in the case of \(E_{\lambda}\) in §1. This completes the proof in the case of (1).

**Case (2)** The proof is quite parallel to the case (1). Note that by Lemma 2, \(\bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset \partial \mathbb{D} \quad (z_0 \in \mathbb{D} \setminus E)\) holds for \(g = \varphi^{-1} \circ f \circ \varphi\) in this case. □

**Case (3)** Since \(g(z) = e^{2\pi i \theta_0}\) with \(\theta_0 \in \mathbb{R} \setminus \mathbb{Q}\), the inverse image of \(\Gamma(t)\) by \(g^{-n}\) is unique and denote it by \(\gamma_n(t)\). Then it is obvious that the end points of \(\gamma_n(t)\) are dense in \(\partial \mathbb{D}\) and \(\varphi\) attains radial limit \(\infty\) there, since \(g(z)\) is an irrational rotation and

\[
\lim_{t \to 1} \varphi(\gamma_n(t)) = \lim_{t \to 1} f^{-1}(\varphi(\Gamma(t))) = \infty.
\]

**Case (4)** In this case we need not assume the accessibility of \(\infty\), because this condition is automatically satisfied ([Ba6]). The set \(\bigcup_{n=0}^{\infty} f^n(C)\) is a
curve which may have self-intersections and tends to $\infty$. It is not difficult to take $L$ satisfying $L \cap \bigcup_{n=0}^{\infty} f^n(C) = \emptyset$. Hence we have $\mathcal{L} \cap \bigcup_{n=0}^{\infty} f^n(C) = \emptyset$. The rest of the proof is quite parallel to the case (1) if the conclusion of Lemma 2 (1) holds for $g$. If we have only the conclusion of Lemma 2 (2), then we can prove that for every arc $A \subset \partial \mathbb{D}$ with $A \cap K \neq \emptyset$, $A \cap \Theta_{\infty} \neq \emptyset$ holds by the similar argument. □

References


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