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Remarks on Maximal $\mathcal{J}$-Trivial Transformation Semigroups on Finite Sets

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Let $T(X)$ be the full transformation semigroup on a finite set X. A semigroup S is $\mathcal{J}$-*trivial* if $a \mathcal{J} b$ implies $a = b$ for every $a, b \in S$. In [3], all maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$ have been determined by the author. In this note, to make the results in [3] more exact, we show the following theorem :

Theorem 1. *Let S and T be maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$. If S and T are isomorphic, then there exists an elemant ξ of the symmetric group on X such that $T = \xi^{-1}S\xi$.*

Of course, the above fact does not hold for any subsemigroups of $T(X)$.

To show the theorem, we give some definitions, notations and results in [3].

Let $E(X)$ and $\Pi(X)$ be the sets of equivalence relations on X and partitions of X, respectively. Then it is well-known that there exists a bijection Φ from $E(X)$ to $\Pi(X)$, where $\Phi(\rho)$ is the set of ρ-classes for $\rho \in E(X)$, and that $E(X)$ is a lattice-ordered set under $\cap$ and $\vee$. Define an order $\leq$ on $\Pi(X)$ by $\Phi(\lambda) \leq \Phi(\rho)$ if $\lambda \supseteq \rho$ for $\lambda, \rho \in E(X)$. Then $\Pi(X)$ is also a lattice-ordered set which is anti-isomorphic to $E(X)$, i. e., $\Phi(\lambda \vee \rho) = \Phi(\lambda) \wedge \Phi(\rho)$ and $\Phi(\lambda \cap \rho) = \Phi(\lambda) \vee \Phi(\rho)$.

For $\alpha \in T(X)$, let $Fix(\alpha) = \{x \in X \mid x\alpha = x\}$ and let $\omega(\alpha) = \{(x, y) \in X \times X \mid x\alpha^s = y\alpha^t$ for some $s, t \geq 0\}$. Then $\omega(\alpha)$ is an equivalece relation on X (see [2]). In this case, $\Phi(\omega(\alpha))$ is denoted by $\Omega(\alpha)$ and each class of $\omega(\alpha)$ is called an *orbit* of α.

Result 1. *A subsemigroup S of $T(X)$ is $\mathcal{J}$-trivial if and only if $Fix(\alpha\beta) = Fix(\alpha) \cap Fix(\beta)$ and $\Omega(\alpha\beta) = \Omega(\alpha) \wedge \Omega(\beta)$ for every $\alpha, \beta \in S$.*

Let $\leq$ be a total order on X. Let $T_{RE}(X, \leq) = \{\alpha \in T(X) \mid x\alpha \leq x$ for all $x \in X\}$. Then $T_{RE}(X, \leq)$ is a subsemigroup of $T(X)$ which is called a *regressive* semigroup. For $\pi = \{X_1, X_2, \ldots, X_r\} \in \Pi(X)$, let $Min(\pi) = \{Min(X_1), Min(X_2), \ldots, Min(X_r)\}$, where $Min(X_i)$ is the minimum element of X_i. A subset P of $\Pi(X)$ is called a *J-subset* with respect to $\leq$ if P is a $\wedge$-semilattice in which $Min(\pi_1 \wedge \pi_2) = Min(\pi_1) \cap Min(\pi_2)$ holds for every $\pi_1, \pi_2 \in P$. For $\pi \in \Pi(X)$, let $J(\pi, \leq) = \{\alpha \in T_{RE}(X, \leq) \mid \Omega(\alpha) = \pi\}$.

Result 2. *A subsemigroup S of $T(X)$ is $\mathcal{J}$-trivial if and only if $S \subseteq T_{RE}(X, \leq)$ for some total order $\leq$ on X and $\Omega(S) = \{\Omega(\alpha) \mid \alpha \in S\}$ is a J-subset of $\Pi(X)$ with respect to $\leq$. A*

$\mathcal{J}$-trivial subsemigroup S of $T(X)$ is maximal if and only if there exists a maximal J-subset P of $\Pi(X)$ with respect to some total order $\leq$ on X such that $\Omega(S) = P$ and $S = \cup \{J(\pi, \leq) \mid \pi \in P\}$.

Let $(X, \leq) = \{1, 2, \dots, n \mid 1 \leq 2 \leq \dots \leq n\}$ be a totally ordered set. For $k \geq 2$, let $\Pi_{(k)} = \{\pi \in \Pi(X) \mid Min(\pi) = \{1, k\}\}$ and $\Pi_{(1)} = \{\pi \in \Pi(X) \mid Min(\pi) = \{1\}\}$. Then, if $\pi \in \Pi_{(1)}$, then $\pi = \{X\}$ and if $\pi \in \Pi_{(k)}$, $k \geq 2$, then $\pi = \{X_1, X_k\}$ with $Min(X_1) = 1$ and $Min(X_k) = k$, so that $|\Pi_{(k)}| = 2^{n-k-1}$.

Let $P_1 = \{\pi_1, \pi_2, \dots, \pi_n\}$, where $\pi_k \in \Pi_{(k)}$ $(k = 1, 2, \dots, n)$. Then P_1 is called an *initial set* of $\Pi(X)$ with respect to $\leq$.

Result 3. *Let $(X, \leq)$ be as above. Every maximal J-subset P of $\Pi(X)$ with respect to $\leq$ contains exact one initial set with respect to $\leq$. Conversely, for any initial set $P_1 = \{\pi_1, \pi_2, \dots, \pi_n\}$ with respect to $\leq$, there exists a maximal J-subset P of $\Pi(X)$ with respect to $\leq$ which contains P_1. In this case, if $\pi \in P$ with $Min(\pi) = Y$, then $\pi = \bigvee_{k \in Y} \pi_k$, where $\pi_k \in P_1$.*

In Result 3, P_1 is called the *initial set* of P. We note that there are $2^{(n-1)(n-2)/2}$ initial subsets with respect to $\leq$, since $|\Pi_{(1)}| = 1$ and $|\Pi_{(k)}| = 2^{n-k-1}$ if $k \geq 2$.

Let S and T be maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$. Then, from Result 2, we have $S \subseteq T_{RE}(X, \leq_S)$ and $T \subseteq T_{RE}(X, \leq_T)$ for some total orders $\leq_S$ and $\leq_T$ on X. Let $(X, \leq_S) = \{1, 2, \dots, n \mid 1 \leq_S 2 \leq_S \dots \leq_S n\}$ and $(X, \leq_T) = \{i(1), i(2), \dots, i(n) \mid i(1) \leq_T i(2) \leq_T \dots \leq_T i(n)\}$. Let ξ be an order-isomorphism from $(X, \leq_S)$ to $(X, \leq_T)$, i. e., $k\xi = i(k)$ $(k = 1, 2, \dots, n)$. Then ξ is an element of the symmetric group on X. Let $P_S = \Omega(S)$ and $P_T = \Omega(T)$. Then, by Result 2, P_S and P_T are maximal J-subsets of $\Pi(X)$ with respect to $\leq_S$ and $\leq_T$, respectively, and $S = \cup \{J(\pi, \leq_S) \mid \pi \in P_S\}$, $T = \cup \{J(\pi, \leq_T) \mid \pi \in P_T\}$. Let $P_{S,1} = \{\pi_1, \pi_2, \dots, \pi_n\}$ and $P_{T,1} = \{\pi_{i(1)}, \pi_{i(2)}, \dots, \pi_{i(n)}\}$ be the initial set of P_S and P_T, respectively, where $Min(\pi_1) = \{1\}$, $Min(\pi_k) = \{1, k\}$ if $k \,{}_S\!\geq 2$ and $Min(\pi_{i(1)}) = \{i(1)\}$, $Min(\pi_{i(k)}) = \{i(1), i(k)\}$ if $i(k) \,{}_T\!\geq i(2)$.

Suppose that S and T are isomorphic. Let $\phi : S \to T$ be an isomorphism.

Hereafter m denotes a sufficiently large integer.

The following lemma is easy to verify :

Lemma 1. *Let $\pi = \{X_1, X_2, \dots, X_r\} \in P_S$ with $Min(X_i) = x_i$ $(i = 1, 2, \dots, r)$. Then :*

(1) *$Fix(\alpha) = Min(\pi)$ for every $\alpha \in J(\pi, \leq_S)$.*

(2) *$\varepsilon \in T(X)$ defined by $X_i\varepsilon = x_i$ $(i = 1, 2, \dots, r)$ is a unique idempotent of $J(\pi, \leq_S)$.*

(3) *For $\alpha \in S$, $\alpha \in J(\pi, \leq_S)$ if and only if $\alpha^m = \varepsilon$, where ε is the unique idempotent $J(\pi, \leq_S)$.*

For $\pi_k \in P_{S,1}$ (resp. $\pi_{i(k)} \in P_{T,1}$), the unique idempotent of $J(\pi_k, \leq_S)$ (resp. $J(\pi_{i(k)}, \leq_T)$) is denoted by ε_k (resp. $\varepsilon_{i(k)}$). Then $X\varepsilon_1 = 1$ (resp. $X\varepsilon_{i(1)} = i(1)$). Therefore ε_1 (resp. $\varepsilon_{i(1)}$) is the zero of S (resp. T), which is denoted by 0_S (resp. 0_T).

Lemma 2. (1) *For* $\mu \in S$, $\mu^{n-1} = 0_S$ *and* $\mu^{n-2} \neq 0_S$ *if and only if* $k\mu = k-1$ *if* $k \;{}_S\!\geq 2$ *and* $1\mu = 1$. (2) *For* $\alpha \in S$, $\Omega(\alpha) \in P_{S,1}$ *if and only if* $(\alpha\beta)^m = 0_S$ or $(\alpha\beta)^m = \alpha^m \neq 0_S$ *for every* $\beta \in S$.

Let μ be as in (1) of Lemma 2. Since $(\mu\phi)^{n-1} = (\mu^{n-1})\phi = 0_S\phi = 0_T$ and $(\mu\phi)^{n-2} = (\mu^{n-2})\phi \neq 0_T$, applying (1) of Lemma 2 to T, we have that $i(k)(\mu\phi) = i(k-1)$ if $i(k) \;{}_T\!\geq i(2)$ and $i(1)(\mu\phi) = i(1)$. By using this fact, we obtain the following lemmas.

Lemma 3. $\varepsilon_k\phi = \varepsilon_{i(k)}$ for every $\pi_k \in P_{S,1}$.

Lemma 4. *Let* $\pi_k = \{X_1, X_k\} \in P_{S,1}$ *with* $Min(X_1) = 1$ *and* $Min(X_k) = k$. *Then* $\pi_{i(k)} = \{X_1\xi, X_k\xi\} \in P_{T,1}$ *with* $Min(X_1\xi) = i(1)$ *and* $Min(X_k\xi) = i(k)$.

Proof of Theorem 1. We show that $i(x)(\alpha\phi) = i(x\alpha)$ for every $x \in X$ and every $\alpha \in S$. Then we can show that $T = \xi^{-1}S\xi$. In fact, since $i(x)\xi^{-1}\alpha\xi = x\alpha\xi = i(x\alpha)$ for every $x \in X$ and for every $\alpha \in S$, we have that $\alpha\phi = \xi^{-1}\alpha\xi$.

If $x = 1$, then the assertion is trivially true.

For $x \;{}_S\!\geq 2$, let $\pi_x = \{X_1, X_x\} \in P_{S,1}$ with $Min(X_1) = 1$ and $Min(X_x) = x$. We first show that if $X_x = \{x\}$ then $i(x)(\alpha\phi) = i(x\alpha)$ for every $\alpha \in S$. Let $x\alpha = k$ and $i(x)(\alpha\phi) = i(j)$ for some $\alpha \in S$. Since $X_x = \{x\}$, $x\varepsilon_x = x$ and $y\varepsilon_x = 1$ if $y \neq x$, so that $x\varepsilon_x\alpha = k$ and $y\varepsilon_x\alpha = 1$. If $k <_S j$, then $x\varepsilon_x\alpha\varepsilon_j = k\varepsilon_j = 1$ and $y\varepsilon_x\alpha\varepsilon_j = 1\,\varepsilon_j = 1$ for every $y \in X$ with $y \neq x$. Thus $\varepsilon_x\alpha\varepsilon_j = 0_S$. But $i(x)(\varepsilon_x\alpha\varepsilon_j)\phi = i(x)\varepsilon_{i(x)}(\alpha\phi)\varepsilon_{i(j)} = i(x)(\alpha\phi)\varepsilon_{i(j)} = i(j)\varepsilon_{i(j)} = i(j) \neq i(1)$, since $i(j) \;{}_T\!> i(k) \;{}_T\!\geq i(1)$. Thus $(\varepsilon_n\alpha\varepsilon_j)\phi \neq 0_T$, a contradiction. If $j <_S k$, then we similarly have $\varepsilon_n\alpha\varepsilon_k \neq 0_S$ but $(\varepsilon_n\alpha\varepsilon_k)\phi = 0_T$, again a contradiction. Thus $k = j$, so that $i(x)(\alpha\phi) = i(k) = i(x\alpha)$.

We now inductively show the above assertion. For $x = n$, let $\pi_n = \{X_1, X_n\}$ with $Min(X_1) = 1$ and $Min(X_n) = n$. Then $X_n = \{n\}$. From the above fact, $i(n)(\alpha\phi) = i(n\alpha)$ for every $\alpha \in S$.

Suppose that, for some $x \in X$ with $x \;{}_S\!\geq 2$, if $r \;{}_S\!> x$, then $i(r)(\alpha\phi) = i(r\alpha)$ for every $\alpha \in S$. Then we show that $i(x)(\alpha\phi) = i(x\alpha)$. Let $x\alpha = t$ and $\pi_x = \{X_1, X_x\}$ with $Min(X_1) = 1$ and $Min(X_x) = x$. If $X_x = \{x\}$, then again by the above fact the assertion is true. Suppose that there exists $r \in X_x$ such that $r \;{}_S\!> x$. Then $r\varepsilon_x\alpha = x\alpha = t$. By the assumption, $i(r)(\varepsilon_x\alpha)\phi = i(r\varepsilon_x\alpha) = i(t)$. Since $\pi_{i(x)} = \{X_1\xi, X_x\xi\}$, we have $i(r) = r\xi \in X_x\xi$, so that $i(r)\varepsilon_{i(x)} = i(x)$. Thus $i(x\alpha) = i(t) = i(r)(\varepsilon_x\alpha)\phi = i(r)\varepsilon_{i(x)}(\alpha\phi) = i(x)(\alpha\phi)$. The proof is complete.

Suppose that $\leq_S = \leq_T$. Then, since $\xi = id_{T(X)}$, we have that $S \cong T$ implies $S = T$, so that $P_S = \Omega(S) = \Omega(T) = P_T$. Consequently, if $P_S \neq P_T$, then S and T are not isomorphic. From Result 3, we have thst $P_S = P_T$ if and only if $P_{1,S} = P_{1,T}$. Since the number of initial sets with respect to a fixed total order $\leq$ on X is $2^{(n-1)(n-2)/2}$, we obtain :

Corollary 2 [3]. *There are* $2^{(n-1)(n-2)/2}$ *maximal* $\mathcal{J}$*-trivial subsemigroups of* $T(X)$ *up to isomorphisms if* $|X| = n$.

References

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