On Termination of One-Rule String Rewriting Systems

天理大学教育研究科
辻（武宮）佳代子（Kayoko Shikishima-Tsuji）
京都産業大学総合学部
勝良 昌司（Masashi Katsura）
東邦大学理学部
小林 弥治（Yuji Kobayashi）

Let $\Sigma$ be a finite alphabet. The free monoid and the free semigroup generated by $\Sigma$ are denoted by $\Sigma^*$ and $\Sigma^+$, respectively. The length of a word $x$ in $\Sigma^*$ is denoted by $|x|$. For $x, y \in \Sigma^*$, we set $\text{OVL}(x, y) = \{z \in \Sigma^3 | x = uz, \ y = vz \text{ for some } u, v \in \Sigma^*\}$. A rewriting system $R$ on $\Sigma$ is a subset of $\Sigma^* \times \Sigma^*$. An element $(l, r)$ in $R$ is denoted by $l \rightarrow r$. If $R$ contains only one element, $R$ is said to be a one-rule rewriting system. A single step reduction relation $\rightarrow^*$ induced by $R$ is the following relation on $\Sigma^*$: For any $x, y \in \Sigma^*$, $x \rightarrow y$ if and only if there exists $(l, r) \in R$ such that $x = ulv$, $y = urv$, for some $u, v \in \Sigma^*$. $\rightarrow^*$ is the reflexive and transitive closure of $\rightarrow$.

A rewriting system $R$ is said to be confluent if for any $w, x, y \in \Sigma^*$, $w \rightarrow^* x \text{ and } w \rightarrow^* y$ imply $x \rightarrow^* z$ and $y \rightarrow^* z$ for some $z \in \Sigma^*$. $R$ is terminating (or noetherian) if there is no infinite sequence $x_1, x_2, \ldots$ such that $x_1 \rightarrow x_2 \rightarrow \ldots$. A confluent and terminating rewriting system is said to be complete.

It is not known whether the completeness is decidable for one-rule rewriting systems. Let $R = \{l \rightarrow r\}$ be a one-rule rewriting system. If $r \in \Sigma^*\Sigma^*$ then $R$ is always non-terminating. If $|l| \geq |r|$ and $l \neq r$ then $R$ is always terminating.

Result 1 [3] It is decidable whether or not a one-rule rewriting system is confluent.

Result 2 [2] For a confluent one-rule rewriting system $R = \{l \rightarrow r\}$ with $|l| < |r|$, we can effectively construct a rewriting system $R' = \{l' \rightarrow r'\}$ such that:

1. $|l'| < |r'|$ and $\text{OVL}(l', l) = \emptyset$.

2. $R'$ is terminating if and only if $R$ is terminating.
Hence the completeness problem for one-rule systems is reduced to the termination problem for one-rule systems \( R = \{ l \rightarrow r \} \) with \( \text{OVL}(l, l) = \emptyset \). It is not difficult to see that if \( \text{OVL}(r, l) = \emptyset \) or \( \text{OVL}(l, r) = \emptyset \) then \( R \) is terminating. In this note, we consider the case where \( \text{OVL}(r, l) = \{ p \} \), a singleton.

For each \( s \in \text{OVL}(l, r) \), we determine \( \bar{s} \in \Sigma^* \) by \( l = \bar{s}s \). The decidability of the terminating problem for such one-rule systems is given as follows.

**Theorem 1.** Let \( R = \{ l \rightarrow r \} \) be a one-rule rewriting system such that \( \text{OVL}(l, l) = \emptyset \) and \( \text{OVL}(r, l) = \{ p \} \). Let \( l = px \), \( r = y\tilde{s}_k \ldots \tilde{s}_1 p \), where \( s_1, ..., s_k \in \text{OVL}(l, r) \) and \( y \in \Sigma^* \) for any \( s \in \text{OVL}(l, r) \).

1. If there is a reduction of length \(|r| \leq 2|l|^2 \) starting with \((y\tilde{s}_k \ldots \tilde{s}_1)^3 \) then \( R \) is non-terminating.

2. Assume that the maximal length of reductions starting with \((y\tilde{s}_k \ldots \tilde{s}_1)^3 \) is \( N \) with \( N < |r|^2 \). If \( |x| > |l| \) and there is a reduction of length \( 2N + 1 \) starting with \((y\tilde{s}_k \ldots \tilde{s}_1)^4 \) then \( R \) is non-terminating, otherwise, \( R \) is terminating.

The exact characterization of non-terminating one-rule systems is given as follows.

**Theorem 2.** Let \( R = \{ l \rightarrow r \} \) be a one-rule rewriting system such that \( \text{OVL}(l, l) = \emptyset \) and \( \text{OVL}(r, l) = \{ p \} \). Then \( R \) is non-terminating if and only if one of the following conditions is satisfied.

\( (k, m, n) \) are positive integers. \( x, y, z, w \in \Sigma^+ \) and \( u, v \in \Sigma^* \).

1. \( l \in \Sigma^* \cdot \Sigma^* \).
2. \( r = s_k u \tilde{s}_k \ldots \tilde{s}_1 p \), \( s_1, ..., s_k \in \text{OVL}(l, r) \cap (xu)^*x \).
3. \( l = p(xu)^n \), \( r = (xu)^{n+m} \tilde{s}_k \ldots \tilde{s}_1 p \), \( s_1, ..., s_k \in \text{OVL}(l, r) \cap (xu)^*x \).
4. \( l = p(xu)^n \), \( r = (xu)^{n+m} \tilde{s}_k \ldots \tilde{s}_1 p \), \( s_1, ..., s_k \in \text{OVL}(l, r) \).

\( s_1, ..., s_{j-1} \in (xu)^*x \), \( s_j \in (xu)^i x \), \( 1 \leq j \leq k \), \( 1 \leq i \leq 2m \).

5. \( l = p(xu)^n \), \( r = y(xyz)^m \tilde{s}_k \ldots \tilde{s}_1 p \), \( s_1, ..., s_k \in \text{OVL}(l, r) \), \( s_1 = y(xyz)^m \), \( xyz = wxy \).

6. \( l = xy((xz^{mk}y)^{m-1}z^{y})^{n+1} \), \( r = y((xz^{mk}y)^{m-1}z^{y})^{n+1}z^{x}x^{y}z^{x} \).

7. \( l = zxyx(z^{2m+n} + 1)xyxv)^{m}x \), \( p = zxyxv, r = zxyx(z^{2m+n} + 1)xyxv)^{m}xux^{2m+n}p. \)

8. \( l = p u((p u)^{k} z)^{2m+n} p u((p u)^{k} z)^{2m+n} p)^{m-1}x \), \( p = zxyx, r = zxyx(p u)^{k} z)^{2m+n} p u((p u)^{k} z)^{2m+n} p)^{m-1}x \).

9. \( l = zy(x^{k-1}y^{m+n+1}x(y x)^{k-1}m)xy, r = x(y x)^{k-1}(y^{m+n+1}x(y x)^{k-1}m)xyz^{m+n+1}x(y x)^{k-1}m. \)
References


1 Tenri University
2 Kyoto Sangyo University
3 Toho University