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Decidability of the equivalence problem of finitely ambiguous finance automata

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Abstract

A finance automaton is a sixtuple $< \Sigma, Q, \delta, S, F, f >$, where $< \Sigma, Q, \delta, S, F >$ is a finite automaton, $f : Q \times \Sigma \times Q \to R \cup \{-\infty\}$ is a finance function, R is the set of real numbers and it holds $f(q, a, q') = -\infty$ iff $q' \notin \delta(q, a)$. The function f is extended to $f : 2^Q \times \Sigma^* \times 2^Q \to R \cup \{-\infty\}$ by the plus-max principle. For any $w \in \Sigma^*$, $f(S, w, F)$ is the profit of w. It is shown that the equivalence problem of finitely ambiguous finance automata is decidable.

1 Introduction

A finance automaton is a finite automaton with a finance function. It may be regarded to be a model describing financial activities. In this paper, we prove decidability of the equivalence problem of finitely ambiguous finance automata by reducing the problem to the problem of finding integer solutions in simultaneous linear inequalities. We also study several subproblems and present a different proof to each subproblem so that the proof is easier if the complexity of the subproblem is easier. The paper consists of five Sections. In Section 2, we present the definition and elementary properties of finance automata. In Section 3, we present the definitions of a union finance automaton and a vector finance automaton each of which is composed from a finitely many deterministic finance automata. In Section 4, first we present a proof showing decidability of the equivalence problem of a union finance automaton and a determininistic finance automaton. Then we show that the inequality problem of a union finance automaton and a deterministic finance automaton is decidable. This implies decidability of the equivalence problem of union finance automata. In Section 5, we present a method for decomposing a finitely ambiguous finance automaton to a union finance automaton. This fact together with results in Section 4 implies decidability of the equivalence problem of finitely ambiguous finance automata.

2 Finance Automata

An alphabet is a nonempty set of symbols. A word is a finite sequence of symbols from the alphabet. A word of zero length is called a null word and denoted by λ. Σ^* denotes the set of all words over an alphabet Σ, and Σ^+ denotes the set of nonnull words. For a word, $w = a_1 a_2 \cdots a_n$ $(n \geq 0, a_1, \cdots, a_n \in \Sigma)$, n is the length of w, and denoted by $|w|$. The cardinality of a set A is denoted by $|A|$. R denotes the set of real numbers. $R_{-\infty}$ denotes $R \cup \{-\infty\}$.

2.1 Basic definitions

Definition 2.1 A finance automaton (in short, an F-automaton) is a sixtuple $A =< \Sigma, Q, \delta, S, F, f >$, where

Σ : an input alphabet, Q : a finite set of states, δ : a transition function
S : the set of initial states, F : the set of final states, f : a finance function

Thus $< \Sigma, Q, \delta, S, F >$ is a finite automatopn. A word $w \in \Sigma^*$ is accepted by A if $\delta(S, w) \in F$. The finance function f is a mapping from $Q \times \Sigma \times Q$ to $R_{-\infty}$ ($f : Q \times \Sigma \times Q \to R_{-\infty}$). f satisfies the following.

$$\forall(p, a, q) \in Q \times \Sigma \times Q,\ q \notin \delta(p, a) \Longleftrightarrow f(p, a, q) = -\infty$$

In the rest of the paper, a finance automaton is called an F-automaton.

Definition 2.2

(1) f is extended to $f : Q \times \Sigma^* \times Q \to R_{-\infty}$ and to $f : 2^Q \times \Sigma^* \times 2^Q \to R_{-\infty}$ in the following way.

(1.1) $\forall p, q \in Q,\ f(p, \lambda, q) = 0$ (if $p = q$), $f(p, \lambda, q) = -\infty$ (if $p \neq q$)

(1.2) $\forall p, q \in Q,\ \forall w \in \Sigma^*,\ \forall a \in \Sigma,$

$$f(p, wa, q) = \max\{f(p, w, q') + f(q', a, q) \mid q' \in Q\}$$

Here, for $\forall i \in R_{-\infty}$, $\max\{i, -\infty\} = i$ and $i + (-\infty) = -\infty$.

(1.3) $\forall t, t' \subset Q,\ \forall w \in \Sigma^*,\ f(t, w, t') = \max\{f(p, w, q) \mid p \in t, q \in t'\}$

(2) The set of words accepted by A is denoted by $L(A)$: $L(A) = \{w \in \Sigma^* \mid \delta(S, w) \cap F \neq \emptyset\}$

(3) Tow F-automata A_1 and A_2 is said to be L-equivalent if $L(A_1) = L(A_2)$.

(4) For $\forall w \in \Sigma^*$, $f(S, w, F)$ is the profit of w (by A). The profit of w is sometimes denoted by $F(w, A)$ ($= f(S, w, F)$).

(5) Let A_1 and A_2 be two F-automata over the input alphabet Σ.If $\forall w \in \Sigma^*$,

(5.1) If for any $w \in \Sigma^*$, it holds $F(w, A_1) = F(w, A_2)$, then A_1 and A_2 are said to be equivalent and write $A_1 \equiv A_2$.

(5.2) If for any $w \in \Sigma^*$, it holds $F(w, A_1) \geq F(w, A_2)$, then A_1 is said to be equal or greater than A_2, and write $A_1 \geq A_2$.

(6) An F-automaton A is deterministic if the following hold.

For $\forall q \in Q,\ \forall a \in \Sigma$, it holds $|\delta(q, a)| \leq 1$ and $|S| \leq 1$.

When A is deterministic, for $\forall q \in Q,\ \forall a \in \Sigma,\ \forall q' \in \delta(q, a)$, we write $\delta(q, a) = q'$.

(7) For any $m \geq 1$, m deterministic F-automata $A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$ are said to be disjoint if for $\forall j, k (1 \leq j < k \leq m)$, $Q_j \cap Q_k = \emptyset$.

It is shown that two deterministic F-automata $A_1 =< \Sigma, Q_1, \delta_1, \{s_1\}, F_1, f_1 >$ and $A_2 =< \Sigma, Q_2, \delta_2, \{s_2\}, F_2, f_2 >$ are equivalent iff for all $w \in \Sigma^*$ with at most $2 \times |Q_1| \times |Q_2|$, $f(\{s_1\}, w, F_1) = f(\{s_2\}, w, F_2)$ [4].

3 Union F-automata and vector F-automata

In this section, from a finite set of deterministic F-automata, we define a union F-automata whose set of states is the union of the set of each deterministic F-automaton, and a vector F-aotomaton whose set of states is the direct product of the set of each deterministic F-automaton.

3.1 Union F-automata

Definition 3.1 For disjoint $m(m \geq 1)$ deterministic F-automata, $A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$, a union F-automaton $A_1 \cup \cdots \cup A_m =< \Sigma, Q, \delta, S, F, f >$ is defined as follows.

(1) $Q = Q_1 \cup \cdots \cup Q_m$, $S = \{s_1, \cdots, s_m\}$, $F = F_1 \cup \cdots \cup F_m$

(2) For $\forall i(1 \leq i \leq m)$, $\forall q \in Q_i$, $\forall a \in \Sigma$,

$$\delta(q,a) = \delta_i(q,a) \text{ and } f(q,a,\delta(q,a)) = f_i(q,a,\delta_i(q,a))$$

Propsition 3.1 Let $A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$ be disjoint $m(m \geq 1)$ deterministic F-automata. Then for $\forall w \in \Sigma^*$, it holds

$$F(w, A_1 \cup \cdots \cup A_m) = \max\{F(w, A_i) \mid 1 \leq i \leq m\}$$

[Proof] For $\forall w \in \Sigma^*$, $F(w, A_1 \cup \cdots \cup A_m) = f(S, w, F)$. By definition, for $\forall t, t' \subset Q$, $\forall w \in \Sigma^*$, it holds $f(t, w, t') = \max\{f(p, w, q) \mid p \in t, q \in t'\}$. Thus

$$\begin{aligned} f(S,w,F) &= \max\{f(s_i, w, p_{in}) \mid s_i \in S, p_{in} \in F_i, 1 \leq i \leq m\} \\ &= \max\{F(w, A_i) \mid 1 \leq i \leq m\} \end{aligned}$$

Hence $F(w, A_1 \cup \cdots \cup A_m) = \max\{F(w, A_i) \mid 1 \leq i \leq m\}$. □

3.2 Vector F-automata

Definition 3.2 For disjoint $m(m \geq 1)$ deterministic F-automata $A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$, a vector F-automaton $V(A_1, \cdots, A_m) =< \Sigma, Q, \delta, S, F, f >$ is defined as follows.

(1) $Q = Q_1 \times \cdots \times Q_m$, $S = \{(s_1, \cdots, s_m)\}$, $F = F_1 \times \cdots \times F_m$

(2) For $\forall (p_1, \cdots, p_m) \in Q_1 \times \cdots \times Q_m, \forall a \in \Sigma$,

$$\begin{aligned} &\delta((p_1, \cdots, p_m), a) = (\delta_1(p_1, a), \cdots, \delta_m(p_m, a)) \\ &f((p_1, \cdots, p_m), a, \delta((p_1, \cdots, p_m), a)) = \max\{f_i(p_i, a, \delta_i(p_i, a)) \mid 1 \leq i \leq m\} \end{aligned}$$

Proposition 3.2 Let $A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$ be disjoint $m(m \geq 1)$ deterministic F-automata. Then for $\forall w \in \Sigma^*$, $w = a_1 a_2 \cdots a_n (n \geq 1, a_1, \cdots, a_n \in \Sigma)$, there exist for each $1 \leq j \leq n$, $1 \leq i \leq m$ and $p_{i,j-1}, p_{i,j} \in Q_i$ with $p_{i\,j} = \delta_i(p_{i\,j-1}, a_j) \in Q_i (1 \leq j \leq n)$ such that it holds $F(w, V(A_1, \cdots, A_m)) = \sum_{j=1}^{n} \max\{f_i(p_{i\,j-1}, a_j, p_{i\,j}) \mid 1 \leq i \leq m\}$.

[Proof] For $\forall w \in \Sigma^*$, $|w| = n$, it holds $F(w, V(A_1, \cdots, A_m)) = f(S, w, F)$. By definition, for $\forall (p_{1\,j-1}, \cdots, p_{m\,j-1}), (p_{1\,j}, \cdots, p_{m\,j}) \in Q_1 \times \cdots \times Q_m$, $\forall a \in \Sigma$,

$$f((p_{1\,j-1}, \cdots, p_{m\,j-1}), a, (p_{1\,j}, \cdots, p_{m\,j})) = \max\{f_i(p_{i\,j-1}, a, p_{i\,j}) \mid 1 \leq i \leq m\} = b_j \quad (1 \leq j \leq n)$$

It also holds for $\forall p, q \in Q$, $\forall v \in \Sigma^*$, $\forall a \in \Sigma$,

$$f(p, va, q) = \max\{f(p, v, q') + f(q', a, q) \mid q' \in Q\}$$

Thus

$$f(S, w, F) = b_1 + \cdots + b_n = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} \max\{f_i(p_{i\,j-1}, a_j, p_{i\,j}) \mid 1 \leq i \leq m\}$$

Hence $F(w, V(A_1, \cdots, A_m)) = \sum_{j=1}^{n} \max\{f_i(p_{i\,j-1}, a_j, p_{i\,j}) \mid 1 \leq i \leq m\}$. □

3.3 Properties of union F-automata and vector F-automata

It is shown [4] that for disjoint $m (m \geq 1)$ deterministic F-automata $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq m)$, the union F-automaton $A_1 \cup \cdots \cup A_m$ and the vector F-automaton $V(A_1, \cdots, A_m)$ are equivalent iff for any $w \in \Sigma^*$ with length less than or equal to $(m+1) \times |Q_1| \times \cdots \times |Q_m|$, it holds

$$F(w, A_1 \cup \cdots \cup A_m) = F(w, V(A_1, \cdots, A_m))$$

4 The equivalence problem of union F-automata

In this section, we show the equivalence problem of union F-automata is decidable.

4.1 The equivalence problem of $A_1 \cup \cdots \cup A_n$ and A_{n+1}

We shall first present an algorithm for deciding whether for $n \geq 1$, a union F-automaton $A_1 \cup \cdots \cup A_n$ and a deterministic F-automaton A_{n+1} are equivalent. Let $n \geq 1$ and for each $1 \leq i \leq n+1$, $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i)$ be a deterministic F-automaton.

Definition 4.1 A finite automaton $\Gamma(A_1, \cdots, A_{n+1}) = < \Sigma, Q, \delta, \{s\}, F >$ is defined as follows.

(1) $Q = Q_1 \times \cdots \times Q_{n+1}$, $S = (s_1, \cdots, s_{n+1})$, $F = F_1 \times \cdots \times F_{n+1}$

(2) for $\forall a \in \Sigma$, $\forall (q_1, \cdots, q_{n+1}) \in Q_1 \times \cdots \times Q_{n+1}$,

$$\delta((q_1, \cdots, q_{n+1}), a) = (\delta_1(q_1, a), \cdots, \delta_{n+1}(q_{n+1}, a))$$

Proposition 4.1 Assume that $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$. Then for $\forall x, z \in \Sigma^*$, $y \in \Sigma^+$, $(q_1, \cdots, q_{n+1}) \in Q$, if $\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_{n+1})$ and $\delta(s, xyz) \in F$, then there exists $1 \leq i \leq n$ for which the following (1)-(3) hold.

(1) $F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1})$

(2) $F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1})$

(3) $f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \leq j \leq n\} = f_{n+1}(q_{n+1}, y, q_{n+1})$

[Proof] We put $a = \max\{f_i(q_i, y, q_i) \mid 1 \leq i \leq n\}$ and $b = F(xyz, A_1 \cup \cdots \cup A_n)$. Define two sets X, Y by $X = \{i \mid 1 \leq i \leq n \textit{ and } f_i(q_i, y, q_i) = a\}$, $Y = \{i \mid 1 \leq i \leq n \textit{ and } F(xyz, A_i) = b\}$. Since $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$, it holds $b = F(xyz, A_{n+1})$. Put $c = \max\{F(xyz, A_i) \mid i \in X\}$, and define a set W by $W = \{i \in X \mid F(xyz, A_i) = c\}$. We consider the following two cases.

(i) $b = c$. It holds $X \cap Y = W$ and for any $i \in X \cap Y$, $a = f_i(q_i, y, q_i)$ and $c = F(xyz, A_i) = F(xyz, A_1 \cup \cdots \cup A_n)$. It is clear that $F(xz, A_i) = c - a = F(xz, A_1 \cup \cdots \cup A_n)$. Moreover $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$. Hence for any $i \in W$, (1)-(3) hold.

(ii) $b \neq c$. Clearly $b > c$. Put $d = \max\{f_i(q_i, y, q_i) \mid i \in Y\}$. Since $b > c$, it holds $d < a$. For each $k > (b - d + a - c)/(a - d)$, consider the word $w = xy^kz$. Then for each $i \in W$, $F(xy^kz, A_1 \cup \cdots \cup A_n) = F(xy^kz, A_i)$. Since $F(xz, A_{n+1}) = F(xz, A_1 \cup \cdots \cup A_n)$, $F(xz, A_{n+1}) > F(xz, A_i)$. Together with this fact and $F(xyz, A_{n+1}) = b$, we have $f_{n+1}(q_{n+1}, y, q_{n+1}) < a$. If we consider a sufficiently large k, it would hold for each $i \in W$, $F(xy^kz, A_{n+1}) < F(xy^kz, A_i) = F(xy^kz, A_1 \cup \cdots \cup A_n)$. This is a contradiction to $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$. Thus the case $b \neq c$ is impossible. □

We shall present a necessary and sufficient condition for$A_1 \cup \cdots \cup A_n$ and A_{n+1} to be equivalent.

Theorem 4.1 Let $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1, n \geq 1)$ be $n+1$ deterministic L-equivalent F-automata, and let $\Gamma(A_1, \cdots, A_{n+1})$ be as in Definition 4.1. Then the following four conditions are equivalent.

(1) $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$

(2) For any $w \in \Sigma^*$, $F(w, A_1 \cup \cdots \cup A_n) = F(w, A_{n+1})$

(3) For any $w \in \Sigma^*$ with length $\leq 3 \times (n+1) \times |Q_1| \times \cdots \times |Q_{n+1}|$, the following (A) holds.

(A): For $\forall x, y, z \in \Sigma^*$, $(q_1, \cdots, q_{n+1}) \in Q$, if $w = xyz$, $\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n)$ and $\delta(s, xyz) \in F$, then there exists $1 \leq i \leq n$ such that the folowing (3.1)- (3.3) hold.

(3.1) $F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1})$

(3.2) $F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1})$

(3.3) $f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \leq j \leq n\} = f_{n+1}(q_{n+1}, y, q_{n+1})$

(4) For any $w \in \Sigma^*$, the following (B) holds.

(B): For $\forall x, y, z \in \Sigma^*$, $(q_1, \cdots, q_{n+1}) \in Q$, if $w = xyz$, $\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n)$ and $\delta(s, xyz) \in F$, then there exists $1 \leq i \leq n$ such that the following (4.1)- (4.3) hold.

(4.1) $F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1})$

(4.2) $F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1})$

(4.3) $f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \leq j \leq n\} = f_{n+1}(q_{n+1}, y, q_{n+1})$

[Proof] (1) $\Leftrightarrow$ (2) is clear. (1) $\Rightarrow$ (3) follows from Proposition 4.1. (3) $\Rightarrow$ (4). Assume (3) holds. We shall prove (4) by induction on $|w|$. Put $c = 3 \times (n+1) \times |Q_1| \times \cdots \times |Q_{n+1}|$. When $|w| \leq c$, (B) holds. Let $k > c$, and assume that for $|w| < k$, (B) holds. Consider the case $|w| = k$. Assume that for $w = xyz$ and $(q_1, \cdots, q_{n+1}) \in Q$, it holds $\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n)$ and $\delta(s, xyz) \in F$. Since $|w| > c$, it holds $|x| \geq c/3$, or $|y| \geq c/3$ or $|z| \geq c/3$.

First consider the case $|y| \geq c/3$. By the definition of c, y has a decomposition $y = y_0 y_1 \cdots y_{n+2}$, $y_0, y_{n+2} \in \Sigma^*$, $y_i \in \Sigma^+ (1 \leq i \leq n+1)$ such that for some $(p_1, \cdots, p_{n+1}) \in Q$, $\delta(s, xy_0) = \delta(s, xy_0 y_1) = \cdots = \delta(s, xy_0 y_1 \cdots y_{n+1})$. For each $1 \leq i \leq n+1$, put $v_i = y_1 \cdots y_{i-1} y_{i+1} \cdots y_{n+1}$.

Consider the case $i = 1$. $v_1 = y_2 y_3 \cdots y_{n+1}$. By the inductive hypothesis, for $x'_1 = xy_0$, $y'_1 = v_1$, $z'_1 = y_{n+2} z$, there exists A_{i_1} to which (B) holds. The set of such A_{i_1} is denoted by X_{11}. Next for $x'_2 = xy_0$, $y'_2 = y_2$, $z'_2 = y_3 y_4 \cdots y_{n+2} z$, there exists A_{i_2} satisfying (B). The set of such A_{i_2} is denoted by X_{12}. Clearly $X_{12} \subseteq X_{11}$. Each A_j belonging to X_{12} has the maximum profits at y_2, $y_3 y_4 \cdots y_{n+1}$, and $xy_0 y_{n+2} z$. Then for $x'_3 = xy_0$, $y'_3 = y_3$, $z'_3 = y_4 y_5 \cdots y_{n+2} z$, there exists A_{i_3} satisfying (B). The set of such A_{i_3} is denoted by X_{13}. Then $X_{13} \supseteq X_{12}$. By continuing this argument, each A_j belonging to X_{12} has the maximum profits at y_2, $y_3 y_4 \cdots y_{n+1}$, and $xy_0 y_{n+2} z$.

In the same way, for each $2 \leq i \leq n$, the set X_{i_2} can be defined. Then there exist $1 \leq p < q \leq n+1$ and $1 \leq r \leq n$ such that $A_r \in X_{p_2} \cap X_{q_2}$ holds. This implies A_r has the maximum profits at $y_1, y_2, \cdots, y_{n+1}$, and $xy_0 y_{n+2} z$, i.e., A_r satisfies (B).

When $|x| \geq c/3$, if $y = \lambda$, then we consider $x = x_0 x_1 \cdots x_{n+1} x_{n+2}$, $\delta(s, x_0) = \delta(s, x_0 x_1) = \cdots = \delta(s, x_0 x_1 \cdots x_{n+1})$ $(x_i \in \Sigma^+, 1 \leq i \leq n+1)$ and the argument is similar to $|y| \geq c/3$. If $y \neq \lambda$, we consider $x = x_0 x_1 \cdots x_n x_{n+1}$, $\delta(s, x_0) = \delta(s, x_0 x_1) = \cdots = \delta(s, x_0 x_1 \cdots x_n)$ $(x_i \in \Sigma^+, 1 \leq i \leq n)$ and y (in sum, $n+1$ subwords), and the argument is similar to $|y| \geq c/3$. The case $|z| \geq c/3$ can be handled in the same way. Thus (3) $\Rightarrow$ (4) is proved. (4) $\Rightarrow$ (2) is clear. This completes the proof of Theorem 4.1. □

The following theorem is now clear.

<u>**Theorem 4.2**</u> Given $n+1$ L-equivalent deterministic F-automata $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1)$, one can decide whether or not $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$ holds.

4.2 The equivalence problem of $A_1 \cup \cdots \cup A_m$ and $A_{m+1} \cup \cdots \cup A_{m+n}$

In this subsection, we shall show decidability of the equivalence problem of union F-automata.

4.2.1 Preriquisites

<u>**Proposition 4.2**</u> For $m + n (m, n \geq 1)$ deterministic F-automata $A_k = < \Sigma, Q_k, \delta_k, \{s_k\}, F_k, f_k > (1 \leq k \leq m+n)$, the following two conditions are equivalent.

(1) $A_1 \cup \cdots \cup A_m \equiv A_{m+1} \cup \cdots \cup A_{m+n}$

(2) For each $1 \leq i \leq m$, $A_i \leq A_{m+1} \cup \cdots \cup A_{m+n}$ and for each $m+1 \leq j \leq m+n$, $A_j \leq A_1 \cup \cdots \cup A_m$

4.2.2 The inequality problem of union F-automata

Definition 4.3 The following problem is called the inequality problem of union F-automata.

Problem input : $n+1 (n \geq 1)$ deterministic F-automata
$A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1)$.
output : if $A_1 \cup \cdots \cup A_n \geq A_{n+1}$, then "yes".
if $A_1 \cup \cdots \cup A_n \not\geq A_{n+1}$, then "no".

Definition 4.4 For $n+1 (n \geq 1)$ deterministic F-automata $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i >$ $(1 \leq i \leq n+1)$, define a deterministic finite automaton $\Pi(A_1, \cdots, A_{n+1}) = < \Sigma, Q, \delta, \{s\}, F >$ as follows.

(1) $Q = Q_1 \times \cdots \times Q_{n+1}$, $s = (s_1, \cdots, s_{n+1})$

(2) For $\forall a \in \Sigma$, $\forall (p_1, \cdots, p_{n+1}) \in Q$,

$$\delta((p_1, \cdots, p_{n+1}), a) = (\delta_1(p_1, a), \cdots, \delta_{n+1}(p_{n+1}, a))$$

(3) $F = (F_1 \times Q_2 \times \cdots \times Q_{n+1}) \cup (Q_1 \times F_2 \times Q_3 \times \cdots \times Q_{n+1}) \cup \cdots \cup (Q_1 \times \cdots \times Q_n \times F_{n+1})$

Proposition 4.3 For any $n+1 (n \geq 1)$ deterministic F-automata $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1)$, it holds $L(\Pi(A_1, \cdots, A_{n+1})) = L(A_1) \cup \cdots \cup L(A_{n+1})$.

Proposition 4.4 For any $n+1 (n \geq 1)$ deterministic F-automata $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1)$, if $A_1 \cup \cdots \cup A_n \geq A_{n+1}$, then it holds $L(\Pi(A_1, \cdots, A_n)) \supseteq L(A_{n+1})$.

Let $n \geq 1$, and $A_i = < \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n+1)$ be $n+1$ deterministic F-automata. As in Definition 4.4, define the deterministic finite automaton $\Pi(A_1, \cdots, A_{n+1}) = < \Sigma, Q, \delta, \{s\}, F >$.

Definition 4.5 For any $w \in \Sigma^+$, define a deterministic finite automaton $A(w) = < \Sigma, Q(w), \delta(w), \{s\}, F(w) >$ as follows.

(1) $Q(w) = \{(p_1, \cdots, p_{n+1}) \mid \textit{for some } x, y \in \Sigma^*, w = xy \textit{ and } (p_1, \cdots, p_{n+1}) = \delta(s, x)\}$

(2) $\delta(w)$ is a mapping from $Q(w) \times \Sigma$ to $Q(w)$ ($\delta(w) : Q(w) \times \Sigma \longrightarrow Q(w)$), and for $\forall (p_1, \cdots, p_{n+1}) \in Q(w)$ and $\forall a \in \Sigma$, define (2.1)-(2.2).

(2.1) If for some $x, y \in \Sigma^*$, it holds $w = xay$ and $(p_1, \cdots, p_{n+1}) = \delta(s, x)$, then

$$\delta(w)((p_1, \cdots, p_{n+1}), a) = \delta((p_1, \cdots, p_{n+1}), a)$$

(2.2) Otherwise $\delta(w)((p_1, \cdots, p_{n+1}), a) = \emptyset$

(3) $F(w) = \{\delta(s, w)\} \cap F$

Definition 4.6 Define a set $S(A_1, \cdots, A_{n+1})$ by : $S(A_1, \cdots, A_{n+1}) = \{A(w) \mid w \in \Sigma^+\}$

Definition 4.7 For $m \geq 1 and 1 \leq r \leq m$, put ${}_mP_r = m(m-1)\cdots(m-(r-1))$.

Definition 4.8 Define an integer $I(A_1, \cdots, A_{n+1})$ as follows, where $m = |Q| \times |\Sigma|$.

$$I(A_1, \cdots, A_{n+1}) = ({}_mP_1 +_m P_2 + \cdots +_m P_m + 1)(|Q| + 1)$$

Definition 4.9 Let $w \in \Sigma^+$ and $A(w) =< \Sigma, Q(w), \delta(w), \{s\}, F(w) >$. For $\forall p \in Q(w)$, $i \geq 1$, $a_1, \cdots, a_i \in \Sigma$, if it holds $\delta(w)(p, a_1 \cdots a_i) = p$ and $i = 1$ or for $0 \leq j < k \leq i$, it holds $\delta(w)(p, a_1 \cdots a_j) \neq \delta(w)(p, a_1 \cdots a_k)$, then $(p, a_1 \cdots a_i, p)$ is called a minimal cycle of $A(w)$.

Lemma 4.1 For any $w \in \Sigma^+$, $A(w)$ has at most ${}_mP_1 +_m P_2 + \cdots +_m P_m$ minimal cycles. Here $m = |Q| \times |\Sigma|$.
[Proof] Let $c = (p, a_1 \cdots a_i, p)$ be a minimal cycle of $A(w)$. If $i = 1$, then the number of such c is at most${}_mP_1 = m$. If $i \geq 2$, then for each c, put $t(c) = ((p, a_1), (\delta(p, a_1), a_2), \cdots, (\delta(p, a_1 \cdots a_{i-1}), a_i))$. Since c is a minimal cycle, each of i pairs of $t(c)$ is distinct. Thus $i \leq m$ and the number of minimal cycles of length i is at most ${}_mP_i$. Hence the total number of minimal cycles is at most ${}_mP_1 +_m P_2 + \cdots +_m P_m$. □

Proposition 4.5 $S(A_1, \cdots, A_{n+1}) = \{A(w) \mid w \in \Sigma^+ \textit{ and } |w| < I(A_1, \cdots, A_{n+1})\}$
[Proof] Put $B = \{A(w) \mid w \in \Sigma^+ \textit{ and } |w| < I(A_1, \cdots, A_{n+1})\}$. We shall prove by induction on $|w|$ that for $\forall w \in \Sigma^+$, $A(w) \in B$. When $|w| < I(A_1, \cdots, A_{n+1})$, the assertion is clear. Assume that for $|w|$ with length less than $k+1$, the assertion holds, and consider the case $|w| = k+1 \geq I(A_1, \cdots, A_{n+1})$. Put $m = |Q| \times |\Sigma|$, $q =_m P_1 +_m P_2 + \cdots +_m P_m + 1$. Since $|w| \geq I(A_1, \cdots, A_{n+1})$, there exists a decomposition of w, $w = w_1 w_2 \cdots w_q w_{q+1}$ with $|w_i| = |Q| + 1 (1 \leq i \leq q)$. Since $|w_i| = |Q| + 1$, in the path of $A(w)$, $(s, w_1 w_2 \cdots w_q w_{q+1}, \delta(s, w))$, in each part of w_i, there exists a minimal cycle (p_i, v_i, p_i). Here w_i is decompsed as $w_i = x_i v_i y_i$. Namely in $A(w)$, there exists a path

$$(s, x_1, p_1, v_1, p_1, y_1 x_2, p_2, v_2, p_2, y_2 x_3, \cdots, y_{q-1} x_q, p_q, v_q, p_q, y_q w_{q+1}, \delta(s, w))$$

The number of minimal cycles is at most $q-1$ by Lemma 4.1. Thus for some $1 \leq i < j \leq q$, it holds $(p_i, v_i, p_i) = (p_j, v_j, p_j)$. Now by putting $w' = w_1 w_2 \cdots w_{j-1} x_j y_j w_{j+1} \cdots w_{q+1}$, we can see $A(w') = A(w)$ by definition of minimal cycles and $A(w)$. By induction, $A(w') \in B$. Hence $A(w) \in B$. □

Definition 4.10 When for $w \in \Sigma^+ \cap L(A_{n+1})$, $A(w)$ has a minimal cycle, define the following (1)-(2).

(1) $MCD(w)$ denotes the set of all sequences

$$(s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$$

which satisfy the following (1.1)-(1.2). Each sequence in $MCD(w)$ is called a minimal cycle decomposition of w.

(1.1) $w = x_1 y_1 x_2 y_2 \cdots x_k y_k x_{k+1}$

(1.2) For each $1 \leq i \leq k$, $p_i = \delta(s, x_1 y_1 x_2 y_2 \cdots x_{i-1} y_{i-1} x_i)$. (p_i, y_i, p_i) is a minimal cycle of $A(w)$ and it holds $|x_i| \leq |Q|$.

(2) Let $MCD(w) = \{\beta_1, \beta_2, \cdots, \beta_t\}(t \geq 1)$. For each $\beta_i(1 \leq i \leq t)$, define a set of linear inequlities $LIS(w, \beta_i)$ as follows.
Let $\beta_i = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. The set of following inequlities over k variables $X_1, X_2, \cdots, X_k$ is $LIS(w, \beta_i)$.

For each $1 \leq i \leq n$,

$$X_j \geq 0 \ \ (1 \leq j \leq k)$$

$$a_i + b_{i\,1}X_1 + b_{i\,2}X_2 + \cdots + b_{i\,k}X_k < a_{n+1} + b_{n+1\,1}X_1 + b_{n+1\,2}X_2 + \cdots + b_{n+1\,k}X_k$$

Here $a_f = F(x_1x_2\cdots x_{k+1}, A_f)(1 \leq f \leq n+1)$, and for each $1 \leq f \leq n+1$, $1 \leq j \leq k$, $b_{f\,j} = e_f(\delta_f(s_f, x_1x_2\cdots x_j), y_j, \delta_f(s_f, x_1x_2\cdots x_j))$.

<u>**Lemma 4.2**</u> The following two conditions are equivalent.

(1) $A_1 \cup \cdots \cup A_n \geq A_{n+1}$

(2) For any $w \in \Sigma^+ \cap L(A_{n+1})$, (2.1) or (2.2) holds.

(2.1) If $A(w)$ has no minimal cycle, then it holds $F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1})$

(2.2) If $A(w)$ has a minimal cycle, then for eacy minimal cycle decomposition of w, $\beta \in MCD(w)$, the simultaneous linear inequlities $LIS(w, \beta)$ have no integer solution.

[Proof] (1) $\Rightarrow$ (2). Assume (1) holds. Clearly (2.1) holds. Assume that for $w \in \Sigma^+ \cap L(A_{n+1})$, $A(w)$ has a minimal cycle, and consider a minimal cycle decomposition of w, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. By definition of β, for any $i_1 \geq 0, \cdots, i_k \geq 0$, the following hold.

$$x_1y_1^{i_1}x_2y_2^{i_2}\cdots x_ky_k^{i_k}x_{k+1} \in L(A_{n+1}) \text{ and}$$
$$F(x_1y_1^{i_1}x_2y_2^{i_2}\cdots x_ky_k^{i_k}x_{k+1}, A_1 \cup \cdots \cup A_n) \geq F(x_1y_1^{i_1}x_2y_2^{i_2}\cdots x_ky_k^{i_k}x_{k+1}, A_{n+1})$$

From this, one can see $LIS(w, \beta)$ has no integer solution.

(2) $\Rightarrow$ (1). We shall prove the contraposition. Assume (1) does not hold. There exists $w \in \Sigma^+ \cap L(A_n + 1)$ such that $F(w, A_1 \cup \cdots \cup A_n) < F(w, A_{n+1})$. If w has no minimal cycle, then (2.1) does not hold. If w has a minimal cycle, consider any minimal cycle decomposition of w, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$, and let $LIS(w, \beta)$ be the following simultaneous linear inequlities.

$$X_j \geq 0 \ \ (1 \leq j \leq k)$$

$$a_i + b_{i\,1}X_1 + b_{i\,2}X_2 + \cdots + b_{i\,k}X_k < a_{n+1} + b_{n+1\,1}X_1 + b_{n+1\,2}X_2 + \cdots + b_{n+1\,k}X_k \ \ (1 \leq i \leq n)$$

Here a_i and $b_{i\,j}$ are as in Definition 4.10. Since $F(w, A_1 \cup \cdots \cup A_n) < F(w, A_{n+1})$, $LIS(w, \beta)$ has an integer solution $X_1 = X_2 = \cdots = X_k = 1$. Hence (2.2) does not hold. □

<u>**Theorem 4.3**</u> The following three conditions are equivalent.

(1) $A_1 \cup \cdots \cup A_n \geq A_{n+1}$

(2) For each $w \in \Sigma^+ \cap L(A_{n+1})$ with length less than $|Q| \times I(A_1, \cdots, A_{n+1})$, the following (2.1) or (2.2) holds.

(2.1) If $A(w)$ has no minimal cycle, then it holds $F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1})$

(2.2) If $A(w)$ has a minimal cycle, then for each minimal cycle decoposition of w, $\beta \in MCD(w)$, the simultaneous linear inequlities $LIS(w, \beta)$ have no integer solution.

(3) For any $w \in \Sigma^+ \cap L(A_{n+1})$, the following (3.1) or (3.2) holds.

(3.1) If $A(w)$ has no minimal cycle, then it holds $F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1})$

(3.2) If $A(w)$ has a minimal cycle, then foe each minima cycle decoposition of w, $\beta \in MCD(w)$, the simultaneous linear inequlities $LIS(w, \beta)$ have no integer solution.

[Proof] (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold from Lemma 4.2. Thus it suffices to prove (2) $\Rightarrow$ (3). Assume (2) holds. We shall prove (3) by induction on $|w|$. When $|w| < |Q| \times I(A_1, \cdots, A_{n+1})$, the assertion is clear. Assume (3) holds for each $w \in \Sigma^*$ with length at most $k \geq |Q| \times I(A_1, \cdots, A_{n+1}) - 1$. Consider $w \in \Sigma^*$ with $|w| = k + 1 \geq |Q| \times I(A_1, \cdots, A_{n+1})$. Since $|w| \geq |Q| \times I(A_1, \cdots, A_{n+1})$, $A(w)$ has a minimal cycle. Consider a minimal cycle decomposition of w, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. It suffices to show the simultaneous inequlites $LIS(w, \beta)$ have no integer solution. Let $LIS(w, \beta)$ be of the following form as in Definition 4.10.

$$X_j \geq 0 \quad (1 \leq j \leq k)$$

$$a_i + b_{i\,1}X_1 + b_{i\,2}X_2 + \cdots + b_{i\,k}X_k < a_{n+1} + b_{n+1\,1}X_1 + b_{n+1\,2}X_2 + \cdots + b_{n+1\,k}X_k \quad (1 \leq i \leq n)$$

Assume these simultaneous inequlities have an integer solution $X_j = r_j (1 \leq j \leq k)$. Since $|w| \geq |Q| \times I(A_1, \cdots, A_{n+1})$, there exists $1 \leq f < g \leq k$ such that the minimal cycle (p_f, y_f, p_f) and the minimal cycle (p_g, y_g, p_g) are the same. Consider the word

$$v = x_1 y_1 x_2 y_2 \cdots x_f y_f x_{f+1} y_{f+1} \cdots x_{g-1} y_{g-1} x_g x_{g+1} y_{g+1} \cdots x_k y_k x_{k+1}$$

The word v has the minimal cycle decomposition

$$\begin{aligned}\beta' = (&s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_f, p_f, y_f, p_f, x_{f+1}, p_{f+1}, y_{f+1}, p_{f+1}, \cdots \\ &\cdots, x_{g-1}, p_{g-1}, y_{g-1}, p_{g-1}, A, p_{g+1}, y_{g+1}, p_{g+1}, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})\end{aligned}$$

Here A is $x_g x_{g+1}$ if $|x_g x_{g+1}| \leq |Q|$, and otherwise the corresponding minimal cycle.

We consider the case $A = x_g x_{g+1}$. In the other case, the proof is similar. The simultaneous linear inequlities $LIS(v, \beta')$ are of the following form.

$$\begin{aligned}&Y_g \geq 0 \quad (1 \leq g \leq k-1) \\ &c_f + d_{f\,1}Y_1 + d_{f\,2}Y_2 + \cdots + d_{f\,k-1}Y_{k-1} < \\ &\qquad c_{n+1} + d_{n+1\,1}Y_1 + d_{n+1\,2}Y_2 + \cdots + d_{n+1\,k-1}Y_{k-1} \quad (1 \leq f \leq n)\end{aligned}$$

By comparing $LIS(w, \beta)$ with $LIS(v, \beta')$, one can see easily that $LIS(v, \beta')$ have the following integer solution.

(i) $Y_j = r_j \quad (1 \leq j < g,\ j \neq f)$

(ii) $Y_f = r_f + r_g$

(iii) $Y_j = r_{j+1} \quad (g \leq j \leq k-1)$

Since $|v| < |w|$, this is a contradiction to the inductive hypothesis. Thus $LIS(w, \beta)$ have no integer solution. Hence (3) holds. □/

By Theorem 4.3, the inequlity problem of union F-automata can be reduced to the problem of solving simultaneous linear inequlities. The latter problem is decidable [3]. Thus the following theorem holds.

Theorem 4.4 The inequlity problem of union F-automata is decidable.

4.2.3 Decidability of the equivalence problem of union F-automata

Definition 4.11 The following propblem is called the equivalence problem of union F-automata.

Problem input : $m(m \geq 1)$deterministic F-automata
$A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq m)$.
$n(n \geq 1)$ deterministic F-automata
$A_{m+j} =< \Sigma, Q_{m+j}, \delta_{m+j}, \{s_{m+j}\}, F_{m+j}, f_{m+j} > (1 \leq j \leq n)$.
output : if $A_1 \cup \cdots \cup A_m \equiv A_{m+1} \cup \cdots \cup A_{m+n}$, then "yes".
if $A_1 \cup \cdots \cup A_m \not\equiv A_{m+1} \cup \cdots \cup A_{m+n}$, then "no".

Theorem 4.5 The equivalence problem of union F-automata is decidable.

[Proof] From Proposition 4.2 and Theorem 4.4, the assertion follows immediately. □

5 Decidability of the equivalence problem of finitely ambiguous F-automata

In this section, we shall prove decidability of the equivalence problem of finitely ambiguous F-automata. To do this, we shall present an algorithm for decomposing a finitely ambiguous F-automaton A to a finite set of deterministic F-automata whose union is equivalent to A.

5.1 Finitely ambiguous F-automata

In this subsection, we prove the problem of decidibg whether a given F-automaton is finitely ambiguous is decidable. This result is already known, but we shall present the proof since the proof seems new.

Definition 5.1 For a nondeterministic F-automaton $A =< \Sigma, Q, \delta, S, F, f >$ and a word $w = a_1 a_2 \cdots a_m \in L(A)$ $(m \geq 1, a_i \in \Sigma)$, a successful path of w is a path spelling w from S to F, i.e.,

$$\text{path } (p_0, a_1, p_1, a_2, p_2, \cdots, p_{m-1}, a_m, p_m)$$

Here $p_i \in Q(0 \leq i \leq m)$, $p_j \in \delta(p_{j-1}, a_j)(1 \leq j \leq m)$, $p_0 \in S$, $p_m \in F$.

Definition 5.2 A nondeterministic F-automaton $A =< \Sigma, Q, \delta, S, F, f >$ is k- ambiguous if there exists a positive integer k such that for any $w \in L(A)$, the number of successful paths of w is at most k. A is finitely ambiguous if for some positive integer k, it is k-ambiguous.

In the following, we shall present a necessary and sufficient condition for a nondeterministic F-automaton $A =< \Sigma, Q, \delta, S, F, f >$ not to be finitely ambiguous.

Definition 5.3 Let $A =< \Sigma, Q, \delta, S, F, f >$ be a nondeterministic F-automaton. For any $w \in \Sigma^+ \cap L(A)$ and a decomposition of w, $w = uv(u, v \in \Sigma^*)$, $SP(A, w, u)$ denotes the number of subpaths appearing at the part from S to $\delta(S, u)$ of all successful paths of w. We denote by $Q(A, w, u)$ the set $\{q \in Q \mid q \in \delta(S, u) \textit{ and } \delta(q, v) \in F\}$.

Theorem 5.1 For any nondeterministic F-automaton $A =< \Sigma, Q, \delta, S, F, f >$, the following three conditions are equivalent.

(1) A is not finitely ambiguous.

(2) There exist $w \in \Sigma^+ \cap L(A)$ with length at most $2^{|Q|+2}$ and a decomposition of w, $w = xyz$, such that it holds $x, z \in \Sigma^*$, $y \in \Sigma^+$, $Q(A, w, x) = Q(A, w, xy)$ and $SP(A, w, x) < SP(A, w, xy)$.

(3) There exists $w \in \Sigma^+ \cap L(A)$ such that the number of successful paths of w is at least $|Q|^{2^{|Q|}}$.

[Proof] Put $m = |Q|$.

(1) $\Rightarrow$ (2). Assume (2) does not hold. Put $n = \max\{SP(A, w, w) \mid w \in \Sigma^+ \cap L(A) \textit{ and } |w| \leq 2^{m+2}\}$. For $w \in \Sigma^+ \cap L(A)$, we shall prove by induction on $|w|$ that it holds $SP(A, w, w) \leq n$, and for a decomposition of $w = xyz$ as (2), it holds $SP(A, w, x) = SP(A, w, xy)$.

When $|w| \leq 2^{m+2}$, the assertion is clear.

Let $|w| > 2^{m+2}$, and assume for any word with length less than $|w|$, the assertion holds. Consider a decomposition of w as (2), $w = xyz$, $x, z \in \Sigma^*$, $y \in \Sigma^+$, $Q(A, w, x) = Q(A, w, xy)$. Since $|w| > 2^{m+2}$, it holds $|x| \geq 2^m$, $|y| \geq 2^{m+1}$ or $|z| \geq 2^m$. We shall present the proof in the case $|x| \geq 2^m$. Other cases can be handled similarly.

Since $|x| \geq 2^m$, there exists a decomposition of x, $x = x_0y_0x_1$, such that $Q(A, w, x_0) = Q(A, w, x_0y_0)$, $x_0, x_1 \in \Sigma^*$, $y_0 \in \Sigma^+$. Put $w_0 = x_0y_0x_1z$ and $w_1 = x_0x_1yz$. By induction, $SP(A, w_0, x_0) = SP(A, w_0, x_0y_0)$, $SP(A, w_1, x_0x_1) = SP(A, w_1, x_0x_1y)$ and $SP(A, w_0, w_0), SP(A, w_1, w_1) \leq n$.

Then on can see the following holds.

$$SP(A, w, x) = SP(A, w, xy) \textit{ and } SP(A, w, w) \leq n$$

Hence A is finitely ambiguous.

(2) $\Rightarrow$ (1). Assume there exist a word w and a decomposition of w, $w = xyz$, satisfying (2). Since $Q(A, w, x) = Q(A, w, xy)$, for any $i \geq 0$, $Q(A, w, xy^i) = Q(A, w, x)$. Let $Q(A, w, x) = \{p_1, p_2, \cdots, p_r\}$. Since $w \in L(A)$, $r \geq 1$ and $SP(A, w, x) < SP(A, w, xy)$, there exists $p_j \in Q(A, w, x)$ such that $|\delta(p_j, y) \cap Q(A, w, x)| \geq 2$. Then for any $i \geq 1$, by comparing $SP(A, w, xy^{i-1})$ and $SP(A, w, xy^i)$, on can see that for each path P of xy^{i-1} form S counted in $SP(A, w, xy^{i-1})$, when y follows xy^{i-1}, a subpath of y after P continues to F. Thus $SP(A, w, xy^i) \geq SP(A, w, xy^{i-1})$.

Moreover in the subpath corresponding to p_j, when y follows xy^{i-1}, the number of subpaths increases. Thus $SP(A, w, xy^i) > SP(A, w, xy^{i-1})$. By induction on i, it holds

$$SP(A, w, xy^i) \geq r + i$$

Hence A is not finitely ambiguous.

(1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). If A is finitely ambiguous, then in the proof of (1) $\Rightarrow$ (2), for any $w \in \Sigma^+ \cap L(A)$, there exists $v \in \Sigma^+ \cap L(A)$ with length at most $2^{|Q|} - 1$ such that the number of successful paths of w is equal to that of v. Thus if $v = a_1 \cdots a_m (a_i \in \Sigma)$, then for each a_i, the number of successful subpaths corresponding to the part of a_i increases at most by factor $|Q|$. Hence (the number of successful paths of w) = (the number of successful paths of v) is at most $|Q|^{2^{|Q|}-1}$.
□

The following theorem holds.

Theorem 5.2 It is solvable to decide whether any given nondeterministic F-automaton is finitely ambiguous.

5.2 Decomposition of finitely ambiguous F-automata

We consider the following problem.

Problem Find an algorithm for decomposing any given finitely ambiguous F-automaton A to a union F-automaton equivalent to A, that is, finding a set of deterministic F-automata A_i $(1 \leq i \leq m, m \geq 1)$ satisfying $A \equiv A_1 \cup \cdots \cup A_m$.

From the following Definition 5.4 to Theorem 5.3, let $A =< \Sigma, Q, \delta, S, F, f >$ be a finitely ambiguous F-automaton.

Definition 5.4 For any $w \in \Sigma^+$ with length at most $|Q|$ and a path $P = (p_0, a_1, p_1, \cdots, p_{m-1}, a_m, p_m)$, define a deterministic F-automaton $A(P) =< \Sigma, Q_P, \delta_P, S_P, F_P, f_P >$ as follows.

(1) Let $X = \{q_0, \cdots, q_m\}$ be a new set with $m + 1$ elements. Then put $Q_P = X$.

(2) For each $0 \leq i \leq m$, define the following.

$\delta_P(q_i, a_i) = q_{i+1}$, $f_P(q_i, a_i, q_{i+1}) = f(p_i, a_i, p_{i+1})$

$S_P = \{q_0\}$, $F_P = \{q_m\}$ if $p_m \in F$ and $F_P = \phi$ otherwise.

Definition 5.5 For any $w \in \Sigma^+$ with length greater than $|Q|$ and a path of w, $P = (p_0, a_1, p_1, a_2, p_2, \cdots, p_{m-1}, a_m, p_m)$, define a deterministic F-automaton $A(P) =< \Sigma, Q_P, \delta_P, S_P, F_P, f_P >$ as follows.

(1) Let $X = \{q_0, \cdots, q_m\}$ be a new set of $m + 1$ elements.

(2) For each $i \geq 0$, define an equivalnece relation $\equiv_i$ over X inductively as follows. where $equiv_{i+1}$ is a refinement of $equiv_i$ for $i \geq 0$.

(2.1) For any $q_j, q_k \in X$, $q_j \equiv_0 q_k$ iff $p_j = p_k$.

(2.2) For $0 \leq r \leq m$, let $B = \{q \in X \mid q \equiv_0 q_r\} = \{q_{j_1}, \cdots, q_{j_k}\}$, $0 \leq j_1 < \cdots < j_k \leq m$. Then $\equiv_1$ over B is the maximal equivalence relation such that for any $1 \leq s < t \leq k$, $q_{j_s} \not\equiv_1 q_{j_t}$ iff $a_{j_s+1} = a_{j_t+1}$ and $p_{j_s+1} \neq p_{j_t+1}$.

(2.3) For any $i \geq 1$ and $q_j, q_k \in X$, $q_j \equiv_i q_k$ iff (i) $q_j \equiv_{i-1} q_k$ and $q_{j+1} \equiv_{i-1} q_{k+1}$ if $k + 1 \neq m$, and (ii) $q_j \equiv_{i-1} q_k$ if $k + 1 = m$.

(3) Let $\equiv'$ be the equivalnece relation over X such that for some $i \geq 0$, $\equiv' = \equiv_i = \equiv_{i+1}$. Define the equivalence relation $\equiv$ by : $\forall 0 \leq j < k \leq m-1$, $q_j \equiv q_k$ iff $q_j \equiv' q_k$ and if for $\exists 0 \leq j \leq m-1$, $q_j \equiv' q_{m-1}$, then $q_m \equiv q_{j+1}$. For each $q \in X$, let $[q]$ denote the equivalnece class of q under $\equiv$.

(4) Define $Q_P = \{[q] \mid q \in X\}$ and for each $0 \leq i < m$ and $a_i \in \Sigma$, define the following.

(4.1) $\delta_P([q_i], a_i) = [q_{i+1}]$

(4.2) $f_P([q_i], a_i, [q_{i+1}]) = f(p_i, a_i, p_{i+1})$

(5) $S = \{[q_0]\}$, $F = \{[q_m]\}$ if $p_m \in F$, and $F = \phi$ otherwise.

The following proposition is clear.

Proposition 5.1 For any $w \in \Sigma^+$ and a path P of w, $A(P)$ is deterministic.

The following proposition is clear since A is finitely ambiguous.

Proposition 5.2 $S(A)$ is a finite set.

The following theorem holds.

Theorem 5.3 For any given finitely ambiguous F-automaton $A =< \Sigma, Q, \delta, S, F, f >$, one can construct a finite set of deterministic F-automata $\{A_1, \cdots, A_m\} (m \geq 1)$ such that $A \equiv A_1 \cup \cdots \cup A_m$.

[Proof] For each $i \geq 1$, construct the set of deterministic F-automata Y_i by $Y_i = \{A(P) \mid \textit{for some } w \in \Sigma^i, \ P \textit{ is a path of } w\}$ untill it holds $Y_k = Y_{k+1}$ for some $k \geq 1$. Such k exists since $S(A)$ is a finite set. Then it holds $S(A) = Y_k$. Let $Y_k = \{A_1, \cdots, A_m\}$. Then it holds $A \equiv A_1 \cup \cdots A_m$. □

From Theorems 4.5 and 5.3, we have the following theorem.

Theorem 5.4 The equivalnec problem of finitely ambiguous F-automata is decidable.

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