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# Low dimensional Homotopy and homology for monoid presentations

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The homotopy theory introduced by Squier [9] gives a very general framework for geometrical approach to monoid presentations. It is considered to include the classical geometrical theory of group presentations ([6], [7]). In this note we report recent results on a relation between the homotopical and the homological finiteness conditions of monoids. The details will be given in [4].

## 1. Homotopy relations on the derivation graphs

Let $\Sigma$ be a finite alphabet and $\Sigma^*$ be the free monoid generated by $\Sigma$. Let E be a irreflexive symmetric relation on $\Sigma^*$. Let $=_E$ denote the congruence on $\Sigma^*$ generated by E, that is, $=_E$ is the smallest compatible equivalence relation containing E. Let $M = \Sigma^*/=_E$ be the quotient monoid. We say M is presented by a presentation $(\Sigma, E)$ and write as $M = M(\Sigma, E)$.

We define a graph $\Gamma = \Gamma(\Sigma, E)$ called the derivation graph of $(\Sigma, E)$ as follows: The set of vertices is $\Sigma^*$, and for $e = (u, v) \in E$ and $x, y \in \Sigma^*$, $(x; u, v; y)$ is an edge from xuy to xvy. An equation $e = (u, v) \in E$ is identified with the naked edge $(1; u, v; 1)$. Since E is symmetric, if $e = (x; u, v; y)$ is an edge, then $(x; v, u; y)$ is also an edge, which is called the reverse of e and written $e^{-1}$.

Let $P(x, y)$ denote the set of (directed) paths from the source x to the target y in $\Gamma$ and $P(\Gamma)$ the set of all the paths in $\Gamma$. The source and the target of a path p are denoted by $\sigma(p)$ and $\tau(p)$, respectively. For two paths p and q such that $\tau(x) = \sigma(y)$, the path $p \circ q$ connecting p and q is defined in the obvious manner. For a path $p = e_1 \circ e_2 \circ \dots \circ e_n$ its reverse $p^{-1}$ =

$e_n^{-1} \circ \ldots \circ e_2^{-1} \circ e_1^{-1}$ is a path from $\tau(p)$ to $\sigma(p)$. A path p with $\sigma(p) = \tau(p)$ is called a closed path with base point $\sigma(p) = \tau(p)$. The set P(x, x) of closed path with base point x is denoted simply by P(x). A <u>cycle</u> is a simple closed path, that is, a closed path which does not pass the same edge twice. The trivial cycle $i_x$ with base point x is the cycle of length 0 at x.

The free monoid $\Sigma^*$ acts on $P(\Gamma)$ on the both sides as follows: Let z, w $\in \Sigma^*$. For an edge e = (x; u, v; y) define

$z \cdot e \cdot w = (zx;\ u,\ v;\ yw),$

and for a path $p = e_1 \circ e_2 \circ \ldots \circ e_n$ define

$z \cdot p \cdot w = (z \cdot e_1 \cdot w) \circ (z \cdot e_2 \cdot w) \circ \ \ldots \ \circ (z \cdot e_n \cdot w).$

Two paths p and q with the same source and the same target are called <u>parallel</u>, and written as $p \| q$.

An equivalence relation $\sim$ on $P(\Gamma)$ is a <u>homotopy relation</u> (Squier, Otto and Kobayashi [9]), if it is contained in $\|$ and satisfies the following four conditions.

(H1) For any $x \in \Sigma^*$ and $e_1 = (u_1, v_1)$, $e_2 = (u_2, v_2) \in E$,

$e_1 \cdot xu_2 \circ v_1 x \cdot e_2 \sim u_1 x \cdot e_2 \circ e_1 \cdot xv_2.$

(H2) For any p, q $\in P(\Gamma)$ and x, y $\in \Sigma^*$,

$p \sim q$ implies $x \cdot p \cdot y \sim x \cdot q \cdot y.$

(H3) For any p, q, r, s $\in P(\Gamma)$ with $\tau(r) = \sigma(p) = \sigma(q)$ and $\sigma(s) = \tau(p) = \tau(q)$,

$p \sim q$ implies $r \circ p \circ s \sim r \circ q \circ s.$

(H4) For any e = (u, v) $\in$ E,

$e \circ e^{-1} \sim i_u.$

Let $\sim$ be a homotopy relation. The equivalence class of p $\in P(\Gamma)$ modulo the relation $\sim$ is written as $[p]_\sim$ or simply as [p]. The quotients $P(\Gamma)/\sim$ is denoted by $\pi_\sim(\Gamma) = \pi_\sim(\Sigma, E)$. The operation $\circ$ of $P(\Gamma)$ induces the operation $\circ$ of $\pi_\sim(\Gamma)$ by $[p] \circ [q] = [p \circ q]$. The action of $\Sigma^*$ on $\pi_\sim(\Gamma)$ is also induced as $x \cdot [p] \cdot y = [x \cdot p \cdot y]$.

We can also consider the quotients $P(x, y)/\sim$ and $P(x)/\sim$, which are denoted by $\pi_\sim(x, y)$ and $\pi_\sim(x)$, respectively. Then,

$\pi_{\sim}(\Gamma) = \bigcup_{x,y\in\Sigma^*} \pi_{\sim}(x, y)$ forms a groupoid and $\pi_{\sim}(x)$ forms a group with operation $\circ$. The group $\pi_{\sim}(x)$ is called the homotopy group at x with respect to $\sim$. In particular, $\pi_0(x)$ denotes the homotopy group at x with respect to $\sim_0$.

The parallelism $\parallel$ is the largest homotopy relation. Since homotopy relations are closed under intersection, for any subset B of $\parallel$, there is the smallest homotopy relation $\sim_B$ containing B, called the homotopy relation generated by B. The homotopy relation $\sim_0$ generated by the empty set is the smallest homotopy relation.
If $p \sim_0 q$, we say two paths p and q are strictly homotopic. If there is a finite subset B of $\parallel$ such that $\sim_B = \parallel$, $(\Sigma, E)$ has finite homotopy type and B is called a finite homotopy base for it. If the empty set generates $\parallel$, that is, $\sim_0 = \parallel$, $(\Sigma, E))$ is strictly aspherical.

## 2. Homotopy reduction systems

In this section $(\Sigma, E)$ is a fixed monoid presentation and $\Gamma = \Gamma(\Sigma, E)$.

A subset B of $\parallel$ is a homotopy reduction system; B is a set of pairs of parallel paths, and an element (p, q) of B is called a rule and written $p \rightsquigarrow q$. For two paths p and q, we write $p \rightsquigarrow_B q$, if there are $x, y \in \Sigma^*$, $p_1, p_2 \in P(\Gamma)$ and $r \rightsquigarrow s \in B$ such that

$$p = p_1 \circ (x \cdot r \cdot y) \circ p_2, \quad q = p_1 \circ (x \cdot s \cdot y) \circ p_2.$$

The reflexive and transitive closure and the reflexive, symmetric transitive closure of $\rightsquigarrow_B$ is denoted by $\rightsquigarrow_B^*$ and $\leftrightsquigarrow_B^*$, respectively.

B is noetherian, if there is no infinite sequence of reductions:

$$p_1 \rightsquigarrow_B p_2 \rightsquigarrow_B \cdots \rightsquigarrow_B p_i \rightsquigarrow_B \cdots .$$

B is confluent, if for any paths p, q and r such that $p \rightsquigarrow_B^* q$ and $p \rightsquigarrow_B^* r$, there is a path s such that $q \rightsquigarrow_B^* s$, $r \rightsquigarrow_B^* s$.

A path p is irreducible with respect to B, if there is no q such that $p \rightsquigarrow_B q$. If B is noetherian and confluent, then for any path p there is a unique irreducible path $\hat{p}$ such that p $\rightsquigarrow_B^* \hat{p}$, which is called the canonical form of p with respect to $\rightsquigarrow_B$.

The relation $\leftrightsquigarrow_B^*$ is contained in the homotopy relation $\sim_B$ generated by B. A system B is called complete, if it is noetherian and confluent and $\leftrightsquigarrow_B^* = \sim_B$. If B is complete, then

$$p \sim_B q \iff \hat{p} = \hat{q}.$$

A system B is reduced, if for any (p, q) $\in$ B, p is irreducible under B - {(p, q)} and q is irreducible under B. B is simple, if any (p, q) $\in$ B cannot be written as

$$p = p' \circ xry \circ p'', \quad q = p' \circ xsy \circ p''$$

with paths p', p" and x, y $\in \Sigma^*$ such that at least one of p', p", x and y is non-trivial. These p', p", x and y are the coats of (p, q).

Two reduction systems $B_1$ and $B_2$ are equivalent, if $\sim_{B_1} = \sim_{B_2}$.

Theorem 2.1 For any complete homotopy reduction system B generating ∥, there is a reduced simple complete system $\overline{B}$ generating B such that $|\overline{B}| \leq |B|$.

Let B be a reduction system. Let r and s be two subpaths of a path p; p = q∘xry∘t = q'∘x'sy'∘t' with paths q, t, q', t' and x, y, x', y' $\in \Sigma^*$. r is left to s in p if the length of q is shorter than that of q'. A sequence of reductions is left-most, if in each step of the reductions a rule is applied to a left-most applicable subpath. If the system B is simple and reduced, a left-most applicable subpath is unique, and so a sequence of the left-most reductions starting with given path is unique.

Let p and q be parallel paths in $\Gamma = \Gamma(\Sigma, E)$. A sequence of two-way reductions from p to q is a sequence

$$h: p_0 \leftrightsquigarrow_B p_2 \leftrightsquigarrow_B \cdots \leftrightsquigarrow_B p_n$$

with $q_0$ = p and $q_n$ = q. $P_i \leftrightsquigarrow P_{i+1}$ is a one-step reductions with respect to $B \cup B^{-1}$, $B^{-1} = \{b^{-1} | b \in B\}$, where $b^{-1}$ is the reverse reduction of b. The path p is the source and q is the target of h and are denoted by $\sigma(h)$ and $\tau(h)$, respectively. h is a (homotopy) reduction cycle, if $\sigma(h) = \tau(h)$.

Let h be a sequence of (two-way) reductions between parallel paths p and q. Let $x, y \in \Sigma^*$ and s and t be paths such that $\tau(s) = x\sigma(p)y$ and $\sigma(t) = x\tau(p)y$. Then $s \circ xpy \circ t$ and $s \circ xqy \circ t$ are parallel path and we have a sequence of reductions $s \circ x \cdot h \cdot y \circ t$ between them, which coincides with h on the subpath p and does not affect the outside. Let g be another sequence of reductions from q to r, then $h \star g$ denotes the conjunction of them; $h \star g$ is the reduction sequence from p to r. We give less priority to the operation $\star$ over $\cdot$ and $\circ$, thus, for example, $(s \circ x \cdot h \cdot y \circ t) \star g$ is written as $s \circ xhy \circ t \star g$. The reverse sequence of reductions of h is denoted by $h^{-1}$.

## 3. Critical pairs

Two paths p and q are coaxal, if there are $x, x', y, y' \in \Sigma^*$ such that $x \cdot p \cdot x' = y \cdot q \cdot y'$. The foursome $(x, x', y, y')$ is an adjuster for p and q. An adjuster with the length $|xx'yy'|$ minimal is unique and called the minimal adjuster for p and q. If $(x, x', y, y')$ is minimal, x or x' and y or y' are empty words.

Let $p_1 \rightsquigarrow q_1$, $p_2 \rightsquigarrow q_2 \in B$. Suppose $p_1 = p_1' \circ t_1$, $p_2 = t_2 \circ p_2'$ with coaxal $t_1$ and $t_2$ and nontrivial $p_1'$ and $p_2'$. In this situation we say that $p_1 \rightsquigarrow q_1$ overlaps with $p_2 \rightsquigarrow q_2$ on the left. Let $(x, x', y, y')$ be the minimal adjuster for $t_1$ and $t_2$. Then we have $x \cdot p_1 \cdot x' \circ y \cdot p_2' \cdot y' = x \cdot p_1' \cdot x' \circ y \cdot p_2 \cdot y'$ (= p) and $p \rightsquigarrow_B x \cdot q_1 \cdot x' \circ y \cdot p_2' \cdot y'$, $p \rightsquigarrow_B x \cdot p_1' \cdot x' \circ y \cdot q_2 \cdot y'$. We call the pair of paths

$$(x \cdot q_1 \cdot x' \circ y \cdot p_2' \cdot y',\ x \cdot p_1' \cdot x' \circ y \cdot q_2 \cdot y')$$

a critical pair of overlapping type. Next suppose $p_1 = p_1' \circ t_1 \circ p_1''$ and $t_1$ and $p_2$ are coaxal. Let $(x, x', y, y')$ be the minimal adjuster for them. Then we have $x \cdot p_1 \cdot x' = x \cdot p_1' \cdot x' \circ y \cdot p_2 \cdot y' \circ x \cdot p_1'' \cdot x'$ (= p) and $p \rightsquigarrow_B x \cdot q_1 \cdot x'$, $p \rightsquigarrow_B x \cdot p_1' \cdot x' \circ y \cdot q_2 \cdot y' \circ x \cdot p_1'' \cdot x'$. We call

$$(x \cdot q_1 \cdot x',\ x \cdot p_1' \cdot x' \circ y \cdot q_2 \cdot y' \circ x \cdot p_1'' \cdot x')$$

a <u>critical pair of inclusion type</u>. If B is reduced, there is no critical pair of inclusion type.

A critical pair (r, s) is <u>resolvable</u>, if there is a path t such that $r \rightsquigarrow_B^* t$ and $s \rightsquigarrow_B^* t$. Since locally confluent noetherian system are confluent ([1], [3]), we have

Theorem 3.1. A noetherian reduction system is confluent, if and only if all the critical pairs are resolvable.

We consider the following subsets of ∥:

$B_0 = \{ e \circ e^{-1} \rightsquigarrow i_u \mid e = (u, v) \in E \}$

and

$B_\ell = \{ u_1 x \cdot e_2 \circ e_1 \cdot x v_2 \rightsquigarrow e_1 \cdot x u_2 \circ v_1 x \cdot e_2 \mid$
$x \in \Sigma^*,\ e_1 = (u_1, v_1),\ e_2 = (u_2, v_2) \in E \}$.

Let B be a homotopy reduction system for (Σ, E) generating the parallelism ∥. We suppose that $B \cap (B_0 \cap B_\ell) = \emptyset$. Let $\bar{C}$ be the set of critical pairs associated with $\bar{B} = B_0 \cup B_\ell \cup B$.

Let $b_1 = (p, q) \in B_\ell$, where

$p = u_1 x e_2 \circ e_1 x v_2,\ q = e_1 x u_2 \circ v_1 x e_2,$

with $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2) \in E$. Let $b_2 = (s, t)$ be a rule in B which overlaps with $b_1$ on the left, that is, $s = w e_1 z \circ s_2$ with $w, z \in \Sigma^*$ such that z is a prefix of $x v_2$ or $x v_2$ is a prefix of z. If z is a prefix of x, we say that they <u>overlap weakly</u>, otherwise we say that they <u>overlap strongly</u>.

If $b_1$ overlaps weakly with $b_2$ on the left, then $x = z x'$ for some $x' \in \Sigma^*$ and $wp \circ s_2 x' v_2 = w u_1 x e_2 \circ s x' v_2$. Then there are one-step reductions

$k_1$: $wp \circ s_2 x' v_2 \rightsquigarrow_{B_\ell} wq \circ s_2 x' v_2$

and

$k_2$: $w u_1 z x' e_2 \circ s x' v_2 \rightsquigarrow_B w u_1 x e_2 \circ t x' v_2$,

and we have a critical pair $(wq \circ s_2 x' v_2,\ w u_1 x e_2 \circ t x' v_2)$. This critical pair is always resolvable because we have the following reductions:

$j_1'$: $wq \circ s_2 x' v_2 = w e_1 x u_2 \circ w v_1 x e_2 \circ s_2 x' v_2$

$\rightsquigarrow_{B_\ell} we_1xu_2\circ s_2x'u_2\circ\tau(s)x'e_2 = sx'u_2\circ\tau(s)x'e_2$

$\rightsquigarrow_B tx'u_2\circ\tau(t)x'e_2$,

and

$j_2'$: $wu_1xe_2\circ tx'v_2 \rightsquigarrow_{B_\ell} tx'u_2\circ\tau(t)x'e_2$.

If $b_1$ weakly overlaps with $b_2$ on the right, that is, $s = s_1\circ ze_2w$ and w is a suffix of x; $x = x'z$. Then we have $u_1x's_1\circ pw = u_1x's\circ e_1xv_2w$ and we have a critical pair $(u_1x't\circ e_1xv_2w,\ u_1x's_1\circ qw)$. This critical pair is also resolvable in a similar way.

Theorem 3.2. If E and B is finite, then there are only a finite number of strong overlappings between $B_\ell$ and B, and there are only a finite number of critical pairs of inclusion type for $\bar{B}$.

Theorem 3.4. Assume that E is non-special and $\bar{B} = B_0 \cup B_\ell \cup B$ is noetherian. Then $\bar{B}$ is complete, if and only if the following four types of critical pairs are all resolvable: critical pairs coming from (1) overlapping between rules from $B_0$ and B, (2) overlapping between rules inside B, (3) strong overlapping between rules from $B_\ell$ and B, and (4) inclusion between rules from $\bar{B}$ and B.

Corollary 3.5. Assume that both E and B is finite and $\bar{B} = B_0 \cup B_\ell \cup B$ is noetherian, then it is decidable whether $\bar{B}$ is complete.

## 4. The (coated) left canonical reduction system

An equation $(u, v) \in E$ is special, if $u = 1$ or $v = 1$. The system E is special, if all the equations in it are special. E is non-special, if every equation in it is not special.

The simplest complete system called the left canonical reduction system is given in the following

Theorem 4.1. If $(\Sigma, E)$ is non-special, then $B = B_0 \cup B_\ell$

is a complete reduction system such that $\sim_B = \sim_0$.

Thus, for a non-special presentation, any path p has the unique canonical form $\hat{p}$ with $p \sim_0 \hat{p}$.

Corollary 4.2. For parallel paths p and q, $p \sim_0 q$ if and only if $\hat{p} = \hat{q}$. In particular, for a closed path p at $x \in \Sigma^*$, $p \sim_0 i_x$ if and only if $\hat{p} = i_x$.

Corollary 4.3. Let $(\Sigma, E)$ be a finite non-special presentation. For given paths p and q it is decidable whether $p \sim_0 q$.

Let $x, y \in \Sigma^*$. We say that x <u>overlaps with y on the left</u> if there is $u \neq 1$ such that $x = x_1u$, $y = uy_1$ for some $x_1$, $y_1$ with $x_1y_1 \neq 1$. Let OVL(x, y) be the set of all such words u. Since it is convenient to include the perfect overlapping, we define the set $\overline{\mathrm{OVL}}(x, y)$ by

$$\overline{\mathrm{OVL}}(x, y) = \{ u \in \Sigma^+ \mid x = x_1u,\ y = uy_1,\ x_1,\ y_1 \in \Sigma^*\}.$$

Let $(\Sigma, E)$ be a non-special presentation. For $e = (u, v) \in E$, let $\lambda = \lambda(e)$ (resp. $\rho = \rho(e)$) be the maximal common prefix (resp. suffix) of u and v. We call $\lambda$ (resp. $\rho$) the <u>left</u> (resp. <u>right</u>) <u>coat</u> of e.

Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be equations from E. Let $\rho$ (resp. $\lambda$) be the right (resp. left) coat of $e_1$ (resp. $e_2$); $u_1 = u_1'\rho$, $v_1 = v_1'\rho$, $u_2 = \lambda u_2'$, $v_2 = \lambda v_2'$. For a word $\gamma \in \overline{\mathrm{OVL}}(\rho, \lambda)$; $\rho = \rho'\gamma$, $\lambda = \gamma\lambda'$, we consider a pair $b_\gamma = (u_1'\rho' \cdot e_2 \circ e_1\lambda' v_2',\ e_1 \cdot \lambda' u_2' \circ v_1'\rho' \cdot e_2)$ of parallel paths in $\Gamma$. Let $B_c$ be the set of all these pairs $b_\gamma$ of paths in $\Gamma$.

An equation of the form (v, u) (or (u, v)) with $u \in \Sigma^*$ and $v \in u\Sigma^* \cap \Sigma^*u$ is called <u>subspecial</u>. A system E is <u>non-subspecial</u> if it contains no subspecial equation.

Theorem 4.4. If E is non-subspecial, then the reduction system $B = B_0 \cup B_\ell \cup B_c$ is complete.

The B in Theorem 4.1 is called the <u>coated left canonical reduction system</u>. If this $B_c$ generates $\parallel$, then E is called <u>coated aspherical</u>.

## 5. Low-dimensional homology

Let a monoid M be given by a presentation $(\Sigma, E)$. For $x \in \Sigma^*$, $\bar{x}$ denotes the image of x in M. Let R be an <u>orientation</u> of E, that is, R is an asymmetric relation such that $E = R \cup R^{-1}$, where $R^{-1} = \{(v, u) | (u, v) \in R\}$. The oriented R is called a rewriting system for M. Let $A = \mathbb{Z}M$ be the monoid algebra of M over $\mathbb{Z}$. The additive group $\mathbb{Z}$ is considered to be a right A-module by the trivial action:

$$1 \cdot x = 1$$

for $x \in M$.

We say that M satisfies the (<u>right</u>) <u>homological finiteness condition</u> $FP_n$, if there is a resolution of $\mathbb{Z}$:

$$F_n \to F_{n-1} \to \dots \to F_0 \to \mathbb{Z}$$

such that $F_i$ are finitely generated free right A-modules for $i = 0, \dots, n$.

The algebra A itself is considered to be the free cyclic right A-module which we denote by $F_0$. There is a natural surjection $\varepsilon: F_0 \to \mathbb{Z}$, which is an A-homomorphism defined by $\varepsilon(1) = 1$.

Let $F_1$ be the free right A-module generated by the set $\Sigma$. We write $\langle a \rangle$ for the generator of $F_1$ corresponding to $a \in \Sigma$. For a word $x = a_1 \cdots a_n \in \Sigma^*$, we define an element $\langle x \rangle$ of $F_1$ by

$$\langle x \rangle = \langle a_1 \rangle \cdot \overline{a_2 \cdots a_n} + \langle a_2 \rangle \cdot \overline{a_3 \cdots a_n} + \dots + \langle a_n \rangle.$$

Let $F_2$ be the free right A-module generated by the set R. By $\langle r \rangle$ we denote the corresponding generator to $r \in R$.

Let B be a homotopy base for $(\Sigma, E)$. Let $F_3$ be the free right A-module generated by B. Again $\langle b \rangle$ denote the generator corresponding to $b \in B$. For $e = (u, v) \in E$ define

$$\langle e \rangle = \begin{cases} \langle r \rangle & \text{if } r = (u, v) \in R \\ -\langle r \rangle & \text{if } r = (v, u) \in R. \end{cases}$$

Let $p = x_1 e_1 y_1 \circ \dots \circ x_n e_n y_n$ be a path in $\Gamma = \Gamma(\Sigma, E)$, where $x_i$,

$y_i \in \Sigma^*$ and $e_i$ are edges. Then, define an element $\langle p\rangle$ of $F_2$ by

$$\langle p\rangle = \langle e_1\rangle\cdot\overline{y_1} + \dots + \langle e_n\rangle\cdot\overline{y_n}.$$

Now, we define A-homomorphisms $\partial_1\colon F_1 \to F_0$, $\partial_2\colon F_2 \to F_1$ and $\partial_3\colon F_3 \to F_2$ by

$$\partial_1(\langle a\rangle) = \overline{a} - \overline{1}$$

for $a \in \Sigma$,

$$\partial_2(\langle r\rangle) = \langle u\rangle - \langle v\rangle$$

for a rule $r = (u, v) \in R$, and

$$\partial_3(\langle b\rangle) = \langle p\rangle - \langle q\rangle$$

for $b = (p, q) \in B$. Easily we find

$$\varepsilon\circ\partial_1 = \partial_1\circ\partial_2 = \partial_2\circ\partial_3 = 0,$$

and hence we have a complex:

$$F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0. \qquad (5.1)$$

Theorem 5.1 ([2], [5], [8]). The complex (5.1) is exact.

Theorem 5.2. If a finitely presented monoid M has finite homotopy type, then it satisfies right and left $FP_3$. If M has a strictly aspherical presentation, then there is a free resolution over $\mathbb{Z}$:

$$0 \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

and M has cohomological dimension at most 2.

Now suppose that B is a homotopy base and $\overline{B} = B_0 \cup B_\ell \cup B$ is a simple reduced complete reduction system. Let $\overline{C}$ be the set of critical pairs associated with $\overline{B}$, and let C ($\subset \overline{C}$) be the set of critical pairs of type (1) - (3) in Theorem 3.4 (since $\overline{B}$ is reduced we need not consider the critical pairs of type (4)).

Let $c = (p, q) \in C$ be a critical pair which comes from path r by one-step reductions $h_1\colon r \rightsquigarrow_B p$ and $h_2\colon r \rightsquigarrow_B q$. Since B is complete, there are sequences of reductions from p to $\hat{r}$ and q to $\hat{r}$, where r is the the canonical form of r. Let $h_3$ and $h_4$ be the left-most reductions from p and q to $\hat{r}$ respectively. Then, we have a cycle of two-way reductions:

$$H(c) = h_1 \star h_3 \star h_4^{-1} \star h_2^{-1},$$

which is called the <u>reduction</u> <u>cycle</u> <u>associated</u> <u>with</u> c.

Now we define a free A-module $F_4$ and a boundary map $\partial_4$: $F_4 \to F_3$ as follows. Let $F_4$ be the free right A-module generated by C, and <c> denotes the generator corresponding to $c \in C$. The A-homomorphism $\partial_4$ is defined by

$$\partial_4(\langle c\rangle) = \langle H(c)\rangle$$

for $c \in C$. We can show that $\partial_3 \circ \partial_4 = 0$, and we have a complex:

$$F_4 \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0. \qquad (5.2)$$

Theorem 5.3. If $\overline{B} = B_0 \cup B_\ell \cup B$ is a complete reduction system, then the complex (5.2) is exact.

Theorem 5.4. If M is given by a finite presentation which admits a finite homotopy base B such that $\overline{B} = B_0 \cup B_\ell \cup B$ (or $\overline{B} = B_0 \cup B_r \cup B$) is a complete system, then M satisfies $FP_4$.

Corollary 5.5. If M is given by a finite non-subspecial presentation that is coated aspherical, then M satisfies $FP_4$.

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