Note on representations of generalized inverse \( \ast \)-semigroups\(^1\)

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Abstract

The Munn representation of an inverse semigroup \( S \), in which the semigroup is represented by isomorphisms between principal ideals of the semilattice \( E(S) \), is not always faithful. By introducing a concept of a presemilattice, Reilly considered of enlarging the carrier set \( E(S) \) of the Munn representation in order to obtain a faithful representation of \( S \) as an inverse subsemigroup of a structure resembling the Munn semigroup \( T_{E(S)} \).

The purpose of this paper is to obtain a generalization of the Reilly’s results for generalized inverse \( \ast \)-semigroups.

1 Introduction

A semigroup \( S \) with a unary operation \( \ast : S \to S \) is called a regular \( \ast \)-semigroup if it satisfies

\[
\begin{align*}
(i) & \quad (x^\ast)^\ast = x, \\
(ii) & \quad (xy)^\ast = y^\ast x^\ast, \\
(iii) & \quad xx^\ast x = x.
\end{align*}
\]

Let \( S \) be a regular \( \ast \)-semigroup. An idempotent \( e \) in \( S \) is called a projection if it satisfies \( e^\ast = e \). For any subset \( A \) of \( S \), denote the sets of idempotents and projections of \( A \) by \( E(A) \) and \( P(A) \), respectively.

Let \( S \) be a regular \( \ast \)-semigroup. It is called a locally inverse \( \ast \)-semigroup if, for any \( e \in E(S) \), \( eSe \) is an inverse subsemigroup of \( S \). If \( E(S) \) is a normal band, then \( S \) is called a generalized inverse \( \ast \)-semigroup.

Let \( S \) and \( T \) be regular \( \ast \)-semigroups. A homomorphism \( \phi : S \to T \) is called a \( \ast \)-homomorphism if \( (a\phi)^\ast = a^\ast \phi \). A congruence \( \sigma \) on \( S \) is called a \( \ast \)-congruence if

\(^1\)This is the abstract and the details will be published elsewhere
(a\sigma)^* = a^*\sigma$. A $*$-congruence $\sigma$ on $S$ is said to be idempotent-separating if $\sigma \subseteq \mathcal{H}$, where $\mathcal{H}$ is one of the Green’s relations. Denote the maximum idempotent-separating $*$-congruence on $S$ by $\mu_S$ or simply by $\mu$. If $\mu_S$ is the identity relation on $S$, $S$ is called fundamental. The following results are well-known, and we use them frequently throughout this paper.

**Result 1.1** [2]. Let $S$ be a regular $*$-semigroup. Then we have the following:

1. $E(S) = P(S)^2$
2. for any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$
3. each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection;
4. $\mu_S = \{(a,b) \in S \times S : a^*ea = b^*eb \text{ and } aea^* = beb^* \text{ for all } e \in P(S)\}$.

For a mapping $\alpha : A \to B$, denote the domain and the range of $\alpha$ by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset $C$ of $A$, $\alpha|_C$ means the restriction of $\alpha$ to $C$.

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse $*$-semigroups in [3], [4] and [5]. In this paper, we follow [5]. A non-empty set $X$ with a reflexive and symmetric relation $\sigma$ is called an $\iota$-set, and denoted by $(X; \sigma)$. If $\sigma$ is transitive, that is, if $\sigma$ is an equivalence relation on $X$, $(X; \sigma)$ is called a transitive $\iota$-set.

Let $(X; \sigma)$ be an $\iota$-set. A subset $A$ of $X$ is called an $\iota$-single subset of $(X; \sigma)$ if it satisfies the following condition:

for any $x \in X$, there exists at most one element $y \in A$ such that $(x, y) \in \sigma$.

We consider the empty set to be an $\iota$-single subset. We remark that if $(X; \sigma)$ is a transitive $\iota$-set, a subset $A$ of $X$ is an $\iota$-single subset if and only if, for $x, y \in A$, $(x, y) \in \sigma$ implies $x = y$. A mapping $\alpha$ in $\mathcal{I}_X$, the symmetric inverse semigroup on $X$, is called a partial one-to-one $\iota$-mapping on $(X; \sigma)$ if $d(\alpha), r(\alpha)$ are both $\iota$-single subsets of $(X; \sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of $\alpha$, respectively. Denote the set of all partial one-to-one $\iota$-mappings of $(X; \sigma)$ by $\mathcal{LI}_{(X;\sigma)}$.

For any $\iota$-single subsets $A$ and $B$ of $(X; \sigma)$, define $\theta_{A,B}$ by

$$\theta_{A,B} = \{(a, b) \in A \times B : (a, b) \in \sigma\} = (A \times B) \cap \sigma.$$ 

Since a subset of an $\iota$-single subset is also an $\iota$-single subset, $\theta_{A,B} \in \mathcal{LI}_{(X;\sigma)}$. For any $\alpha, \beta \in \mathcal{LI}_{(X;\sigma)}$, define $\theta_{\alpha,\beta}$ by $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{(X;\sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{LI}_{(X;\sigma)}$ as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$
where the multiplication of the right side of the first equality is that of $I_X$. Denote $(\mathcal{LI}(X;\sigma), \circ, \ast)$ by $\mathcal{LI}(X;\sigma)(M)$ or simply by $\mathcal{LI}(X;\sigma)$. In this paper, we use $\mathcal{LI}(X;\sigma)$ rather than $\mathcal{LI}(X;\sigma)(M)$.

**Result 1.2** [5]. For any $\iota$-set $(X;\sigma)$, $\mathcal{LI}(X;\sigma)$, defined above, is a locally inverse $\ast$-semigroup. If $(X;\sigma)$ is a transitive $\iota$-set, then $\mathcal{LI}(X;\sigma)$ is a generalized inverse $\ast$-semigroup. In this case, we denote it by $\mathcal{GI}(X;\sigma)$ instead of $\mathcal{LI}(X;\sigma)$.

Moreover, if $\sigma$ is the identity relation on $X$, then $\mathcal{LI}(X;\sigma)$ is the symmetric inverse semigroup $I_X$ on $X$.

We call $\mathcal{LI}(X;\sigma)[\mathcal{GI}(X;\sigma)]$ the $\iota$-symmetric locally [generalized] inverse $\ast$-semigroup on the $\iota$-set [the transitive $\iota$-set] $(X;\sigma)$ with the structure sandwich set $M$.

Let $S$ be a regular $\ast$-semigroup, and define a relation $\Omega$ on $S$ as follows:

$$(x, y) \in \Omega \iff \text{there exists } e \in E(S) \text{ such that } x\rho_e = y,$$

where $\rho_a(a \in S)$ is the mapping of $Sa^*$ onto $Sa$ defined by $x\rho_a = xa$.

**Result 1.3** [5]. Let $S$ be a locally inverse $\ast$-semigroup. For each $a \in S$, let

$$\rho_a : x \mapsto xa \quad (x \in d(\rho_a) = Sa^*).$$

Then a mapping

$$\rho : a \mapsto \rho_a$$

is a $\ast$-monomorphism of $S$ into $\mathcal{LI}(S;\Omega)(M)$.

For a partial groupoid $X$, if there exist a semilattice $Y$, a partition $\pi : X \sim \sum \{X_e : e \in Y\}$ of $X$ and mappings $\varphi_{e,f} : X_e \rightarrow X_f$ ($e \geq f$ in $Y$) such that

1. for any $e \in Y$, $\varphi_{e,e} = 1_{X_e}$,
2. if $e \geq f \geq g$, then $\varphi_{e,f}\varphi_{f,g} = \varphi_{e,g}$,
3. for $x \in X_e$, $y \in X_f$, $xy$ is defined in $X$ if and only if $x\varphi_{e,f} = y\varphi_{f,e}$, and in this case $xy = x\varphi_{e,f}$,

then $X$ is called a strong $\pi$-groupoid with mappings $\{\varphi_{e,f} : e, f \in Y, e \geq f\}$, and it is denoted by $X(\pi; Y; \{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong $\pi$-groupoid. A subset $A$ of $X$ is called a $\pi$-singleton subset of $X(\pi; Y; \{\varphi_{e,f}\})$, if there exists $e \in Y$ such that
\[ |A \cap X_f| = \begin{cases} 1 & \text{if } f \in \langle e \rangle, \\ 0 & \text{otherwise}, \end{cases} \]

\((A \cap X_f) \varphi_{f,g} = A \cap X_g\) for any \(f, g \in \langle e \rangle\) such that \(f \geq g\),

where \(\langle e \rangle\) is the principal ideal of \(Y\) generated by \(e\). In this case, we sometimes denote the \(\pi\)-singleton subset \(A\) by \(A(e)\). If \(A(e)\) is a \(\pi\)-singleton subset, then \(|A \cap X_f| = 1\) for any \(f \in \langle e \rangle\). We denote the only one element of \(A \cap X_f\) by \(a_f\). We remark that, for any \(\pi\)-singleton subset \(A(e), A(e) = \{a_e \varphi_{e,f} : f \in \langle e \rangle\}\).

Let \(X(\pi; Y; \{\varphi_{e,f}\})\) be a strong \(\pi\)-groupoid. Define an equivalence relation \(\mathcal{U}\) on \(\mathcal{X}\) by

\[ \mathcal{U} = \{(A(e), B(f)) \in \mathcal{X} \times \mathcal{X} : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)}\}. \]

For \((A(e), B(f)) \in \mathcal{U}\), let \(T_{A(e), B(f)}\) be the set of all \(\pi\)-isomorphisms of \(A(e)\) onto \(B(f)\), and let

\[ T_{X(\pi)} = \bigcup_{(A(e), B(f)) \in \mathcal{U}} T_{A(e), B(f)}. \]

For any \(\alpha, \beta \in T_{X(\pi)}\), define a mapping \(\theta_{\alpha, \beta}\) as follows:

\[ d(\theta_{\alpha, \beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\}, \]

\[ r(\theta_{\alpha, \beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\}, \]

\[ a \theta_{\alpha, \beta} = b \text{ if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}. \]

Then \(\theta_{\alpha, \beta} \in T_{X(\pi)}\). Let \(\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\pi)}\}\), and define a multiplication \(\circ\) and a unary operation \(*\) on \(T_{X(\pi)}\) by

\[ \alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta, \]

\[ \alpha^* = \alpha^{-1}. \]

Then \(T_{X(\pi)}(\circ, *)\) is a regular \(*\)-semigroup. We denote it by \(T_{X(\pi)}(\mathcal{M})\).
Result 1.4 [4]. A regular *-semigroup $T_{X(\pi)}(\mathcal{M})$ is a generalized inverse *-semigroup whose set of projections is partially isomorphic to $X$.

Let $S$ be a generalized inverse *-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of $E$, and let $P = P(E_i)$. Then $\pi : P \sim \sum\{P_i : i \in I\}$ is a partition of $P$. For any $i, j \in I$ ($i \geq j$), define a mapping $\varphi_{i,j} : P_i \rightarrow P_j$ by

$$e\varphi_{i,j} = efe \quad \text{for some (any) } f \in P_j.$$

Then $P(\pi; I; \{\varphi_{i,j}\})$ is a strong $\pi$-groupoid.

Result 1.5 [4]. Let $S$ be a generalized inverse *-semigroup. For each $a \in S$, let

$$\tau_a : e \mapsto a^*ea \quad (e \in d(\tau_a) = P(Sa^*)).$$

Then a mapping $\tau : a \mapsto \tau_a$ is a *-homomorphism of $S$ into $T_{P(\pi)}(\mathcal{M})$ such that $\tau \circ \tau^{-1} = \mu$.

A regular *-subsemigroup $T$ of a regular *-semigroup $S$ is said to be $\mathcal{P}$-full if $P(T) = P(S)$.

Result 1.6 [4]. A generalized inverse *-semigroup $S$ is fundamental if and only if it is *-isomorphic to a $\mathcal{P}$-full generalized inverse *-subsemigroup of $T_{X(\pi)}(\mathcal{M})$ on a strong $\pi$-groupoid $X(\pi; I; \{\varphi_{i,j}\})$ such that $P(T_{X(\pi)}(\mathcal{M}))$ is partially isomorphic to $P(S)$.

In § 2, by introducing the concept of partially ordered $\varrho$-set $(X(\vartriangleleft); \{\phi_\vartriangleleft\})$, we construct a fundamental generalized inverse *-semigroup $T_{X(\vartriangleleft)}(\mathcal{M})$. Also, we shall see that $T_{X(\vartriangleleft)}(\mathcal{M})$ has similar properties with $T_{X(\varpi)}(\mathcal{M})$, where $T_{X(\varpi)}(\mathcal{M})$ has been given by T. Imaoka, I. Inata and H. Yokoyama [4]. And we shall show that two concepts, strong $\pi$-groupoids and partially ordered $\varrho$-sets, are equivalent.

In § 3, we shall introduce the notion of $\omega$-set $(X(\lt); \sigma)$, and construct a generalized inverse *-semigroup $T_{X(\lt); \sigma}(\mathcal{M})$. Furthermore, let $S$ be a generalized inverse *-semigroup with the set of projections $P$, we shall make two generalized inverse *-semigroups $T_{P(\vartriangleleft)}(\mathcal{M})$ and $T_{S(\vartriangleleft); \Omega}(\mathcal{M})$, where the former is obtained in § 2, and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.
2 Fundamental generalized inverse $*$-semigroups

2.1 $T_{X(\triangleleft)}(\mathcal{M})$

Let $X(\triangleleft)$ be a partially ordered set and, for each $x \in X$, consider an order-preserving mapping $\phi_x : X \to X$. If a relation $\rho = \{(x, y) \in X \times X : y\phi_x = x, x\phi_y = y\}$ is an equivalence relation on $X$ such that

(P1) $x \leq y \implies$ for each $y' \in y\rho$, there exists $x' \in x\rho$ such that $x' \leq y'$,

(P2) a relation $\leq = \{(x\rho, y\rho) \in X/\rho \times X/\rho :$ there exists $x' \in x\rho$ such that $x' \leq y\}$ is a partial order and $X/\rho(\leq)$ is a semilattice,

(P3) $x_1 \leq y, x_2 \leq y$ and $x_1\rho \leq x_2\rho \implies x_1 \leq x_2$,

then $(X(\triangleleft); \{\phi_x\})$ is called a partially ordered $\rho$-set.

Let $(X(\triangleleft); \{\phi_x\})$ be a partially ordered $\rho$-set. Define an equivalence relation $\mathcal{U}$ on $X$ by

$$\mathcal{U} = \{(\langle a\rangle, \langle b\rangle) \in X \times X : \langle a\rangle \simeq \langle b\rangle \text{(order isomorphic)}\},$$

where $X$ is the set of all principal ideals of $(X(\triangleleft); \{\phi_x\})$. For $((a\rangle, \langle b\rangle) \in \mathcal{U}$, let $T_{(a\rangle},(b\rangle}$ be the set of all (order) isomorphisms of $\langle a\rangle$ onto $\langle b\rangle$, and let

$$T_{X(\triangleleft)} = \bigcup_{((a\rangle, \langle b\rangle) \in \mathcal{U}} T_{(a\rangle},(b\rangle}.$$

For any $\alpha, \beta \in T_{X(\triangleleft)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \rho\},$$

where $\rho$ is defined in $(X(\triangleleft); \{\phi_x\})$.

Then $\theta_{\alpha,\beta} \in T_{X(\triangleleft)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\triangleleft)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\triangleleft)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha,\beta} \beta,$$

$$\alpha^* = \alpha^{-1}.$$

Then it is clear that $T_{X(\triangleleft)}(\circ, *)$ is a regular $*$-subsemigroup of the $\iota$-symmetric generalized inverse $*$-semigroup $GI_{X(\triangleleft)}(\mathcal{M})$. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{X(\triangleleft)}(\mathcal{M})$.

Let $S$ be a generalized inverse $*$-semigroup and $P = P(S)$. We consider $P$ as a partially ordered set with respect to the natural order. Now, we have the following results.
Theorem 2.1 A regular $*$-semigroup $T_{X(\preceq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup whose set of projections is order isomorphic to $X(\preceq)$.

Corollary 2.2 A partially ordered set $X$ is order isomorphic to the set of projections of a generalized inverse $*$-semigroup if and only if it is a partially ordered $\phi$-set.

2.2 Representations

Let $S$ be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of $E$, and let $P_i = P(E_i)$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by
\[ f \phi_e = efe. \]
Let $e, f \in P$, define a relation $\preceq$ on $P$ by
\[ e \preceq f \iff e = fef, \]
that is, $\preceq$ is the restriction of natural order on $S$ to $P$.

Lemma 2.3 The set $(P(\preceq); \{\phi_e\})$, defined above, is a partially ordered $\phi$-set.

Now, we can consider the generalized inverse $*$-semigroup $T_{P(\preceq)}(\mathcal{M})$, where $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha$ and $\beta$ are order isomorphisms among principal ideals of $(P(\preceq); \{\phi_e\})\}$.

Lemma 2.4 For any $a \in S$, $P(Sa) (= P(Sa^*a))$ is a principal ideal of $(P(\preceq); \{\phi_e\})$.

For any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by
\[ e\tau_a = a^*ea, \]
where $e \in \langle aa^* \rangle$. It follows from [4] that $\tau_a \in T_{S(\preceq)}$ and $\tau_a^* = \tau_a$. Moreover, for any $a, b \in S$, $\theta_{\alpha, \beta} \tau_a = \tau_{a^*ab^*}$. And we have the following theorem.

Theorem 2.5 Let $S$ be a generalized inverse $*$-semigroup such that $E(S) = E$ and $P(S) = P$. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of $E$ and $P_i = P(E_i)$. Denote the restriction of the natural order on $S$ to $P$ by $\preceq$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by $f \phi_e = efe$. Then $(P(\preceq); \{\phi_e\})$ is a partially ordered $\phi$-set and $T_{P(\preceq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup.

Moreover, for any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by $e\tau_a = a^*ea$. Then a mapping $\tau : S \to T_{P(\preceq)}(\mathcal{M}) (a \mapsto \tau_a)$ is a $*$-homomorphism and the kernel of $\tau$ is the maximum idempotent-separating $*$-congruence on $S$.

Now, we have the following theorem.
Theorem 2.6 A generalized inverse \(*\)-semigroup \(S\) is fundamental if and only if it is \(*\)-isomorphic to a \(P\)-full generalized inverse \(*\)-subsemigroup of \(T_{X(\leq)}(\mathcal{M})\) on a partially ordered \(\rho\)-set \((X(\leq); \{\phi_x\})\) such that \(P(T_{X(\leq)}(\mathcal{M}))\) is order isomorphic to \(P(S)\).

Denote the sets of all partially ordered \(\rho\)-sets and the set of all strong \(\pi\)-groupoids by \(\mathbb{P}\) and \(\mathbb{S}\), respectively.

Remark 2.7 Let \((X(\leq); \{\phi_x\})\) be any element of \(\mathbb{P}\). For any \(x_\rho, y_\rho \in X/\rho\) \((x_\rho \geq y_\rho)\), define a mapping \(\overline{\varphi}_{x_\rho,y_\rho} : X_{x_\rho} \rightarrow X_{y_\rho}\) by

\[ x'_{\overline{\varphi}_{x_\rho,y_\rho}} = y', \text{ where } y' \in y_\rho \text{ such that } y' \preceq x'. \]

Moreover, we define a partial product on \(X\) as follows:

\[ xy = \begin{cases} x_{\overline{\varphi}_{x_\rho,y_\rho}} & \text{if } x_{\overline{\varphi}_{x_\rho,y_\rho}} = y_{\overline{\varphi}_{y_\rho,x_\rho}} \\ \text{undefined} & \text{otherwise.} \end{cases} \]

Then \((X(\leq); \{\phi_x\})\lambda = X(\pi_\rho; X/\rho; \{\overline{\varphi}_{x_\rho,y_\rho}\})\) is a strong \(\pi\)-groupoid, where \(\pi_\rho\) is the partition of \(X\) induced by \(\rho\).

Conversely, let \((X(\pi; Y; \{\varphi_{e,f}\})\) be any element of \(\mathbb{S}\). For any \(x \in X\), define a mapping \(\tilde{\phi}_x : X \rightarrow X\) by

\[ y_{\tilde{\phi}_x} = x_{\varphi_{e,f}}, \]

where \(x \in X_e\) and \(y \in X_f\). If we define \(\blacktriangle = \{(x, y) \in X \times X : x_{\tilde{\phi}_y} = x\}\), then \(X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\blacktriangle); \{\tilde{\phi}_x\})\) is a partially ordered \(\rho\)-set.

Hence the mappings \(\lambda, \mu\) from \(\mathbb{P}\) to \(\mathbb{S}\) and from \(\mathbb{S}\) to \(\mathbb{P}\), respectively, are well-defined. Moreover \(\mu\lambda = 1_\mathbb{S}\), and for any \((X(\leq); \{\phi_x\})\in \mathbb{P}\), if \((X(\leq); \{\phi_x\})\lambda \mu = (X(\blacktriangle); \{\tilde{\phi}_x\})\), then \(\leq = \blacktriangle\).

By the above argument, for any \((X(\leq); \{\phi_x\})\) in \(\mathbb{P}\), without loss of generality, we can consider \((X(\leq); \{\phi_x\})\) as a member of \(\mathbb{P}\lambda\mu\).

Now, let \((X(\pi; Y; \{\varphi_{e,f}\})\) be any element of \(\mathbb{S}\). If \((X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\leq); \{\phi_x\})\). Then we can construct two generalized inverse \(*\)-semigroups \(T_{X(\pi)}(\mathcal{M})\) and \(T_{X(\leq)}(\mathcal{M})\). In this case, these two generalized inverse \(*\)-semigroups are \(*\)-isomorphic.
3  Extensions of $T_{X(\leq)}(\mathcal{M})$

3.1  $T_{(X(\leq),\sigma)}(\mathcal{M})$

By a pre-order on a set $X$ we shall mean a reflexive and transitive relation. Let $X(\leq)$ be a pre-ordered set and let $\nu = \{(a,b) \in X \times X : a \leq b \text{ and } b \leq a\}$. Then $\nu$ is an equivalence relation on $X$ and $X/\nu$ is a partially ordered set with respect to the induced relation

$$(C1) \quad av \leq bv \text{ if and only if } a \leq b.$$ 

We call $\leq$ the naturally induced order on $X/\nu$ from $\leq$. Clearly $\nu$ is the smallest equivalence relation on $X$ for which $(C1)$ defines a partial order on $X/\nu$. We call $\nu$ the minimum partial order congruence (mpo-congruence) on $X$ from $\leq$.

A subset $A$ of $X$ is an ideal of $X$ provided that $x \leq y$ and $y \in A$ implies $x \in A$. For $a \in X$, we call $\{x \in X : x \leq a\}$ the principal ideal generated by $a$ and denote it by $(a)$.

A bijection $\alpha$ of one pre-ordered set $X$ onto another $Y$ will be called an isomorphism provided that, for $a,b \in X$, $a \leq b$ if and only if $a\alpha \leq b\alpha$. In particular, if $\nu_X$ and $\nu_Y$ denote the respective mpo-congruences then $(a,b) \in \nu_X$ if and only if $(a\alpha,b\alpha) \in \nu_Y$.

Let $X(\leq)$ be a pre-ordered set and $\nu$ the mpo-congruence from $\leq$. Then $X$ is a partially pre-ordered $\mathcal{G}$-set if and only if $X/\nu$ is a partially ordered $\mathcal{G}$-set with respect to the naturally induced order $\leq$ from $\leq$.

Let $X(\leq)$ be a partially pre-ordered $\mathcal{G}$-set and $\sigma$ an equivalence relation on $X$ such that

1. For any $x$ in $X$, $(x)$ is an $\nu$-single subset with respect to $\sigma$,
2. For $x,y$ in $X$, if $(x,y) \in \sigma$ then $(x\nu,y\nu) \in \mathcal{G}$,
3. For $x,y,z$ in $X$, if $(x\nu)\mathcal{G} \land (y\nu)\mathcal{G} = (z\nu)\mathcal{G}$, $z_1\nu \leq x\nu$ and $z_2\nu \leq y\nu$, then for any $a \in (z_i)$, there exists $b \in (z_j)$ such that $(a,b) \in \sigma$, where $1 \leq i,j \leq 2$.

Then $(X(\leq);\sigma)$ is called an $\omega$-set.

Let $(X(\leq),\sigma)$ be an $\omega$-set and let $T_{(X(\leq),\sigma)}$ denote the set of all isomorphisms from a principal ideal onto another one.

For any $\alpha, \beta \in T_{(X(\leq),\sigma)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(a,b) \in r(\alpha) \times d(\beta) : (a,b) \in \sigma\}.$$
Then $\theta_{\alpha, \beta} \in T_{(X(\preceq); \sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{(X(\preceq); \sigma)}\}$, and denote a multiplication $\circ$ and a unary operation $*$ on $T_{(X(\preceq); \sigma)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta,$$

$$\alpha^* = \alpha^{-1}.$$

Clearly, $\alpha \circ \beta$ is an isomorphism from $\langle z_1 \alpha^{-1} \rangle$ onto $\langle z_2 \beta \rangle$. It is obvious that $T_{(X(\preceq); \sigma)}(\circ, *$) is a regular $*$-semigroup. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{(X(\preceq); \sigma)}(\mathcal{M})$.

**Theorem 3.1** A regular $*$-semigroup $T_{(X(\preceq); \sigma)}(\mathcal{M})$ is a generalized inverse $*$-subsemigroup of $\mathcal{G}T_{(X(\sigma))}(\mathcal{M})$ whose set of projections is order isomorphic to $X/\nu$.

**Remark 3.2** In $T_{(X(\preceq); \sigma)}(\mathcal{M})$, if $\preceq = \preceq$ and $\sigma = \emptyset$ then $T_{(X(\preceq); \emptyset)}(\mathcal{M}) = T_{X(\preceq)}(\mathcal{M})$.

Let $(X(\preceq); \sigma)$ be an $\omega$-set and let $Y = X/\nu$, where $\nu$ is the mpo-congruence from $\preceq$. For any element $\alpha$ in $T_{(X(\preceq); \sigma)}$, assume that $d(\alpha) = \langle a \rangle$. Then we can define a new mapping $\alpha' \in T_{Y(\preceq)}$ as follows:

$$d(\alpha') = \{x\nu : x \in d(\alpha)\},$$

$$(x\nu)\alpha' = (x\alpha)\nu.$$

Then $\alpha' \in T_{Y(\preceq)}$. Now, define a mapping $\xi : T_{(X(\preceq); \sigma)}(\mathcal{M}) \rightarrow T_{Y(\preceq)}(\mathcal{M})$ by $\alpha\xi = \alpha'$. Then, it is easy to see that $\xi$ is a $*$-homomorphism.

**Proposition 3.3** The mapping $\xi : \alpha \mapsto \alpha'$ of $T_{(X(\preceq); \sigma)}(\mathcal{M})$ into $T_{Y(\preceq)}(\mathcal{M})$ is a $*$-homomorphism of $T_{(X(\preceq); \sigma)}(\mathcal{M})$ onto a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{Y(\preceq)}(\mathcal{M})$ such that $\xi \circ \xi^{-1} = \mu$, where $\mu$ is the maximum idempotent separating $*$-congruence on $T_{(X(\preceq); \sigma)}(\mathcal{M})$.

Hereafter, we shall refer to $\xi$ as the natural projection of $T_{(X(\preceq); \sigma)}(\mathcal{M})$ to $T_{Y(\preceq)}(\mathcal{M})$.

### 3.2 Inflated representations

Let $S$ be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Define a relation $\preceq$ on $S$ by:

$$a \preceq b \text{ if and only if } a^*a \leq b^*b,$$
for \(a, b \in S\). Then clearly \(\preceq\) is a pre-order on \(S\) for which the mpo-congruence from \(\preceq\) is \(\nu = \mathcal{L}\). Hence \(S/\mathcal{L} = S/\nu\), under the naturally induced order \(\preceq\) from \(\preceq\), is just the set of \(\mathcal{L}\)-classes of \(S\) under the usual partial ordering of the \(\mathcal{L}\)-classes of a generalized inverse \(*\)-semigroup and so is order isomorphic to the partially ordered \(q\)-set \(P\) of \(S\). Hence \(S\) is a partially pre-ordered \(q\)-set under \(\preceq\). Then \(q = \mathcal{J}^E|_P\) and hence \((av)\rho(bv) \iff a^*aJ^E b^*b\). Hereafter, for any \(a \in S\), we think \(av = L_{a^*a}\) as \(a^*a\).

For any \(a \in S\), define a mapping \(\rho_a : Sa^* \to Sa\) as follows:

\[
d(\rho_a) = Sa^* = Saa^*, \quad x\rho_a = xa.
\]

Let \(\rho : S \to \mathcal{G}(S;\Omega)(\mathcal{M})\) by \(a\rho = \rho_a\), where the relation \(\Omega\) defined by: for \(x, y \in S\),

\[
(x, y) \in \Omega \iff x\rho_e = y \quad \text{for some } e \in E.
\]

Since \(S\) is a regular \(*\)-semigroup, the representation \(\rho\) is faithful. Moreover, it follows from [6, Lemma 3.3] that it is a \(*\)-monomorphism.

**Lemma 3.4** The set \((S(\preceq);\Omega)\), defined above, is an \(\omega\)-set.

Again, we consider \(\rho_a : Sa^* \to Sa\). By Lemma 3.4, \(d(\rho_a) = \langle a^* \rangle\) and \(r(\rho_a) = \langle a \rangle\).

For \(x, y \in d(\rho_a)\), \(x^*x, y^*y \leq a^*a\). Now \(x \preceq y\) if and only if \(x^*x \leq y^*y\) while \(xa \preceq ya\) if and only if \(a^*x^*xa = (xa)^*(xa) \leq (ya)^*(ya) = a^*y^*ya\). But, since \(x^*x, y^*y \leq a^*a\) it follows that \(x^*x \leq y^*y\) if and only if \(a^*x^*xa \leq a^*y^*ya\). Therefore \(x \preceq y\) if and only if \(xa \preceq ya\). Thus \(\rho_a\) is an isomorphism of \(\langle a^* \rangle\) onto \(\langle a \rangle\), and hence \(S\rho \subseteq T(S(\preceq);\Omega)(\mathcal{M})\).

Now, we have the following theorem.

**Theorem 3.5** Let \(S\) be a generalized inverse \(*\)-semigroup and define the relation \(\preceq\) on \(S\) by \(a \preceq b\) if and only if \(a^*a \leq b^*b\). Then \(\preceq\) is a pre-order on \(S\) with respect to which \(S\) is a partially pre-ordered \(q\)-set, moreover \((S(\preceq);\Omega)\) is an \(\omega\)-set. The faithful representation \(\rho\) of \(S\) embeds \(S\) as a \(\mathcal{P}\)-full generalized inverse \(*\)-subsemigroup of \(T(S(\preceq);\Omega)(\mathcal{M})\).

If \(\nu\) is the mpo-congruence on \(S\) from \(\preceq\), then \(\nu = \mathcal{L}\) and \(S/\nu\) is order isomorphic to the partially ordered \(q\)-set \(P\) of \(S\). Moreover, \(\rho\xi = \tau\), where \(\xi\) is the natural projection and \(\tau\) is the representation which is defined in Theorem 2.5.

**References**


