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On the Boundary of unbounded invariant Fatou Components of Entire Functions

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1 Definitions and Results

Let $f$ be a transcendental entire function and $f^n$ denote the $n$-th iterate of $f$. Recall that the Fatou set $F_f$ and the Julia set $J_f$ of $f$ are defined as follows:

\[ F_f := \{ z \in \mathbb{C} \mid \{f^n\}_{n=1}^\infty \text{ is a normal family in a neighborhood of } z \}, \]
\[ J_f := \mathbb{C} \setminus F_f. \]

It is possible to consider the Julia set to be a subset of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by adding the point of infinity $\infty$ to it. This definition is mainly adopted in the case of meromorphic functions (for example, see [Ber]) and also there are some researches on convergence phenomena of Julia sets as subsets of $\hat{\mathbb{C}}$ ([Ki], [Kr], [KrK]). In this setting, $J_f$ is compact in $\hat{\mathbb{C}}$ and hence $J_f$ is rather easy to handle. But for a transcendental entire function the suitable phase space as a dynamical system is the complex plane $\mathbb{C}$, not the Riemann sphere $\hat{\mathbb{C}}$, because $\infty$ is an essential singularity of $f$ and there seems to be no reasonable way to define the value at $\infty$. So it is more natural to regard $J_f$ as a subset of $\mathbb{C}$ rather than of $\hat{\mathbb{C}}$ and hence we define $J_f$ as above and write $J_f \cup \{\infty\}$ when we consider $J_f$ to be a subset of $\hat{\mathbb{C}}$.

A connected component $U$ of $F_f$ is called a Fatou component. A Fatou component is called a wandering domain if $f^m(U) \cap f^n(U) = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$. If there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$, $U$ is called a periodic component and it is well known that there are following four possibilities:
1. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $|(f^{n_0})'(z_0)| < 1$ and every point $z \in U$ satisfies $f^{n_0k}(z) \to z_0$ as $k \to \infty$. The point $z_0$ is called an attracting periodic point and the domain $U$ is called an attracting basin.

2. There exists a point $z_0 \in \partial U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i \theta}$ ($\theta \in \mathbb{Q}$) and every point $z \in U$ satisfies $f^{n_0k}(z) \to z_0$ as $k \to \infty$. The point $z_0$ is called a parabolic periodic point and the domain $U$ is called a parabolic basin.

3. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i \theta}$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$) and $f^{n_0}|U$ is conjugate to an irrational rotation of a unit disk. The domain $U$ is called a Siegel disk.

4. For every $z \in U$, $f^{n_0k}(z) \to \infty$ as $k \to \infty$. The domain $U$ is called a Baker domain.

In particular, if $n_0 = 1$, $U$ is called an invariant component. $U$ is called completely invariant if $U$ satisfies $f^{-1}(U) \subseteq U$. $U$ is called a preperiodic component if $f^m(U)$ is a periodic component for an $m \geq 1$. $U$ is called eventually periodic if $U$ is periodic or preperiodic. It is known that eventually periodic components of a transcendental entire function are simply connected ([Ber], [EL1]) while a wandering domain can be multiply-connected ([Ba1], [Ba2], [Ba5]).

The boundary of unbounded periodic Fatou component can be extremely complicated. For example, consider the exponential family $E_{\lambda}(z) := \lambda e^z$. If $\lambda$ satisfies $0 < \lambda < \frac{1}{e}$, $E_{\lambda}(z)$ has a unique attracting fixed point $p_{\lambda}$ with an unbounded simply connected completely invariant basin $\Omega(p_{\lambda})$ and the Fatou set $F_{E_{\lambda}}$ is equal to this basin ([DG]). Let $\varphi : \mathbb{D} \to \Omega(p_{\lambda})$ be a Riemann map of $\Omega(p_{\lambda})$ from a unit disk $\mathbb{D}$, then the radial limit $\lim_{r \nearrow 1} \varphi(re^{i\theta})$ exists for all $e^{i\theta} \in \partial \mathbb{D}$ and moreover the set

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}$$

is dense in $\partial \mathbb{D}$ ([DG]). This implies that the Riemann map is highly discontinuous and hence the boundary of $\Omega(p_{\lambda})$, which is equal to $J_{E_{\lambda}}$, is extremely complicated. From this fact, it follows that $J_{E_{\lambda}}$ is disconnected in $\mathbb{C}$, since $\varphi$ is conformal the set

$$\varphi(\{re^{i\theta_1} \mid 0 \leq r < 1\} \cup \{re^{i\theta_2} \mid 0 \leq r < 1\}) \subset U \quad (\theta_1, \theta_2 \in \Theta_\infty, \ \theta_1 \neq \theta_2)$$
is a Jordan arc in \( \mathbb{C} \) and this separates \( J_{E\lambda} \) into two disjoint relatively open subsets.

Taking these facts into account, we shall investigate the set \( \Theta_{\infty} \) for a general unbounded periodic component \( U \) and also consider the following problem

**Problem**: When is the Julia set of a transcendental entire function \( f \) connected or disconnected as a subset of \( \mathbb{C} \)?

If \( f \) is a polynomial, the following criterion is well known. (For example, see [Bea] or [M]).

**Proposition A** Let \( f \) be a polynomial of degree \( d \geq 2 \). Then the Julia set \( J_f \) is connected if and only if no finite critical values of \( f \) tend to \( \infty \) by the iterates of \( f \).

Here, a critical value is a point \( p := f(c) \) for a point \( c \) with \( f'(c) = 0 \). This is a singularity of \( f^{-1} \). For polynomials we have only to consider this type of singularities but there can be another type of singularities called an asymptotic value for the transcendental case. A point \( p \) is called an asymptotic value if there exists a continuous curve \( L(t) \) \((0 \leq t < 1)\) called an asymptotic path with

\[
\lim_{t \to 1} L(t) = \infty \quad \text{and} \quad \lim_{t \to 1} f(L(t)) = p.
\]

A point \( p \) is called a singular value if it is either a critical or an asymptotic value and we denote the set of all singular values as \( \text{sing}(f^{-1}) \).

If \( f \) is transcendental, however, the above criterion does not hold. For example, let us consider the exponential family \( E_{\lambda}(z) := \lambda e^z \) again. If \( \lambda \) satisfies \( 0 < \lambda < \frac{1}{e} \), the unique singular value \( z = 0 \) (this is an asymptotic value) is attracted to the fixed point \( p_{\lambda} \) and hence does not tend to \( \infty \) but the Julia set \( J_{E\lambda} \) is disconnected as we mentioned above.

For other values of \( \lambda \), for example \( \lambda > \frac{1}{e} \), the singular value \( z = 0 \) may tend to \( \infty \). If \( f \) is a polynomial all of whose critical values tend to \( \infty \), then \( J_f \) is a Cantor set and especially disconnected. But on the other hand in this case \( J_f \) is equal to the entire plain \( \mathbb{C} \) ([D]) and hence connected.

Before considering the connectivity of \( J_f \) in \( \mathbb{C} \), we investigate the connectivity of \( J_f \cup \{\infty\} \) in \( \mathring{\mathbb{C}} \). In this situation compactness of \( J_f \cup \{\infty\} \) in \( \mathring{\mathbb{C}} \) makes the problem easier. Actually we can prove the following:
Theorem 1 Let \( f \) be a transcendental entire function. Then the set \( J_f \cup \{\infty\} \) in \( \hat{\mathbb{C}} \) is connected if and only if \( F_f \) has no multiply-connected wandering domains.

Corollary 1 Under one of the following conditions, \( J_f \cup \{\infty\} \) in \( \hat{\mathbb{C}} \) is connected.

1. \( f \in B := \{ f \mid \text{sing}(f^{-1}) \text{ is bounded}\} \).
2. \( F_f \) has an unbounded component.
3. There exists a curve \( \Gamma(t) \) \( (0 \leq t < 1) \) with \( \lim_{t \to 1} \Gamma(t) = \infty \) such that \( f|\Gamma \) is bounded. Especially \( f \) has a finite asymptotic value.

Then how about \( J_f \) in \( \mathbb{C} \) itself? The results depend on whether \( F_f \) admits an unbounded component or not. In the case when \( F_f \) admits no unbounded components, we obtain the following:

Theorem 2 Let \( f \) be a transcendental entire function. If all the components of \( F_f \) are bounded and simply connected, then \( J_f \) is connected.

The following is an easy consequence from Theorem 1 and 2.

Corollary 2 Let \( f \) be a transcendental entire function. If all the components of \( F_f \) are bounded, then \( J_f \) is connected in \( \mathbb{C} \) if and only if \( J_f \cup \{\infty\} \) is connected in \( \hat{\mathbb{C}} \).

As we mentioned before, for the unbounded component \( \Omega(p_\lambda) \) of \( F_{E_\lambda} \) the set of all angles where the Riemann map \( \varphi : \mathbb{D} \to \Omega(p_\lambda) \) admits the radial limit \( \infty \) is dense in \( \partial \mathbb{D} \) and this leads to the disconnectivity of \( J_{E_\lambda} \).

The Main result of this paper is the generalization of this fact. Under some conditions this result holds for various kinds of unbounded periodic Fatou components. Here, a point \( p \in \partial U \) is accessible if there exists a continuous curve \( L(t) \) \( (0 \leq t < 1) \) in \( U \) with \( \lim_{t \to 1} L(t) = p \).

Main Theorem Let \( U \) be an unbounded periodic Fatou component of a transcendental entire function \( f \), \( \varphi : \mathbb{D} \to U \) be a Riemann map of \( U \) from a unit disk \( \mathbb{D} \), and

\[
P_{f_{n_0}} := \bigcup_{n=0}^{\infty} (f_{n_0})^n(\text{sing}((f_{n_0})^{-1})).
\]

We assume one of the following four conditions:

1. \( U \) is an attracting basin of period \( n_0 \) and \( \infty \in \partial U \) is accessible. There
exists a finite point \( q \in \partial U \) with \( q \notin P_{f^n_0} \), \( m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \( (0 \leq t \leq 1) \) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

(2) \( U \) is a parabolic basin of period \( n_0 \) and \( \infty \in \partial U \) is accessible. There exists a finite point \( q \in \partial U \) with \( q \notin P_{f^n_0} \), \( m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \( (0 \leq t \leq 1) \) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

(3) \( U \) is a Siegel disk of period \( n_0 \) and \( \infty \in \partial U \) is accessible.

(4) \( U \) is a Baker domain of period \( n_0 \) and \( f^{m_0} | U \) is not univalent. There exists a finite point \( q \in \partial U \) with \( q \notin P_{f^n_0} \), \( m_0 \in \mathbb{N} \) and a continuous curve \( C(t) \subset U \) \( (0 \leq t \leq 1) \) with \( C(1) = q \) and satisfies \( f^{m_0}(C) \supset C \).

Then the set

\[
\Theta_{\infty} := \{ e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty \}
\]

is dense in \( \partial \mathbb{D} \) in the case of (1), (2) or (3). In the case of (4), the closure \( \overline{\Theta_{\infty}} \) contains a certain perfect set in \( \partial \mathbb{D} \). In particular, \( J_f \) is disconnected in all cases.

In the case of the exponential family, Devaney and Goldberg ([DG]) obtained the explicit expression

\[
\varphi^{-1} \circ E_\lambda \circ \varphi(z) = \exp \left( \frac{\mu + \mu z}{1 + z} \right), \quad \mu \in \{ z \mid \text{Im } z > 0 \}
\]

for a suitable Riemann map \( \varphi \) which was crucial to show the density of \( \Theta_{\infty} \) in \( \partial \mathbb{D} \). In general, of course, we cannot obtain the explicit form of \( \varphi^{-1} \circ f^{m_0} \circ \varphi(z) \) so instead of it we take advantage of a property of inner functions. In general analytic function \( g : \mathbb{D} \rightarrow \mathbb{D} \) is called an inner function if the radial limit \( g(e^{i\theta}) := \lim_{r \nearrow 1} g(re^{i\theta}) \) exists for almost every \( e^{i\theta} \in \partial \mathbb{D} \) and satisfies \( |g(e^{i\theta})| = 1 \). It is easy to see that \( \varphi^{-1} \circ f^{m_0} \circ \varphi \) is an inner function. It is known that an inner function \( g \) has a unique fixed point \( p \in \mathbb{D} \) called a Denjoy-Wolff point and \( g^n(z) \) tends to \( p \) locally uniformly on \( \mathbb{D} \) ([DM]). The following is an important lemma for the proof of the Main Theorem.

**Lemma 1** Let \( g : \mathbb{D} \rightarrow \mathbb{D} \) be an inner function which is not a Möbius transformation and \( p \) its Denjoy-Wolff point.

(1) If \( p \in \mathbb{D} \), then \( \bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset \partial \mathbb{D} \) holds for every \( z_0 \in \mathbb{D} \setminus E \) where \( E \) is a certain exceptional set of logarithmic capacity zero.

(2) If \( p \in \partial \mathbb{D} \), then \( \bigcup_{n=1}^{\infty} g^{-n}(z_0) \supset K \) holds for every \( z_0 \in \mathbb{D} \setminus E \) where \( E \) is a certain exceptional set of logarithmic capacity zero and \( K \) is a certain perfect set in \( \partial \mathbb{D} \).
If $U$ is either an attracting basin or a parabolic basin and $g = \varphi^{-1} \circ f^{n_0} \circ \varphi$, we can say more about the set $\bigcup_{n=1}^{\infty} g^{-n}(z_0)$.

**Lemma 2** Let $U$ be either an attracting basin or a parabolic basin (not necessarily unbounded) and $g = \varphi^{-1} \circ f^{n_0} \circ \varphi$. Then there exists a set $E \subset \mathbb{D}$ of logarithmic capacity zero such that

$$\frac{\sigma_n(z_0, A)}{\sigma_n(z_0, \partial \mathbb{D})} \rightarrow \frac{\text{meas} A}{2\pi} \quad (n \rightarrow \infty)$$

holds for every $z_0 \in \mathbb{D} \setminus E$ and every arc $A$ in $\partial \mathbb{D}$, where $\sigma_n(z_0, A) = \sum (1 - |\zeta|^2)$ and sum is taken over all $\zeta = |\zeta|e^{i\theta}$ with $g^n(\zeta) = z_0$ and $e^{i\theta} \in A$.

The conclusion of Lemma 2 is stronger than that of Lemma 1 (1), because it implies not only that the inverse images $g^{-n}(z_0)$ accumulate on all over $\partial \mathbb{D}$ but also that their distribution is uniform on $\partial \mathbb{D}$. We shall not give the definition of logarithmic capacity here (see [P2]). But we recall that a set of logarithmic capacity zero is extremely thin: it cannot contain a connected set with more than one point and its Hausdorff dimension is zero ([DM], [P2]).

In §2 we prove Theorem 1 and Corollary 1. §3 consists of three subsections. In §3.1 we prove Theorem 2 and make some remarks on the sufficient conditions for $f$ to admit no unbounded Fatou components. In §3.2 we prove Lemma 1 and Lemma 2 which are keys for the proof of the Main Theorem. In §3.3 we prove the Main Theorem.

## 2 Connectivity of $J_f \cup \{\infty\}$ in $\widehat{\mathbb{C}}$

**Proposition B** Let $K$ be a compact subset in $\widehat{\mathbb{C}}$. Then $K$ is connected if and only if each component of the complement $K^c$ is simply connected.

Since $J_f \cup \{\infty\}$ is compact in $\widehat{\mathbb{C}}$, we can apply Proposition B. As we mentioned in §1, eventually periodic components are simply connected. So if a Fatou component $U$ is not simply connected, then $U$ is necessarily
a wandering domain which is not simply connected. This completes the proof.

(Proof of Corollary 1): Under the condition (1), $f^n$ cannot tend to $\infty$ through $F_f$ ([EL2]). On the other hand, $f^n$ tends to $\infty$ on any multiply-connected wandering domains ([Ba4], [EL1]). So all the Fatou components are simply connected in this case. Under the condition (2) or (3), it is known that all the Fatou components must be simply connected ([Ba4], [EL1], p.620 Corollary 1, 2).

Remark 1 (1) Let $S := \{ f \mid \# \text{sing}(f^{-1}) < \infty \} \subset B$. Then there is even no wandering domain in $F_f$ for $f \in S$ ([GK]). For $f \in B$, $F_f$ may admit a wandering domain $U$ but $U$ must be simply connected as we mentioned above. Under an additional condition

$$J_f \cap \left( \text{derived set of } \bigcup_{n=0}^{\infty} f^n(\text{sing}(f^{-1})) \right) = \emptyset,$$

$f \in B$ has also no wandering domain ([BHKMT]).

(2) We can conclude that in general if $J_f \cup \{ \infty \}$ is disconnected, all the Fatou components are bounded and some of which are multiply-connected wandering domains.

3 Connectivity of $J_f$ in $\mathbb{C}$

3.1 The case when all the Fatou components are bounded

Suppose that a closed connected subset $K$ in $\mathbb{C}$ is bounded. Then all the components of the complement $K^c$ other than the unique unbounded component $V$ are simply connected. (Of course, $V \cup \{ \infty \} \subset \overline{\mathbb{C}}$ is simply connected). If $K$ is unbounded, then all the components of $K^c$ are simply connected, but the converse is false as the example $J_{E_1}(0 < \lambda < \frac{1}{e})$ shows. (Compare with the Proposition B). But note that $J_{E_\lambda} \cup \{ \infty \}$ is connected in $\overline{\mathbb{C}}$. For the connectivity of a closed subset in $\mathbb{C}$, the following criterion holds.

Proposition 1 Let $K$ be a closed subset of $\mathbb{C}$. Then $K$ is connected if and only if the boundary of each component $U$ of the complement $K^c$ is connected.
(Proof): For the 'only if' part, see [New]. Suppose that $K$ is disconnected. Then there exist two closed sets $K_1$ and $K_2$ with $K = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$. Take a point $z_0$ with $d(z_0, K_1) = d(z_0, K_2)$ where $d$ denotes the Euclidian distance in $\mathbb{C}$. Then $z_0 \in K^c$ and so let $U_0$ be the connected component of $K^c$ containing $z_0$. Since $\partial U_0$ is connected by the assumption, either $\partial U_0 \subset K_1$ or $\partial U_0 \subset K_2$. Without loss of generality we can assume $\partial U_0 \subset K_1$. On the other hand denote $r_0 := d(z_0, K_1) = d(z_0, K_2)$ and let $D_{r_0}(z_0) := \{z \mid |z - z_0| < r_0\}$. Then $D_{r_0}(z_0) \subset U_0$ and there exists a point $w \in K_2$ with $w \in \overline{U_0}$. Since $w \in K_2 \subset K$, we have $w \in \partial U_0$ but this is a contradiction since $\partial U_0 \subset K_1$ and $K_1 \cap K_2 = \emptyset$. This completes the proof.

(Proof of Theorem 2): By Proposition 1, it is sufficient to to show that the boundary $\partial U$ is connected for each Fatou component $U$. Since $U$ is bounded, the boundary of $U$ as a subset of $\mathbb{C}$ and the one as the subset of $\mathbb{C}$ coincide. Hence $U$ is simply connected if and only if $\partial U$ is connected ([Bea], p.81, Proposition 5.1.4). This completes the proof.

Remark 2 (1) Since a non-simply connected Fatou component is necessarily a wandering domain, the assumption of Theorem 2 is equivalent to that all the components of $F_f$ are bounded and $F_f$ admits no multiply-connected wandering domains.

(2) Several sufficient conditions are known for a transcendental entire function $f$ to admit no unbounded Fatou components as follows:

(i) ([Ba3]) $\log M(r) = O((\log r)^p)$ (as $r \to \infty$) where $M(r) = \sup_{|z|=r} |f(z)|$ and $1 < p < 3$.

(ii) ([S]) There exists $\varepsilon \in (0, 1)$ such that $\log \log M(r) < \frac{(\log r)^{\frac{1}{2}}}{(\log \log r)^{\varepsilon}}$ for large $r$.

(iii) ([S]) The order of $f$ is less than $\frac{1}{2}$ and $\frac{\log M(2r)}{\log M(r)} \to c$ (finite constant) as $r \to \infty$.

Note that the condition (ii) includes the condition (i).

3.2 A property of inner functions

(Proof of Lemma 1): If $g$ is a finite Blaschke product, then $g$ is a rational function of degree $d \geq 2$. It is well known that in general the
closure of the set of all the inverse images of $z_0$ by a rational function $R$ of degree $d \geq 2$ contains its Julia set $J_R$ for any $z_0$ which is not a Fatou exceptional point ([Bea], p.79, Theorem 4.2.7). If the Denjoy-Wolff point $p$ is in $\mathbb{D}$, then $J_g = \partial \mathbb{D}$ and if the Denjoy-Wolff point $p$ is in $\partial \mathbb{D}$, then $J_g = \partial \mathbb{D}$ or at least $J_g$ is a perfect set in $\partial \mathbb{D}$. In any cases all the inverse images of $z_0$ are in $\mathbb{D}$ for every $z_0 \in \mathbb{D}$. So our assertion holds for

$$E = E(g) := \{z \mid z \text{ is a Fatou exceptional point for } g\}$$

and we have $\#E(g) \leq 2$, which implies that $E$ is a set of logarithmic capacity zero.

If $g$ is not a finite Blaschke product, then by Frostman’s theorem ([G], p.79, Theorem 6.4) there exists a set $E_1 \in \mathbb{D}$ of capacity zero such that $T_a \circ g$ is a Blaschke product for every $a \in \mathbb{D} \setminus E_1$, where $T_a(z) := \frac{z-a}{1-\overline{a}z}$. Therefore $B := T_a \circ g \circ T_a^{-1}$ is also a Blaschke product. By applying Frostman’s theorem to each $g^n$, we obtain the set $\bigcup_{n=1}^{\infty} E_i$ of logarithmic capacity zero such that $T_a \circ g^n \circ T_a^{-1} = B^n$ holds and each $B^n$ is a Blaschke product for every $a \in \mathbb{D} \setminus (\bigcup_{n=1}^{\infty} E_i)$. Now it is sufficient to prove our lemma for $B$, so we concentrate on a fixed $a \in \mathbb{D} \setminus (\bigcup_{n=1}^{\infty} E_i)$ and corresponding Blaschke product $B = T_a \circ g \circ T_a^{-1}$. Let $A_n \subset \partial \mathbb{D}$ be the set of accumulation points of $B^{-n}(0)$, then $A_n$ is closed and $B^n$ can be analytically continued to a meromorphic function on $\overline{\mathbb{C}} \setminus A_n$ by the reflection principle ([G], p.75, Theorem 6.1). In other words, $A_n$ is equal to the set of singularities of $B^n$ (that is, points at which $B^n(z)$ does not extend analytically). There exists a set $E_{B_n}$ of logarithmic capacity zero such that $A_n$ is equal to the set of accumulation points of $B^{-n}(p)$ for $p \in \mathbb{D} \setminus E_{B_n}$ ([G], Theorem 6.6). Let $E := \bigcup_{n=1}^{\infty} E_{B_n}$, then $E$ is a set of capacity zero and for every $z_0 \in \mathbb{D} \setminus E$ we have $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset \bigcup_{n=1}^{\infty} A_n$.

First let us consider the case when the Denjoy-Wolff point $p$ is in $\mathbb{D}$. Suppose that $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset \partial \mathbb{D}$ does not hold for a $z_0 \in \mathbb{D} \setminus E$, then there exists a open set $V$ with $V \cap \partial \mathbb{D} \neq \emptyset$ such that $B^n$ can be defined on $V$ for every $n \in \mathbb{N}$ and $V \cap \left(\bigcup_{n=1}^{\infty} B^{-n}(z_0)\right) = \emptyset$. We take $V$ as the maximal set satisfying this property. Let $W := V \cap \partial \mathbb{D}$. Since $B$ is not a finite Blaschke product, we have $\#\{B^{-1}(0)\} = \infty$ and so $A_1 \neq \emptyset$. Hence for a $z_0 \in \mathbb{D} \setminus E$ we have $W \neq \partial \mathbb{D}$. So there exists a point $\alpha \in \partial \mathbb{D} \setminus W$. Since $B^n$ cannot take the values $z_0, \frac{1}{z_0}$ and $\alpha$, $\{B^n[V]\}_{n=1}^{\infty}$ is a normal family. Then by the dynamics of $B$ on $\mathbb{D}$, we have $B^n[V] \longrightarrow p$ locally uniformly. But on the other hand $B^n|(V \cap (\mathbb{D})^c) \longrightarrow \frac{1}{p}$ by the construction of the extension,
which is a contradiction. Hence $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset \partial \mathbb{D}$ holds in this case.

Next we consider the case when the Denjoy-Wolff point $p$ is on the boundary of $\mathbb{D}$. Let $K := \bigcup_{n=1}^{\infty} A_n$ and suppose that $K \neq \partial \mathbb{D}$. Then $B^n$ is defined on $\mathbb{C} \setminus K$ for every $n \in \mathbb{N}$. Obviously $K$ is closed. If $K$ consists of a single point, say $\beta$, then we have $B(\partial \mathbb{D} \setminus \{\beta\}) \subset \partial \mathbb{D} \setminus \{\beta\}$ and $B|(\partial \mathbb{D} \setminus \{\beta\})$ is one to one since $B$ is extended by the reflection principle. It follows that $B$ is a Möbius transformation, which is a contradiction. By the similar argument, we can prove $\#K \geq 3$. Then $K$ cannot have an isolated point. If this is not the case, let $\beta \in K$ be an isolated point. Then $\beta$ is an essential singularity and hence by Picard’s theorem, $B$ takes all but exceptional two values in $\mathbb{C}$ infinitely often. This contradicts the fact that $B(\mathbb{C} \setminus K) \subset \mathbb{C} \setminus K$ and $\#K \geq 3$. Therefore it follows that $K$ is a perfect set. Since $\bigcup_{n=1}^{\infty} B^{-n}(z_0) \supset K$ holds for every $z_0 \in \mathbb{D} \setminus E$, this completes the proof. $\square$

(Proof of Lemma 2): In the case when $U$ is an attracting basin, the result is a special case of Theorem 3 in [P1]. In the case when $U$ is a parabolic basin, the result follows by combining the series of theorems in [DM] (Theorem 6.1, Theorem 4.2, Corollary 4.3, Theorem 3.1) together with the Theorem 3 in [P1]. $\square$

3.3 In the case when $F_f$ admits an unbounded component — On the Boundary of unbounded invariant Fatou Components

(Proof of Main Theorem): In what follows we assume that $n_0 = 1$ (that is, $U$ is an invariant component) and $m_0 = 1$ for simplicity. This causes no loss of generality, because we have only to consider $f^{m_0}$ instead of $f$ in general cases.

Case (1) Since $\infty$ is accessible, there exists a continuous curve $L(t)$ ($0 \leq t < 1$) in $U$ with $\lim_{t \to 1} L(t) = \infty$. By deforming $L(t)$ slightly, we construct a new curve $\mathcal{L}(t)$ satisfying the following condition.

Lemma 3 There exists a curve $\mathcal{L}(t)$ ($0 \leq t < 1$) with $\lim_{t \to 1} \mathcal{L}(t) = \infty$ such that every branch of $f^{-n}$ can be analytically continued along it for every $n \in \mathbb{N}$.

(Proof): We may assume that $L(0) \notin P_f$, since $q \notin P_f$ we have $U \notin P_f$. Let $p_0 := L(0), p_1, p_2, \ldots$ be points on $L$ such that all the piecewise linear line segments connecting $p_0, p_1, p_2, \ldots$ lie in $U$. Let $F_n^{(1)}, F_n^{(2)}, \ldots, F_n^{(m)}, \ldots$
be all the branches of $f^{-n}$ which take values on $U$. The range of the suffix $m$ may be finite or infinite. Define

$$\Theta_n^{(m)}(p_0) := \{e^{i\theta} \mid F_n^{(m)}\text{ can be analytically continued along the ray} \quad \text{from } p_0 \text{ in the direction } \theta\} \quad (n = 1, 2, \ldots).$$

Then by the next Gross's Star Theorem ([Nev]), it follows that $\Theta_n^{(m)}(p_0)$ has full measure in $\partial \mathbb{D}$.

**Lemma C (Gross's Star Theorem)** Let $f$ be an entire function and $F$ a branch of $f^{-1}$ defined in the neighborhood of $p_0 \in \mathbb{C}$. Then $F$ can be analytically continued along almost all rays from $p_0$ in the direction $\theta$.

Then the set

$$\Theta(p_0) := \bigcap_{n \geq 1, m \geq 1} \Theta_n^{(m)}(p_0)$$

has also full measure in $\partial \mathbb{D}$. Hence by changing $p_1$ slightly to a point $p'_1$, the segments $p_0p'_1$ and $p'_1p_2$ lie in $U$ and all the branches $F_n^{(m)}$ ($n \geq 1, m \geq 1$) can be analytically continued along $\overline{p_0p'_1}$. By the same method, we can find a point $p'_2$ close to $p_2$ such that the segment $\overline{p'_1p'_2}$ lies in $U$ and has the same property as above. By repeating this argument, we can prove the Lemma 3. \qed

Let $l_n^{(m)}(t) := F_n^{(m)}(\mathcal{L}(t))$ then we have $\lim_{t \to 1} l_n^{(m)}(t) = \infty$. For suppose this is false, then there exist an increasing sequence of parameter values $t_1 < t_2 < \cdots < t_k < \cdots$ and a finite point $\alpha$ with $\lim_{k \to \infty} l_n^{(m)}(t_k) = \alpha \neq \infty$. Then it follows that $\lim_{k \to \infty} \mathcal{L}(t_k) = f^n(\alpha) \neq \infty$ and this contradicts the fact $\lim_{k \to \infty} \mathcal{L}(t_k) = \infty$.

Let $\varphi : \mathbb{D} \to U$ be a Riemann map of $U$. Then

$$\Gamma(t) := \varphi^{-1}(\mathcal{L}(t)) \quad \text{and} \quad \gamma_n^{(m)}(t) := \varphi^{-1}(l_n^{(m)}(t))$$

are curves in $\mathbb{D}$ landing at a point in $\partial \mathbb{D}$. This fact is not so trivial but follows from the proposition in [P2] (p.29, Proposition 2.14). We may assume that $\Gamma(t)$ lands at $z = 1 \in \partial \mathbb{D}$ for simplicity. If $\lim_{t \to 1} \gamma_n^{(m_0)}(t) = e^{i\theta_0}$, then since $\lim_{t \to 1} \varphi(\gamma_n^{(m_0)}(t)) = \lim_{t \to 1} l_n^{(m)}(t) = \infty$, it follows that there exists the radial limit $\lim_{r \to 1} \varphi(re^{i\theta_0})$ and this is equal to $\infty$. This fact follows from the theorem in [P2] (p.34, Theorem 2.16). Therefore it is sufficient to show that the set of all the landing points of $\gamma_n^{(m)}(t)$ ($n \geq 1, m \geq 1$) is dense in $\partial \mathbb{D}$.
Let $g := \varphi^{-1} \circ f \circ \varphi : \mathbb{D} \rightarrow \mathbb{D}$. Then by Fatou's theorem $\varphi$ has radial limit $\varphi(e^{i\theta}) = \lim_{r \uparrow 1} \varphi(re^{i\theta}) \in \partial U$ and non-constant for almost every $e^{i\theta} \in \partial \mathbb{D}$. Hence $f \circ \varphi(re^{i\theta})$ is a curve landing at a point in $\partial U \setminus \{\infty\}$ for almost every $e^{i\theta} \in \partial \mathbb{D}$. Therefore it follows that $\lim_{r \uparrow 1} \varphi^{-1} \circ f \circ \varphi(re^{i\theta}) \in \partial \mathbb{D}$ a.e. and thus $g$ is an inner function. Let $\hat{C} := \varphi^{-1}(C)$ then by the same reason for $\Gamma(t)$, $\hat{C}$ is a curve in $\mathbb{D}$ with an end point $\check{q} \in \partial U$ satisfying $\check{g}(\hat{C}) \supset \hat{C}$.

From the dynamics of $g : \mathbb{D} \rightarrow \mathbb{D}$, it follows that the set $\bigcup_{n=0}^{\infty} g^n(\hat{C}) \cup \{\check{p}, \check{q}\}$ is compact in $\overline{\mathbb{D}}$ where $\check{p} = \varphi^{-1}(p)$ and $\check{p}$ is an attracting fixed point of $\check{g}$ and the distance between this set and $z = 1$ is positive. Hence there exists $\varepsilon_0 > 0$ such that

$$U_{\varepsilon_0}(1) \cap \left\{ \bigcup_{n=0}^{\infty} g^n(\hat{C}) \cup \{\check{p}, \check{q}\} \right\} = \emptyset$$

(1)

Since $\Gamma(t)$ lands at $z = 1$, there exists $t_0 \in [0,1)$ such that $\Gamma|[t_0,1) \subset U_{\varepsilon_0}(1)$. So by rewriting $\Gamma|[t_0,1)$ to $\Gamma(t)$ ($0 \leq t < 1$) we may assume that $\Gamma(t) \subset U_{\varepsilon_0}(1)$ for $0 \leq t < 1$). Let $K := \{z \mid |z| \leq 1 - \varepsilon_0\}$ then since every point in $\mathbb{D}$ tends to $\check{p}$ under $g^n$ and $K$ is compact, there exists $n_1 \in \mathbb{N}$ such that for every $N \geq n_1$ we have $g^N(K) \subset U_{\varepsilon}(\check{p})$. Then we have $\gamma_{N}^{(m)}(t) \subset K^c$ for every $N \geq n_1$. For suppose that $\gamma_{N}^{(m)}(t) \cap K \neq \emptyset$, then by operating $f^N$ we have $\Gamma(t) \cap K \neq \emptyset$ which contradicts $\Gamma(t) \subset U_{\varepsilon_0}(1)$.

Now suppose that the conclusion does not hold. Then there exists $(\theta_1, \theta_2) := \{e^{i\theta} \mid \theta_1 < \theta < \theta_2\} \subset \partial \mathbb{D}$ with $\Theta_{\infty} \cap (\theta_1, \theta_2) = \emptyset$.

By changing the starting point $\Gamma(0)$ slightly, if necessary, we may assume that the points $\gamma_{n}^{(m)}(0)$ ($n, m = 1, 2, \cdots$) accumulate to all over $\partial \mathbb{D}$ by Lemma 1 (1) while the end points $\gamma_{n}^{(m)}(1) := \lim_{t\rightarrow 1} \gamma_{n}^{(m)}(t)$ ($n, m = 1, 2, \cdots$) are not in $(\theta_1, \theta_2)$. Therefore there exists $\gamma_{n_1}^{(m_1)}(t)$ such that $\gamma_{n_1}^{(m_1)}(t) \subset K^c$ and $\gamma_{n_1}^{(m_1)}(1) \in \partial \mathbb{D} \setminus (\theta_1, \theta_2)$

On the other hand there exist inverse images $g^{-n}(\hat{C})$ which have limit points on $(\theta_1, \theta_2)$ densely. The reason is as follows: Since $q \notin P_f$, there exists a neighborhood $V$ of $q$ such that all the branches $F_{n_1}^{(1)}, F_{n_2}^{(2)}, \ldots, F_{n}^{(m)}$, \ldots can be defined. Let $V_0 \subset V$ is a neighborhood of $q$ with $\overline{V_0} \subset V$. We may assume that $C \subset V_0$. Define

$$c_{n}^{(m)}(t) := F_{n}^{(m)}(C(t)), \quad \tilde{c}_{n}^{(m)}(t) := \varphi^{-1}(c_{n}^{(m)}(t)).$$

Then $c_{n}^{(m)}(t)$ is a curve in $U$ landing at a point in $\partial U$ and $\tilde{c}_{n}^{(m)}(t)$ is a curve in $\mathbb{D}$ landing at a point in $\partial \mathbb{D}$ by the same reason as before. Let
$(\theta_3, \theta_4) \subset (\theta_1, \theta_2)$. By changing the starting point $C(0)$ slightly, if necessary, we may assume that the points $c_n^{(m)}(0)$ accumulate to $(\theta_3, \theta_4)$ by Lemma 1 (1). Since radial limits of $\phi$ exist and non-constant almost everywhere, by changing $\theta_3$ and $\theta_4$ slightly if necessary, we may assume that there exist the finite values $\phi(e^{i\theta_3})$ and $\phi(e^{i\theta_4})$ with $\phi(e^{i\theta_3}) \neq \phi(e^{i\theta_4})$. Then $c_n^{(m)}(0)$ accumulate on $\partial U \cap \phi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})$. In general the family of single-valued analytic branch of $f^{-n} (n = 1, 2, \ldots)$ on a domain $U_0$ is normal and furthermore if $U_0 \cap J \neq \emptyset$, any local uniform limit of a subsequence in the family is constant ([Bea], p.193, Theorem 9.2.1, Lemma 9.2.2). So the family $\{F_n^{(m)}\}_{\mathbb{V}_0}$ is normal and all its limit functions are constant and hence for a suitable subsequence the diameter of $c_n^{(m)}(t)$ tends to zero, that is, $c_n^{(m)}(t)$ must land at a point in $\partial U \cap \phi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})$ if the constant limit is finite. Therefore $c_n^{(m)}(t)$ must land at a point in $(\theta_3, \theta_4)$. If the constant limit is $\infty$, for large enough $n_k$ the curves $c_n^{(m)}$ cannot intersect both $\phi(re^{i\theta_3})$ and $\phi(re^{i\theta_4})$ which are bounded set, since the convergence is uniform on $V_0$. Hence again we can conclude that $c_n^{(m)}(t)$ must land at a point in $\partial U \cap \phi(\{re^{i\theta} \mid \theta_3 < \theta < \theta_4, 0 \leq r \leq 1\})$ and therefore $c_n^{(m)}(t)$ must land at a point in $(\theta_3, \theta_4)$. This proves the assertion.

Then there exists $c_{N_1}^{(M_1)}$ such that $\gamma_{n_1}^{(m_1)} \cap c_{N_1}^{(M_1)} \neq \emptyset$. We may assume that $n_1 > N_1$. Let $u \in \gamma_{n_1}^{(m_1)} \cap c_{N_1}^{(M_1)}$, then since $u \in \gamma_{n_1}^{(m_1)}$, we have $g^{n_1}(u) \in U_{\varepsilon_0}(1)$. On the other hand since $u \in c_{N_1}^{(M_1)}$ and $n_1 > N_1$, we have $g^{n_1}(u) \in \bigcup_{n=0}^{\infty} g^n(C)$ which contradicts (1). Therefore $\Theta_{\infty}$ is dense in $\partial \mathbb{D}$. Disconnectivity of $J_f$ easily follows by the same argument as in the case of $E_{\lambda}$ in §1. This completes the proof in the case of (1).

Case (2) The proof is quite parallel to the case (1). Note that by Lemma 2, $\bigcup_{n=1}^{\infty} g^{-n}(z_0) \subset \partial \mathbb{D}$ $(z_0 \in \mathbb{D} \setminus E)$ holds for $g = \varphi^{-1} \circ f \circ \varphi$ in this case. □

Case (3) Since $g(z) = e^{2\pi i \theta_0}$ with $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$, the inverse image of $\Gamma(t)$ by $g^{-n}$ is unique and denote it by $\gamma_n(t)$. Then it is obvious that the end points of $\gamma_n(t)$ are dense in $\partial \mathbb{D}$ and $\varphi$ attains radial limit $\infty$ there, since $g(z)$ is an irrational rotation and

$$\lim_{t \to 1} \varphi(\gamma_n(t)) = \lim_{t \to 1} f^{-1}(\varphi(\Gamma(t))) = \infty.$$ □

Case (4) In this case we need not assume the accessibility of $\infty$, because this condition is automatically satisfied ([Ba6]). The set $\bigcup_{n=0}^{\infty} f^n(C)$ is a
curve which may have self-intersections and tends to $\infty$. It is not difficult to take $L$ satisfying $L \cap (\bigcup_{n=0}^{\infty} f^n(C)) = \emptyset$. Hence we have $\mathcal{L} \cap (\bigcup_{n=0}^{\infty} f^n(C)) = \emptyset$. The rest of the proof is quite parallel to the case (1) if the conclusion of Lemma 2 (1) holds for $g$. If we have only the conclusion of Lemma 2 (2), then we can prove that for every arc $A \subset \partial \mathbb{D}$ with $A \cap K \neq \emptyset$, $A \cap \Theta_{\infty} \neq \emptyset$ holds by the similar argument. \qed

References


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