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Remarks on Maximal $\mathcal{J}$-Trivial Transformation Semigroups
on Finite Sets

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Let $T(X)$ be the full transformation semigroup on a finite set $X$. A semigroup $S$ is $\mathcal{J}$-trivial if $a \mathcal{J} b$ implies $a = b$ for every $a, b \in S$. In [3], all maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$ have been determined by the author. In this note, to make the results in [3] more exact, we show the following theorem:

**Theorem 1.** Let $S$ and $T$ be maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$. If $S$ and $T$ are isomorphic, then there exists an element $\xi$ of the symmetric group on $X$ such that $T = \xi^{-1}S\xi$.

Of course, the above fact does not hold for any subsemigroups of $T(X)$.

To show the theorem, we give some definitions, notations and results in [3].

Let $E(X)$ and $\Pi(X)$ be the sets of equivalence relations on $X$ and partitions of $X$, respectively. Then it is well-known that there exists a bijection $\Phi$ from $E(X)$ to $\Pi(X)$, where $\Phi(\rho)$ is the set of $\rho$-classes for $\rho \in E(X)$, and that $E(X)$ is a lattice-ordered set under $\cap$ and $\vee$. Define an order $\leq$ on $\Pi(X)$ by $\Phi(\lambda) \leq \Phi(\rho)$ if $\lambda \supseteq \rho$ for $\lambda, \rho \in E(X)$. Then $\Pi(X)$ is also a lattice-ordered set which is anti-isomorphic to $E(X)$, i.e., $\Phi(\lambda \vee \rho) = \Phi(\lambda) \wedge \Phi(\rho)$ and $\Phi(\lambda \cap \rho) = \Phi(\lambda) \vee \Phi(\rho)$.

For $\alpha \in T(X)$, let $\text{Fix}(\alpha) = \{x \in X \mid x\alpha = x\}$ and let $\omega(\alpha) = \{(x, y) \in X \times X \mid x\alpha^t = y\alpha^t \text{ for some } s, t \geq 0\}$. Then $\omega(\alpha)$ is an equivalence relation on $X$ (see [2]). In this case, $\Phi(\omega(\alpha))$ is denoted by $\Omega(\alpha)$ and each class of $\omega(\alpha)$ is called an orbit of $\alpha$.

**Result 1.** A subsemigroup $S$ of $T(X)$ is $\mathcal{J}$-trivial if and only if $\text{Fix}(\alpha\beta) = \text{Fix}(\alpha) \cap \text{Fix}(\beta)$ and $\Omega(\alpha\beta) = \Omega(\alpha) \wedge \Omega(\beta)$ for every $\alpha, \beta \in S$.

Let $\leq$ be a total order on $X$. Let $T_{\mathcal{R}E}(X, \leq) = \{\alpha \in T(X) \mid x\alpha \leq x \text{ for all } x \in X\}$. Then $T_{\mathcal{R}E}(X, \leq)$ is a subsemigroup of $T(X)$ which is called a regressive semigroup. For $\pi = \{X_1, X_2, \ldots, X_r\} \subseteq \Pi(X)$, let $\text{Min}(\pi) = \{\text{Min}(X_1), \text{Min}(X_2), \ldots, \text{Min}(X_r)\}$, where $\text{Min}(X_i)$ is the minimum element of $X_i$. A subset $P$ of $\Pi(X)$ is called a $J$-subset with respect to $\leq$ if $P$ is a $\wedge$-semilattice in which $\text{Min}(\pi_1 \wedge \pi_2) = \text{Min}(\pi_1) \cap \text{Min}(\pi_2)$ holds for every $\pi_1, \pi_2 \in P$. For $\pi \in \Pi(X)$, let $J(\pi, \leq) = \{\alpha \in T_{\mathcal{R}E}(X, \leq) \mid \Omega(\alpha) = \pi\}$.

**Result 2.** A subsemigroup $S$ of $T(X)$ is $\mathcal{J}$-trivial if and only if $S \subseteq T_{\mathcal{R}E}(X, \leq)$ for some total order $\leq$ on $X$ and $\Omega(S) = \{\Omega(\alpha) \mid \alpha \in S\}$ is a $J$-subset of $\Pi(X)$ with respect to $\leq$. A
\$J\$-trivial subsemigroup \(S\) of \(T(X)\) is maximal if and only if there exists a maximal \(J\)-subset \(P\) of \(\Pi(X)\) with respect to some total order \(\leq\) on \(X\) such that \(\Omega(S) = P\) and \(S = \bigcup \{J(\pi, \leq) | \pi \in P\}\).

Let \((X, \leq) = \{1, 2, ..., n \mid 1 \leq 2 \leq ... \leq n\}\) be a totally ordered set. For \(k \geq 2\), let \(\Pi_{(k)} = \{\pi \in \Pi(X) | \text{Min}(\pi) = \{1, k\}\}\) and \(\Pi_{(1)} = \{\pi \in \Pi(X) | \text{Min}(\pi) = \{1\}\}\). Then, if \(\pi \in \Pi_{(1)}\), then \(\pi = \{X\}\) and if \(\pi \in \Pi_{(k)}\), \(k \geq 2\), then \(\pi = \{X_{1}, X_{k}\}\) with \(\text{Min}(X_{1}) = 1\) and \(\text{Min}(X_{k}) = k\), so that \(|\Pi_{(k)}| = 2^{n-k}\).

Let \(P_{1} = \{\pi_{1}, \pi_{2}, ..., \pi_{n}\}\), where \(\pi_{k} \in \Pi_{(k)}\) \((k = 1, 2, ..., n)\). Then \(P_{1}\) is called an initial set of \(\Pi(X)\) with respect to \(\leq\).

Result 3. Let \((X, \leq)\) be as above. Every maximal \(J\)-subset \(P\) of \(\Pi(X)\) with respect to \(\leq\) contains exactly one initial set with respect to \(\leq\). Conversely, for any initial set \(P_{1} = \{\pi_{1}, \pi_{2}, ..., \pi_{n}\}\) with respect to \(\leq\), there exists a maximal \(J\)-subset \(P\) of \(\Pi(X)\) with respect to \(\leq\) which contains \(P_{1}\). In this case, if \(\pi \in P\) with \(\text{Min}(\pi) = Y\), then \(\pi = V_{k \in Y} \pi_{k}\), where \(\pi_{k} \in P_{1}\).

In Result 3, \(P_{1}\) is called the initial set of \(P\). We note that there are \(2^{(n-1)(n-2)/2}\) initial subsets with respect to \(\leq\), since \(|\Pi_{(1)}| = 1\) and \(|\Pi_{(k)}| = 2^{n-k}\) if \(k \geq 2\).

Let \(S\) and \(T\) be maximal \(J\)-trivial subsemigroups of \(T(X)\). Then, from Result 2, we have \(S \subseteq T_{RE}(X, \leq_{S})\) and \(T \subseteq T_{RE}(X, \leq_{T})\) for some total orders \(\leq_{S}\) and \(\leq_{T}\) on \(X\). Let \((X, \leq_{S}) = \{1, 2, ..., n \mid 1 \leq 2 \leq ... \leq n\}\) and \((X, \leq_{T}) = \{i(1), i(2), ..., i(n) \mid i(1) \leq_{T} i(2) \leq_{T} ... \leq_{T} i(n)\}\). Let \(\xi\) be an order-isomorphism from \((X, \leq_{S})\) to \((X, \leq_{T})\), i.e., \(\xi_{k} = i(k)\) \((k = 1, 2, ..., n)\). Then \(\xi\) is an element of the symmetric group on \(X\). Let \(P_{S} = \Omega(S)\) and \(P_{T} = \Omega(T)\). Then, by Result 2, \(P_{S}\) and \(P_{T}\) are maximal \(J\)-subsets of \(\Pi(X)\) with respect to \(\leq_{S}\) and \(\leq_{T}\), respectively, and \(S = \bigcup \{J(\pi, \leq_{S}) | \pi \in P_{S}\}\), \(T = \bigcup \{J(\pi, \leq_{T}) | \pi \in P_{T}\}\). Let \(P_{S,1} = \{\pi_{1}, \pi_{2}, ..., \pi_{n}\}\) and \(P_{T,1} = \{\pi_{i(1)}, \pi_{i(2)}, ..., \pi_{i(n)}\}\) be the initial set of \(P_{S}\) and \(P_{T}\), respectively, where \(\text{Min}(\pi_{1}) = \{1\}\), \(\text{Min}(\pi_{k}) = \{1, k\}\) if \(k \geq 2\) and \(\text{Min}(\pi_{i(1)}) = \{i(1)\}\), \(\text{Min}(\pi_{i(k)}) = \{i(1), i(k)\}\) if \(i(k) \geq i(2)\).

Suppose that \(S\) and \(T\) are isomorphic. Let \(\phi : S \rightarrow T\) be an isomorphism.

Hereafter \(m\) denotes a sufficiently large integer.

The following lemma is easy to verify:

Lemma 1. Let \(\pi = \{X_{1}, X_{2}, ..., X_{r}\} \in P_{S}\) with \(\text{Min}(X_{i}) = x_{i}\) \((i = 1, 2, ..., r)\). Then:

1. Fix(\(\alpha\)) = Min(\(\pi\)) for every \(\alpha \in J(\pi, \leq_{S})\).
2. \(e \in T(X)\) defined by \(x_{i} = x_{i}\) \((i = 1, 2, ..., r)\) is a unique idempotent of \(J(\pi, \leq_{S})\).
3. For \(\alpha \in S, \alpha \in J(\pi, \leq_{S})\) if and only if \(\alpha^{m} = e\), where \(e\) is the unique idempotent \(J(\pi, \leq_{S})\).
For $\pi_k \in P_{S,1}$ (resp. $\pi_{i(k)} \in P_{T,1}$), the unique idempotent of $J(\pi_k, \leq_S)$ (resp. $J(\pi_{i(k)}, \leq_T)$) is denoted by $\varepsilon_k$ (resp. $\varepsilon_{i(k)}$). Then $X_{i1} = 1$ (resp. $X_{i(1)} = i(1)$). Therefore $\varepsilon_1$ (resp. $\varepsilon_{i(1)}$) is the zero of $S$ (resp. $T$), which is denoted by $0_S$ (resp. $0_T$).

**Lemma 2.** (1) For $\mu \in S$, $\mu^{n-1} = 0_S$ and $\mu^{n-2} \neq 0_S$ if and only if $k\mu = k-1$ if $k \geq 2$ and $1\mu = 1$. (2) For $\alpha \in S$, $\Omega(\alpha) \in P_{S,1}$ if and only if $(\alpha\beta)^m = 0_S$ or $(\alpha\beta)^m = \alpha^m \neq 0_S$ for every $\beta \in S$.

Let $\mu$ be as in (1) of Lemma 2. Since $(\mu\phi)^{n-1} = (\mu^{n-1})\phi = 0_S\phi = 0_T$ and $(\mu\phi)^{n-2} = (\mu^{n-2})\phi \neq 0_T$, applying (1) of Lemma 2 to $T$, we have that $i(k)(\mu\phi) = i(k-1)$ if $i(k) \geq i(2)$ and $i(1)(\mu\phi) = i(1)$. By using this fact, we obtain the following lemmas.

**Lemma 3.** $\varepsilon_k\phi = \varepsilon_{i(k)}$ for every $\pi_k \in P_{S,1}$.

**Lemma 4.** Let $\pi_k = \{X_1, X_k\} \in P_{S,1}$ with $\text{Min}(X_1) = 1$ and $\text{Min}(X_k) = k$. Then $\pi_{i(k)} = \{X_1\xi, X_k\xi\} \in P_{T,1}$ with $\text{Min}(X_1\xi) = i(1)$ and $\text{Min}(X_k\xi) = i(k)$.

**Proof of Theorem 1.** We show that $i(x)(\alpha\phi) = i(x\alpha)$ for every $x \in X$ and every $\alpha \in S$. Then we can show that $T = \xi^{-1}S\xi$. In fact, since $i(x)(\xi^{-1}\alpha\xi) = \alpha\xi\xi = i(x\alpha)$ for every $x \in X$ and for every $\alpha \in S$, we have that $\alpha\phi = \xi^{-1}\alpha\xi$.

If $x = 1$, then the assertion is trivially true.

For $x \geq 2$, let $\pi_x = \{X_1, X_x\} \in P_{S,1}$ with $\text{Min}(X_1) = 1$ and $\text{Min}(X_x) = x$. We first show that if $X_x = \{x\}$ then $i(x)(\alpha\phi) = i(x\alpha)$ for every $\alpha \in S$. Let $x\alpha = k$ and $i(x)(\alpha\phi) = i(j)$ for some $\alpha \in S$. Since $X_x = \{x\}$, $x\epsilon_x = x$ and $y\epsilon_x = 1$ if $y \neq x$, so that $x\epsilon_x\alpha = k$ and $y\epsilon_x\alpha = 1$. If $k < j$, then $x\epsilon_x\alpha\epsilon_j = j\epsilon_j = 1$ and $y\epsilon_x\alpha\epsilon_j = j\epsilon_j = 1$ for every $y \in X$ with $y \neq x$. Thus $\varepsilon_x\alpha\epsilon_j = 0_S$. But $i(x)(\epsilon_x\alpha\epsilon_j)\phi = i(x)\epsilon_x\alpha\epsilon_j(\alpha\phi)\epsilon_i(j) = i(x)(\epsilon_x\alpha\epsilon_j(\alpha\phi))\epsilon_i(j) = i(x)(\epsilon_x\alpha\epsilon_j)\phi \neq 0_S$, a contradiction. Thus $i(x)(\epsilon_x\alpha\epsilon_j)\phi = 0_S$, a contradiction. If $j < k$, then we similarly have $\epsilon_n\alpha\epsilon_k = 0_S$ and $(\epsilon_n\alpha\epsilon_k)\phi = 0_T$, again a contradiction. Thus $k = j$, so that $i(x)(\alpha\phi) = i(k) = i(x\alpha)$.

We now inductively show the above assertion. For $x = n$, let $\pi_n = \{X_1, X_n\}$ with $\text{Min}(X_1) = 1$ and $\text{Min}(X_n) = n$. Then $X_n = \{n\}$. From the above fact, $i(n)(\alpha\phi) = i(n\alpha)$ for every $\alpha \in S$.

Suppose that, for some $x \in X$ with $x \geq 2$, if $r \succ x$, then $i(r)(\alpha\phi) = i(r\alpha)$ for every $\alpha \in S$. Then we show that $i(x)(\alpha\phi) = i(x\alpha)$. Let $x\alpha = t$ and $\pi_x = \{X_1, X_x\}$ with $\text{Min}(X_1) = 1$ and $\text{Min}(X_x) = x$. If $X_x = \{x\}$, then again by the above fact the assertion is true. Suppose that there exists $r \in X_x$ such that $r \succ x$. Then $r\epsilon_x\alpha = x\alpha = t$. By the assumption, $i(r)(\epsilon_x\alpha)\phi = i(r\epsilon_x\alpha) = i(t)$. Since $\pi_{i(t)} = \{X_1\xi, X_t\xi\}$, we have $i(r) = r\xi \in X_t\xi$, so that $i(r)(\epsilon_t\alpha) = i(x)$. Thus $i(x\alpha) = i(t)(\epsilon_x\alpha)\phi = i(r)(\epsilon_x\alpha)\phi = i(x)(\alpha\phi)$. The proof is complete.
Suppose that $\leq_S = \leq_T$. Then, since $\xi = id_{T(X)}$, we have that $S \cong T$ implies $S = T$, so that $P_S = \Omega(S) = \Omega(T) = P_T$. Consequently, if $P_S \neq P_T$, then $S$ and $T$ are not isomorphic. From Result 3, we have that $P_S = P_T$ if and only if $P_{1,S} = P_{1,T}$. Since the number of initial sets with respect to a fixed total order $\leq \mathrm{o}n X$ is $2^{(n-1)(n-2)/2}$, we obtain:

**Corollary 2** [3]. There are $2^{(n-1)(n-2)/2}$ maximal $\mathcal{J}$-trivial subsemigroups of $T(X)$ up to isomorphisms if $|X| = n$.

**References**


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