Decidability of the equivalence problem of finitely ambiguous finance automata

Kosaburo Hashiguchi (橋口攻三郎) and Kenichi Ishiguro (石黒賢一)
Department of Information Technology, Faculty of Engineering, Okayama University

Abstract

A finance automaton is a sixtuple $< \Sigma, Q, \delta, S, F, f >$, where $< \Sigma, Q, \delta, S, F >$ is a finite automaton, $f : Q \times \Sigma \times Q \rightarrow R \cup \{-\infty\}$ is a finance function, $R$ is the set of real numbers and it holds $f(q, a, q') = -\infty$ iff $q' \notin \delta(q, a)$. The function $f$ is extended to $f : 2^Q \times \Sigma^* \times 2^Q \rightarrow R \cup \{-\infty\}$ by the plus-max principle. For any $w \in \Sigma^*$, $f(S, w, F)$ is the profit of $w$. It is shown that the equivalence problem of finitely ambiguous finance automata is decidable.

1 Introduction

A finance automaton is a finite automaton with a function. It may be regarded to be a model describing financial activities. In this paper, we prove decidability of the equivalence problem of finitely ambiguous finance automata by reducing the problem to the problem of finding integer solutions in simultaneous linear inequalities. We also study several subproblems and present a different proof to each subproblem so that the proof is easier if the complexity of the subproblem is easier. The paper consists of five Sections. In Section 2, we present the definition and elementary properties of finance automata. In Section 3, we present the definitions of a union finance automaton and a vector finance automaton each of which is composed from a finitely many deterministic finance automata. In Section 4, first we present a proof showing decidability of the equivalence problem of a union finance automaton and a deterministic finance automaton. Then we show that the inequality problem of a union finance automaton and a deterministic finance automaton is decidable. This implies decidability of the equivalence problem of union finance automata. In Section 5, we present a method for decomposing a finitely ambiguous finance automaton to a union finance automaton. This fact together with results in Section 4 implies decidability of the equivalence problem of finitely ambiguous finance automata.

2 Finance Automata

An alphabet is a nonempty set of symbols. A word is a finite sequence of symbols from the alphabet. A word of zero length is called a null word and denoted by $\lambda$. $\Sigma^*$ denotes the set of all words over an alphabet $\Sigma$, and $\Sigma^+$ denotes the set of nonnull words. For a word, $w = a_1a_2\cdots a_n$ ($n \geq 0, a_1, \ldots, a_n \in \Sigma$), $n$ is the length of $w$, and denoted by $|w|$. The cardinality of a set $A$ is denoted by $|A|$. $R$ denotes the set of real numbers. $R_{-\infty}$ denotes $R \cup \{-\infty\}$.
2.1 Basic definitions

**Definition 2.1** A finance automaton (in short, an F-automaton) is a sixtuple $A = <\Sigma, Q, \delta, S, F, f>$, where

- $\Sigma$ : an input alphabet, $Q$ : a finite set of states, $\delta$ : a transition function
- $S$ : the set of initial states, $F$ : the set of final states, $f$ : a finance function

Thus $<\Sigma, Q, \delta, S, F>$ is a finite automaton. A word $w \in \Sigma^*$ is accepted by $A$ if $\delta(S, w) \in F$. The finance function $f$ is a mapping from $Q \times \Sigma \times Q$ to $R_\infty$ ($f : Q \times \Sigma \times Q \rightarrow R_\infty$). $f$ satisfies the following.

$$\forall (p, a, q) \in Q \times \Sigma \times Q, q \notin \delta(p, a) \iff f(p, a, q) = -\infty$$

In the rest of the paper, a finance automaton is called an F-automaton.

**Definition 2.2**

1. $f$ is extended to $f : Q \times \Sigma^* \times Q \rightarrow R_\infty$ and to $f : 2^Q \times \Sigma^* \times 2^Q \rightarrow R_\infty$ in the following way.

   1.1 $\forall p, q \in Q, f(p, \lambda, q) = 0$ (if $p = q$), $f(p, \lambda, q) = -\infty$ (if $p \neq q$)
   1.2 $\forall p, q \in Q, \forall w \in \Sigma^*, \forall a \in \Sigma,$
   $$f(p, wa, q) = \max\{f(p, w, q') + f(q', a, q) | q' \in Q\}$$

   Here, for $\forall i \in R_\infty$, $\max\{i, -\infty\} = i$ and $i + (-\infty) = -\infty$.

   1.3 $\forall t, t' \in Q, \forall w \in \Sigma^*, f(t, w, t') = \max\{f(p, w, q) | p \in t, q \in t'\}$

2. The set of words accepted by $A$ is denoted by $L(A) : L(A) = \{w \in \Sigma^* | \delta(S, w) \cap F \neq \emptyset\}$

3. Tow F-automata $A_1$ and $A_2$ is said to be L-equivalent if $L(A_1) = L(A_2)$.

4. For $\forall w \in \Sigma^*$, $f(S, w, F)$ is the profit of $w$ (by $A$). The profit of $w$ is sometimes denoted by $F(w, A)$ ($= f(S, w, F)$).

5. Let $A_1$ and $A_2$ be two F-automata over the input alphabet $\Sigma$. If $\forall w \in \Sigma^*$,

   5.1 If for any $w \in \Sigma^*$, it holds $F(w, A_1) = F(w, A_2)$, then $A_1$ and $A_2$ are said to be equivalent and write $A_1 \equiv A_2$.
   5.2 If for any $w \in \Sigma^*$, it holds $F(w, A_1) \geq F(w, A_2)$, then $A_1$ is said to be equal or greater than $A_2$, and write $A_1 \geq A_2$.

6. An F-automaton $A$ is deterministic if the following hold.

   For $\forall q \in Q, \forall a \in \Sigma$, it holds $|\delta(q, a)| \leq 1$ and $|S| \leq 1$.

   When $A$ is deterministic, for $\forall q \in Q, \forall a \in \Sigma, \forall q' \in \delta(q, a)$, we write $\delta(q, a) = q'$.

7. For any $m \geq 1$, $m$ deterministic F-automata $A_i = <\Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq m)$ are said to be disjoint if for $\forall j, k(1 \leq j < k \leq m), Q_j \cap Q_k = \emptyset$.

It is shown that two deterministic F-automata $A_1 = <\Sigma, Q_1, \delta_1, \{s_1\}, F_1, f_1 >$ and $A_2 = <\Sigma, Q_2, \delta_2, \{s_2\}, F_2, f_2 >$ are equivalent iff for all $w \in \Sigma^*$ with at most $2 \times |Q_1| \times |Q_2|$, $f(\{s_1\}, w, F_1) = f(\{s_2\}, w, F_2)$ [4].
3 Union $F$-automata and vector $F$-automata

In this section, from a finite set of deterministic $F$-automata, we define a union $F$-automaton whose set of states is the union of the set of each deterministic $F$-automaton, and a vector $F$-automaton whose set of states is the direct product of the set of each deterministic $F$-automaton.

3.1 Union $F$-automata

Definition 3.1 For disjoint $m(m \geq 1)$ deterministic $F$-automata, $A_i = \langle \Sigma_i, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle (1 \leq i \leq m)$, a union $F$-automaton $A_1 \cup \cdots \cup A_m = \langle \Sigma, Q, \delta, S, F, f \rangle$ is defined as follows.

1. $Q = Q_1 \cup \cdots \cup Q_m$, $S = \{s_1, \ldots, s_m\}$, $F = F_1 \cup \cdots \cup F_m$
2. For $\forall i(1 \leq i \leq m)$, $\forall q \in Q_i$, $\forall a \in \Sigma$, $\delta(q, a) = \delta_i(q, a)$ and $f(q, a, \delta(q, a)) = f_i(q, a, \delta_i(q, a))$

Proposition 3.1 Let $A_i = \langle \Sigma_i, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle (1 \leq i \leq m)$ be disjoint $m(m \geq 1)$ deterministic $F$-automata. Then for $\forall w \in \Sigma^*$, it holds

$F(w, A_1 \cup \cdots \cup A_m) = \max\{F(w, A_i) | 1 \leq i \leq m\}$

Proof] For $\forall w \in \Sigma^*$, $F(w, A_1 \cup \cdots \cup A_m) = f(S, w, F)$. By definition, for $\forall t, t' \subset Q$, $\forall w \in \Sigma^*$, it holds $f(t, w, t') = \max\{f(p, w, q) | p \in t, q \in t'\}$. Thus

$f(S, w, F) = \max\{f(s_i, w, p_{in}) | s_i \in S, \forall p_{in} \in F_i, 1 \leq i \leq m\}$

$= \max\{F(w, A_i) | 1 \leq i \leq m\}$

Hence $F(w, A_1 \cup \cdots \cup A_m) = \max\{F(w, A_i) | 1 \leq i \leq m\}$. $\square$

3.2 Vector $F$-automata

Definition 3.2 For disjoint $m(m \geq 1)$ deterministic $F$-automata $A_i = \langle \Sigma_i, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle (1 \leq i \leq m)$, a vector $F$-automaton $V(A_1, \cdots, A_m) = \langle \Sigma, Q, \delta, S, F, f \rangle$ is defined as follows.

1. $Q = Q_1 \times \cdots \times Q_m$, $S = \{(s_1, \ldots, s_m)\}$, $F = F_1 \times \cdots \times F_m$
2. For $\forall (p_1, \cdots, p_m) \in Q_1 \times \cdots \times Q_m, \forall a \in \Sigma$,

$\delta([p_1, \ldots, p_m], a) = (\delta_1(p_1, a), \ldots, \delta_m(p_m, a))$

$f([p_1, \ldots, p_m], a, \delta([p_1, \ldots, p_m], a)) = \max\{f_i(p_i, a, \delta_i(p_i, a)) | 1 \leq i \leq m\}$

Proposition 3.2 Let $A_i = \langle \Sigma_i, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle (1 \leq i \leq m)$ be disjoint $m(m \geq 1)$ deterministic $F$-automata. Then for $\forall w \in \Sigma^*$, $w = a_1a_2 \cdots a_n (n \geq 1, a_1, \ldots, a_n \in \Sigma)$, there exist for each $1 \leq j \leq n$, $1 \leq i \leq m$ and $p_{i,j-1}, p_{i,j} \in Q_i$ with $p_{i,j} = \delta(p_{i,j-1}, a_j) \in Q_i(1 \leq j \leq n)$ such that it holds $F(w, V(A_1, \cdots, A_m)) = \sum_{j=1}^{n} \max\{f_i(p_{i,j-1}, a_j, p_{i,j}) | 1 \leq i \leq m\}$.
For $\forall w \in \Sigma^*$, $|w| = n$, it holds $F(w, V(A_1, \cdots, A_m)) = f(S, w, F)$. By definition, for $\forall (p_{1,j-1}, \cdots, p_{m,j-1}), (p_{1,j}, \cdots, p_{m,j}) \in Q_1 \times \cdots \times Q_m$, $\forall a \in \Sigma$,

$$f((p_{1,j-1}, \cdots, p_{m,j-1}), a, (p_{1,j}, \cdots, p_{m,j})) = \max \{f_i(p_{i,j-1}, a, p_{i,j}) \mid 1 \leq i \leq m\} = b_j \ (1 \leq j \leq n)$$

It also holds for $\forall p, q \in Q$, $\forall v \in \Sigma^*$, $\forall a \in \Sigma$,

$$f(p, va, q) = \max \{f(p, v, q') + f(q', a, q) \mid q' \in Q\}$$

Thus

$$f(S, w, F) = b_1 + \cdots + b_n = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} \max \{f_i(p_{i,j-1}, a, p_{i,j}) \mid 1 \leq i \leq m\}$$

Hence $F(w, V(A_1, \cdots, A_m)) = \sum_{j=1}^{n} \max \{f_i(p_{i,j-1}, a, p_{i,j}) \mid 1 \leq i \leq m\}$. \hfill \qed

### 3.3 Properties of union $F$-automata and vector $F$-automata

It is shown [4] that for disjoint $m(m \geq 1)$ deterministic $F$-automata $A_i = <\Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i>$ $(1 \leq i \leq m)$, the union $F$-automaton $A_1 \cup \cdots \cup A_m$ and the vector $F$-automaton $V(A_1, \cdots, A_m)$ are equivalent iff for any $w \in \Sigma^*$ with length less than or equal to $(m + 1) \times |Q_1| \times \cdots \times |Q_m|$, it holds

$$F(w, A_1 \cup \cdots \cup A_m) = F(w, V(A_1, \cdots, A_m))$$

### 4 The equivalence problem of union $F$-automata

In this section, we show the equivalence problem of union $F$-automata is decidable.

#### 4.1 The equivalence problem of $A_1 \cup \cdots \cup A_n$ and $A_{n+1}$

We shall first present an algorithm for deciding whether for $n \geq 1$, a union $F$-automaton $A_1 \cup \cdots \cup A_n$ and a deterministic $F$-automaton $A_{n+1}$ are equivalent. Let $n \geq 1$ and for each $1 \leq i \leq n + 1$, $A_i = <\Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i>$ be a deterministic $F$-automaton.

**Definition 4.1** A finite automaton $\Gamma(A_1, \cdots, A_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$ is defined as follows.

1. $Q = Q_1 \times \cdots \times Q_{n+1}$, $S = (s_1, \cdots, s_{n+1})$, $F = F_1 \times \cdots \times F_{n+1}$

2. for $\forall a \in \Sigma$, $\forall (q_1, \cdots, q_{n+1}) \in Q_1 \times \cdots \times Q_{n+1}$,

   $$\delta((q_1, \cdots, q_{n+1}), a) = (\delta_1(q_1, a), \cdots, \delta_{n+1}(q_{n+1}, a))$$

**Proposition 4.1** Assume that $A_1 \cup \cdots \cup A_n \equiv A_{n+1}$. Then for $\forall x, z \in \Sigma^*$, $y \in \Sigma^+$, $(q_1, \cdots, q_{n+1}) \in Q$, if $\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_{n+1})$ and $\delta(s, xyz) \in F$, then there exists $1 \leq i \leq n$ for which the following (1)-(3) hold.
(1) \( F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1}) \)

(2) \( F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1}) \)

(3) \( f_i(q_i, y, q_i) = \max \{ f_j(q_j, y, q_j) \mid 1 \leq j \leq n \} = f_{n+1}(q_{n+1}, y, q_{n+1}) \)

[Proof] We put \( a = \max \{ f_i(q_i, y, q_i) \mid 1 \leq i \leq n \} \) and \( b = F(xyz, A_1 \cup \cdots \cup A_n) \). Define two sets \( X, Y \) by \( X = \{ i \mid 1 \leq i \leq n \text{ and } f_i(q_i, y, q_i) = a \}, Y = \{ i \mid 1 \leq i \leq n \text{ and } F(xyz, A_i) = b \}. \) Since \( A_1 \cup \cdots \cup A_n \equiv A_{n+1} \), it holds \( b = F(xyz, A_{n+1}) \). Put \( c = \max \{ F(xyz, A_i) \mid i \in X \} \), and define a set \( W \) by \( W = \{ i \in X \mid F(xy, A_i) = c \}. \) We consider the following two cases.

(i) \( b = c \). It holds \( X \cap Y = W \) and for any \( i \in X \cap Y, a = f_i(q_i, y, q_i) \) and \( c = F(xyz, A_i) = F(xy, A_1 \cup \cdots \cup A_n) \). It is clear that \( F(xz, A_i) = c - a = F(xz, A_1 \cup \cdots \cup A_n) \). Moreover \( A_1 \cup \cdots \cup A_n \equiv A_{n+1} \). Hence for any \( i \in W, (1)-(3) \) hold.

(ii) \( b \neq c \). Clearly \( b > c \). Put \( d = \max \{ f_i(q_i, y, q_i) \mid i \in Y \} \). Since \( b > c \), it holds \( d < a \). For each \( i \in W, F(xy^kz, A_1 \cup \cdots \cup A_n) = F(xy^kz, A_i) \). Since \( F(xz, A_{n+1}) = F(xz, A_1 \cup \cdots \cup A_n), F(xz, A_{n+1}) > F(xz, A_i) \). Together with this fact and \( F(xyz, A_{n+1}) = b \) we have \( f_{n+1}(q_{n+1}, y, q_{n+1}) < a \). If we consider a sufficiently large \( k \), it would hold for each \( i \in W, F(xy^kz, A_{n+1}) < F(xy^kz, A_i) = F(xy^kz, A_1 \cup \cdots \cup A_n) \). This is a contradiction to \( A_1 \cup \cdots \cup A_n \equiv A_{n+1} \). Thus the case \( b \neq c \) is impossible. \( \Box \)

We shall present a necessary and sufficient condition for \( A_1 \cup \cdots \cup A_n \) and \( A_{n+1} \) to be equivalent.

**Theorem 4.1** Let \( A_i = \langle \Sigma, Q, \delta, \{ s_i \}, F, f_i \rangle (1 \leq i \leq n+1, n \geq 1) \) be \( n+1 \) deterministic \( L \)-equivalent \( F \)-automata, and let \( \Gamma(A_1, \cdots, A_{n+1}) \) be as in Definition 4.1. Then the following four conditions are equivalent.

(1) \( A_1 \cup \cdots \cup A_n \equiv A_{n+1} \)

(2) For any \( w \in \Sigma^* \), \( F(w, A_1 \cup \cdots \cup A_n) = F(w, A_{n+1}) \)

(3) For any \( w \in \Sigma^* \) with length \( \leq 3 \times (n + 1) \times |Q_1| \times \cdots \times |Q_{n+1}| \), the following (A) holds.

(A): For \( \forall x, y, z \in \Sigma^*, (q_1, \cdots, q_{n+1}) \in Q \), if \( w = xyz, \delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n) \) and \( \delta(s, xyz) \in F \), then there exists \( 1 \leq i \leq n \) such that the following (3.1) - (3.3) hold.

(3.1) \( F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1}) \)

(3.2) \( F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1}) \)

(3.3) \( f_i(q_i, y, q_i) = \max \{ f_j(q_j, y, q_j) \mid 1 \leq j \leq n \} \) for \( 1 \leq j \leq n \) \)

(4) For any \( w \in \Sigma^* \), the following (B) holds.

(B): For \( \forall x, y, z \in \Sigma^*, (q_1, \cdots, q_{n+1}) \in Q \), if \( w = xyz, \delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n) \) and \( \delta(s, xyz) \in F \), then there exists \( 1 \leq i \leq n \) such that the following (4.1) - (4.3) hold.

(4.1) \( F(xyz, A_1 \cup \cdots \cup A_n) = F(xyz, A_i) = F(xyz, A_{n+1}) \)

(4.2) \( F(xz, A_1 \cup \cdots \cup A_n) = F(xz, A_i) = F(xz, A_{n+1}) \)

(4.3) \( f_i(q_i, y, q_i) = \max \{ f_j(q_j, y, q_j) \mid 1 \leq j \leq n \} \)
\[(4.3) \quad f_i(q_i, y, q_i) = \max\{f_j(q_j, y, q_j) \mid 1 \leq j \leq n\} = f_{n+1}(q_{n+1}, y, q_{n+1})\]

**Proof**  
(1) \(\Leftrightarrow\) (2) is clear. (1) \(\Rightarrow\) (3) follows from Proposition 4.1. (3) \(\Rightarrow\) (4). Assume (3) holds. We shall prove (4) by induction on \(|w|\). Put \(c = 3 \times (n + 1) \times |Q_1| \times \cdots \times |Q_{n+1}|\). When \(|w| \leq c\), (B) holds. Let \(k > c\) and assume that for \(|w| < k\), (B) holds. Consider the case \(|w| = k\). Assume that for \(w = xyz\) and \((q_1, \cdots, q_{n+1}) \in Q\), it holds \(\delta(s, x) = \delta(s, xy) = (q_1, \cdots, q_n)\) and \(\delta(s, xz) \in F\). Since \(|w| > c\), it holds \(|x| \geq c/3\), or \(|y| \geq c/3\) or \(|z| \geq c/3\).

First consider the case \(|y| \geq c/3\). By the definition of \(c\), \(y\) has a decomposition \(y = y_0y_1 \cdots y_{n+2}. y_0, y_{n+2} \in \Sigma^*, y_i \in \Sigma^+ (1 \leq i \leq n + 1)\) such that for some \((p_1, \cdots, p_{n+1}) \in Q\), \(\delta(s, xyo) = \delta(s, xyo) \cdots = \delta(s, xyo)\). For each \(1 \leq i \leq n + 1\), put \(v_i = y_1 \cdots y_{i-1}y_i+1 \cdots y_{n+1}\).

Consider the case \(i = 1. \) \(v_1 = y_2y_3 \cdots y_{n+2}\). By the inductive hypothesis, for \(x' = xy_0\), \(y_1 = v_1, z_1 = y_{n+2}\), there exists \(A_{i_1}\) to which (B) holds. The set of such \(A_{i_1}\) is denoted by \(X_{i_1}\). Next for \(x'_2 = xyo\), \(y'_2 = y_2, z'_2 = y_3y_4 \cdots y_{n+2}\), there exists \(A_{i_2}\) satisfying (B). The set of such \(A_{i_2}\) is denoted by \(X_{i_2}\). Clearly \(X_{i_2} \subseteq X_{i_1}\). Each \(A_j\) belonging to \(X_{i_2}\) has the maximum profits at \(y_2, y_3y_4 \cdots y_{n+1}, \) and \(xy_0y_{n+2}\) for each \(2 \leq i \leq n\). the set \(X_{i_2}\) can be defined. Then there exist \(1 \leq q \leq n + 1 \) \(1 \leq r \leq n\) such that \(A_{r} \in X_{q} \cap X_{\tau}\) holds. This implies \(A_{r}\) has the maximum profits at \(y_1, y_2, \cdots, y_{n+1}\), and \(xy_0y_{n+2}\), i.e., \(A_{r}\) satisfies (B).

When \(|z| \geq c/3\), if \(y = \lambda\), then we consider \(x = x_0x_1 \cdots x_{n+1}x_{n+2}, \delta(s, x_0) = \delta(s, x_0x_1) = \cdots = \delta(s, x_0x_{n+1}) (x_i \in \Sigma^+, 1 \leq i \leq n + 1)\) and the argument is similar to \(|y| \geq c/3\). If \(y \neq \lambda\), we consider \(x = x_0x_1 \cdots x_{n+1}, \delta(s, x_0) = \delta(s, x_0x_1) = \cdots = \delta(s, x_0x_{n+1}) (x_i \in \Sigma^+, 1 \leq i \leq n)\) and \(y\) (in sum, \(n + 1\) subwords), and the argument is similar to \(|y| \geq c/3\). The case \(|z| \geq c/3\) can be handled in the same way. Thus (3) \(\Rightarrow\) (4) is proved. (4) \(\Rightarrow\) (2) is clear. This completes the proof of Theorem 4.1. \(\square\)

The following theorem is now clear.

**Theorem 4.2**  
Given \(n + 1\) \(L\)-equivalent deterministic \(F\)-automata \(A_i =< \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i > (1 \leq i \leq n + 1)\), one can decide whether or not \(A_1 \cup \cdots \cup A_n \equiv A_{n+1}\) holds.

4.2 The equivalence problem of \(A_1 \cup \cdots \cup A_m\) and \(A_{m+1} \cup \cdots \cup A_{m+n}\)

In this subsection, we shall show decidability of the equivalence problem of union \(F\)-automata.

4.2.1 Prerequisites

**Proposition 4.2**  
For \(m + n (m, n \geq 1)\) deterministic \(F\)-automata \(A_k =< \Sigma, Q_k, \delta_k, \{s_k\}, F_k, f_k > (1 \leq k \leq m + n)\), the following two conditions are equivalent.

1. \(A_1 \cup \cdots \cup A_m \equiv A_{m+1} \cup \cdots \cup A_{m+n}\)

2. For each \(1 \leq i \leq m, A_i \leq A_{m+1} \cup \cdots \cup A_{m+n}\) and for each \(m + 1 \leq j \leq m + n, A_j \leq A_1 \cup \cdots \cup A_m\)
4.2.2 The inequality problem of union $F$-automata

**Definition 4.3** The following problem is called the inequality problem of union $F$-automata.

**Problem**

- **input**: $n + 1(n \geq 1)$ deterministic $F$-automata $A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ ($1 \leq i \leq n + 1$).
- **output**: if $A_1 \cup \cdots \cup A_n \geq A_{n+1}$, then "yes".
  if $A_1 \cup \cdots \cup A_n \not\geq A_{n+1}$, then "no".

**Definition 4.4** For $n + 1(n \geq 1)$ deterministic $F$-automata $A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ ($1 \leq i \leq n + 1$), define a deterministic finite automaton $\Pi(A_1, \cdots, A_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$ as follows.

1. $Q = Q_1 \times \cdots \times Q_{n+1}, s = (s_1, \cdots, s_{n+1})$
2. For $\forall a \in \Sigma, \forall (p_1, \cdots, p_{n+1}) \in Q$, 
$$\delta((p_1, \cdots, p_{n+1}), a) = (\delta_1(p_1, a), \cdots, \delta_{n+1}(p_{n+1}, a))$$
3. $F = (F_1 \times Q_2 \times \cdots \times Q_{n+1}) \cup (Q_1 \times F_2 \times Q_3 \times \cdots \times Q_{n+1}) \cup \cdots \cup (Q_1 \times \cdots \times Q_n \times F_{n+1})$

**Proposition 4.3** For any $n + 1(n \geq 1)$ deterministic $F$-automata $A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ ($1 \leq i \leq n + 1$), it holds $L(\Pi(A_1, \cdots, A_{n+1})) = L(A_1) \cup \cdots \cup L(A_{n+1})$.

**Proposition 4.4** For any $n + 1(n \geq 1)$ deterministic $F$-automata $A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ ($1 \leq i \leq n + 1$), if $A_1 \cup \cdots \cup A_n \geq A_{n+1}$, then it holds $L(\Pi(A_1, \cdots, A_n)) \supseteq L(A_{n+1})$.

Let $n \geq 1$, and $A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle$ ($1 \leq i \leq n + 1$) be $n + 1$ deterministic $F$-automata. As in Definition 4.4, define the deterministic finite automaton $\Pi(A_1, \cdots, A_{n+1}) = \langle \Sigma, Q, \delta, \{s\}, F \rangle$.

**Definition 4.5** For any $w \in \Sigma^+$, define a deterministic finite automaton $A(w) = \langle \Sigma, Q(w), \delta(w), \{s\}, F(w) \rangle$ as follows.

1. $Q(w) = \{(p_1, \cdots, p_{n+1}) | \text{for some } x, y \in \Sigma^*, w = xy \text{ and } (p_1, \cdots, p_{n+1}) = \delta(s, x)\}$
2. $\delta(w)$ is a mapping from $Q(w) \times \Sigma$ to $Q(w)$ ($\delta(w) : Q(w) \times \Sigma \rightarrow Q(w)$), and for $\forall (p_1, \cdots, p_{n+1}) \in Q(w)$ and $\forall a \in \Sigma$, define (2.1)-(2.2).

**(2.1)** If for some $x, y \in \Sigma^*$, it holds $w = xy$ and $(p_1, \cdots, p_{n+1}) = \delta(s, x)$, then 
$$\delta(w)((p_1, \cdots, p_{n+1}), a) = \delta((p_1, \cdots, p_{n+1}), a)$$

**(2.2)** Otherwise $\delta(w)((p_1, \cdots, p_{n+1}), a) = \emptyset$

3. $F(w) = \{\delta(s, w)\} \cap F$
**Definition 4.6** Define a set $S(A_1, \cdots, A_{n+1})$ by: $S(A_1, \cdots, A_{n+1}) = \{ A(w) \mid w \in \Sigma^+ \}$

**Definition 4.7** For $m \geq 1$ and $1 \leq r \leq m$, put $m_P = m(m-1)\cdots(m-(r-1))$.

**Definition 4.8** Define an integer $I(A_1, \cdots, A_{n+1})$ as follows, where $m = |Q| \times |\Sigma|$.

$$I(A_1, \cdots, A_{n+1}) = (m_P + m_P + \cdots + m_P + 1)(|Q| + 1)$$

**Definition 4.9** Let $w \in \Sigma^+$ and $A(w) = \langle \Sigma, Q(w), \delta(w), \{ s \}, F(w) \rangle$. For all $p \in Q(w)$, $i \geq 1$. $a_1, \cdots, a_i \in \Sigma$, if it holds $\delta(w)(p, a_1 \cdots a_i) = p$ and $i = 1$ or for $0 \leq j < k \leq i$, it holds $\delta(w)(p, a_1 \cdots a_j) \neq \delta(w)(p, a_1 \cdots a_k)$, then $(p, a_1 \cdots a_i, p)$ is called a minimal cycle of $A(w)$.

**Lemma 4.1** For any $w \in \Sigma^+$, $A(w)$ has at most $m_P + m_P + \cdots + m_P$ minimal cycles. Here $m = |Q| \times |\Sigma|$. 

**Proof** Let $c = (p, a_1 \cdots a_i, p)$ be a minimal cycle of $A(w)$. If $i = 1$, then the number of such $c$ is at most $m_P = m$. If $i \geq 2$, then for each $c$, put $t(c) = ((p, a_1), (\delta(p, a_1), a_2), \cdots, (\delta(p, a_1 \cdots a_{i-1}), a_i))$. Since $c$ is a minimal cycle, each of $i$ pairs of $t(c)$ is distinct. Thus $i \leq m$ and the number of minimal cycles of length $i$ is at most $m_P$. Hence the total number of minimal cycles is at most $m_P + m_P + \cdots + m_P$. □

**Proposition 4.5** $S(A_1, \cdots, A_{n+1}) = \{ A(w) \mid w \in \Sigma^+ \text{ and } |w| < I(A_1, \cdots, A_{n+1}) \}$

**Proof** Put $B = \{ A(w) \mid w \in \Sigma^+ \text{ and } |w| < I(A_1, \cdots, A_{n+1}) \}$. We shall prove by induction on $|w|$ that for all $w \in \Sigma^+$, $A(w) \in B$. When $|w| < I(A_1, \cdots, A_{n+1})$, the assertion is clear. Assume that for $|w|$ with length less than $k + 1$, the assertion holds, and consider the case $|w| = k + 1 \geq I(A_1, \cdots, A_{n+1})$. Put $m = |Q| \times |\Sigma|$, $q = m_P + m_P + \cdots + m_P + 1$. Since $|w| \geq I(A_1, \cdots, A_{n+1})$, there exists a decomposition of $w$, $w = w_1w_2 \cdots w_qw_{q+1}$ with $|w_i| = |Q| + 1(1 \leq i \leq q)$. Since $|w_i| = |Q| + 1$, in the path of $A(w)$, $(s, w_1w_2 \cdots w_qw_{q+1}, \delta(s, w))$, in each part of $w_i$, there exists a minimal cycle $(p_i, v_i, p_i)$. Here $w_i$ is decomposed as $w_i = x_iv_iv$. Namely in $A(w)$, there exists a path

$$(s, x_1, p_1, v_1, p_1, y_1x_2, p_2, v_2, p_2, y_2x_3, \cdots, y_{q-1}x_q, p_q, v_q, p_q, y_qw_{q+1}, \delta(s, w))$$

The number of minimal cycles is at most $q - 1$ by Lemma 4.1. Thus for some $1 \leq i \leq j \leq q$, it holds $(p_i, v_i, p_i) = (p_j, v_j, p_j)$. By putting $w' = w_1w_2 \cdots w_{j-1}x_jy_jw_{j+1} \cdots w_{q+1}$, we can see $A(w') = A(w)$ by definition of minimal cycles and $A(w)$. By induction, $A(w') \in B$. Hence $A(w) \in B$. □

**Definition 4.10** When for $w \in \Sigma^+ \cap L(A_{n+1})$, $A(w)$ has a minimal cycle, define the following (1)-(2).

1. $MCD(w)$ denotes the set of all sequences

$$(s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$$

which satisfy the following (1.1)-(1.2). Each sequence in $MCD(w)$ is called a minimal cycle decomposition of $w$.

1.1 $w = x_1y_1x_2y_2 \cdots x_ky_kx_{k+1}$

1.2 For each $1 \leq i \leq k$, $p_i = \delta(s, x_1y_1x_2y_2 \cdots x_{i-1}y_{i-1}x_i)$. $(p_i, y_i, p_i)$ is a minimal cycle of $A(w)$ and it holds $|x_i| \leq |Q|$. 


(2) Let \( MCD(w) = \{ \beta_1, \beta_2, \cdots, \beta_t \} (t \geq 1) \). For each \( \beta_i (1 \leq i \leq t) \), define a set of linear inequalities \( LIS(w, \beta_i) \) as follows.

Let \( \beta_i = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1}) \). The set of following inequalities over \( k \) variables \( X_1, X_2, \cdots, X_k \) is \( LIS(w, \beta_i) \).

For each \( 1 \leq i \leq n \),

\[
X_j \geq 0 \quad (1 \leq j \leq k)
\]

\[
a_i + b_{i1}X_1 + b_{i2}X_2 + \cdots + b_{ik}X_k < a_{n+1} + b_{n+11}X_1 + b_{n+12}X_2 + \cdots + b_{n+1k}X_k
\]

Here \( a_f = F(x_1 x_2 \cdots x_{k+1}, A_f) (1 \leq f \leq n + 1) \), and for each \( 1 \leq f \leq n + 1, 1 \leq j \leq k \),

\[
b_{fj} = \epsilon_f(s_f)\delta_f(s_f, X_j) \epsilon_f(x_f, X_{j-1}) \epsilon_f(y_f, X_{j+1}) \epsilon_f(\beta_f(s_f, X_j)) \epsilon_f(x_f, X_{j-1}) \epsilon_f(y_f, X_{j+1}) \epsilon_f(\beta_f(s_f, X_j))
\]

**Lemma 4.2** The following two conditions are equivalent.

(1) \( A_1 \cup \cdots \cup A_n \geq A_{n+1} \)

(2) For any \( w \in \Sigma^+ \cap L(A_{n+1}) \), (2.1) or (2.2) holds.

(2.1) If \( A(w) \) has no minimal cycle, then it holds \( F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1}) \)

(2.2) If \( A(w) \) has a minimal cycle, then for each minimal cycle decomposition of \( w, \beta \in MCD(w) \), the simultaneous linear inequalities \( LIS(w, \beta) \) have no integer solution.

**Proof**  
(1) \( \Rightarrow \) (2). Assume (1) holds. Clearly (2.1) holds. Assume that for \( w \in \Sigma^+ \cap L(A_{n+1}) \), \( A(w) \) has a minimal cycle, and consider a minimal cycle decomposition of \( w, \beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1}) \). By definition of \( \beta \), for any \( i_1 \geq 0, \cdots, i_k \geq 0 \), the following hold.

\[
x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1} \in L(A_{n+1}) \text{ and}
\]

\[
F(x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1}, A_1 \cup \cdots \cup A_n) \geq F(x_1 y_1^{i_1} x_2 y_2^{i_2} \cdots x_k y_k^{i_k} x_{k+1}, A_{n+1})
\]

From this, one can see \( LIS(w, \beta) \) has no integer solution.

(2) \( \Rightarrow \) (1). We shall prove the contraposition. Assume (1) does not hold. There exists \( w \in \Sigma^+ \cap L(A_{n+1}) \) such that \( F(w, A_1 \cup \cdots \cup A_n) < F(w, A_{n+1}) \). If \( w \) has no minimal cycle, then (2.1) does not hold. If \( w \) has a minimal cycle, consider any minimal cycle decomposition of \( w, \beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1}) \), and let \( LIS(w, \beta) \) be the following simultaneous linear inequalities.

\[
X_j \geq 0 \quad (1 \leq j \leq k)
\]

\[
a_i + b_{i1}X_1 + b_{i2}X_2 + \cdots + b_{ik}X_k < a_{n+1} + b_{n+11}X_1 + b_{n+12}X_2 + \cdots + b_{n+1k}X_k \quad (1 \leq i \leq n)
\]

Here \( a_i \) and \( b_{ij} \) are as in Definition 4.10. Since \( F(w, A_1 \cup \cdots \cup A_n) < F(w, A_{n+1}) \), \( LIS(w, \beta) \) has an integer solution \( X_1 = X_2 = \cdots = X_k = 1 \). Hence (2.2) does not hold.

**Theorem 4.3** The following three conditions are equivalent.

(1) \( A_1 \cup \cdots \cup A_n \geq A_{n+1} \)

(2) For each \( w \in \Sigma^+ \cap L(A_{n+1}) \) with length less than \( |Q| \times I(A_1, \cdots, A_{n+1}) \), the following (2.1) or (2.2) holds.
(2.1) If $A(w)$ has no minimal cycle, then it holds $F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1})$

(2.2) If $A(w)$ has a minimal cycle, then for each minimal cycle decomposition of $w$, $\beta \in MCD(w)$, the simultaneous linear inequalities $LIS(w, \beta)$ have no integer solution.

(3) For any $w \in \Sigma^+ \cap L(A_{n+1})$, the following (3.1) or (3.2) holds.

(3.1) If $A(w)$ has no minimal cycle, then it holds $F(w, A_1 \cup \cdots \cup A_n) \geq F(w, A_{n+1})$

(3.2) If $A(w)$ has a minimal cycle, then for each minimal cycle decomposition of $w$, $\beta \in MCD(w)$, the simultaneous linear inequalities $LIS(w, \beta)$ have no integer solution.

[Proof] (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold from Lemma 4.2. Thus it suffices to prove (2) $\Rightarrow$ (3). Assume (2) holds. We shall prove (3) by induction on $|w|$. When $|w| < |Q| \times I(A_1, \cdots, A_{n+1})$, the assertion is clear. Assume (3) holds for each $w \in \Sigma^*$ with length at most $k \geq |Q| \times I(A_1, \cdots, A_{n+1}) - 1$. Consider $w \in \Sigma^*$ with $|w| = k + 1 \geq |Q| \times I(A_1, \cdots, A_{n+1})$. Since $|w| \geq |Q| \times I(A_1, \cdots, A_{n+1})$, $A(w)$ has a minimal cycle. Consider a minimal cycle decomposition of $w$, $\beta = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$. It suffices to show the simultaneous inequalities $LIS(w, \beta)$ have no integer solution. Let $LIS(w, \beta)$ be of the following form as in Definition 4.10.

$$X_j \geq 0 \quad (1 \leq j \leq k)$$

$$a_1 + b_1 X_1 + b_2 X_2 + \cdots + b_k X_k < a_{n+1} + b_{n+1} X_1 + b_{n+2} X_2 + \cdots + b_{n+k} X_k \quad (1 \leq i \leq n)$$

Assume these simultaneous inequalities have an integer solution $x_j = r_j (1 \leq j \leq k)$. Since $|w| \geq |Q| \times I(A_1, \cdots, A_{n+1})$, there exists $1 \leq f < g \leq k$ such that the minimal cycle $(p_f, y_f, p_f)$ and the minimal cycle $(p_g, y_g, p_g)$ are the same. Consider the word

$$v = x_1 y_1 x_2 y_2 \cdots x_f y_f x_{f+1} y_{f+1} \cdots x_{g-1} y_{g-1} x_g y_g + 1 x_{g+1} y_{g+1} \cdots x_k y_k x_{k+1}$$

The word $v$ has the minimal cycle decomposition

$$\beta' = (s, x_1, p_1, y_1, p_1, x_2, p_2, y_2, p_2, \cdots, x_f, p_f, y_f, p_f, x_{f+1}, p_{f+1}, y_{f+1}, p_{f+1}, \cdots, x_g, p_g, y_g, p_g, x_{g+1}, y_{g+1}, p_{g+1}, x_{g+2}, \cdots, x_k, p_k, y_k, p_k, x_{k+1}, p_{k+1})$$

Here $A$ is $x_g x_{g+1}$ if $|x_g x_{g+1}| \leq |Q|$, and otherwise the corresponding minimal cycle.

We consider the case $A = x_g x_{g+1}$. In the other case, the proof is similar. The simultaneous linear inequalities $LIS(v, \beta')$ are of the following form.

$$Y_g \geq 0 \quad (1 \leq g \leq k-1)$$

$$c_f + d_f Y_1 + d_{f+1} Y_2 + \cdots + d_{k-1} Y_{k-1} < c_{n+1} + d_{n+1} Y_1 + d_{n+2} Y_2 + \cdots + d_{n+k-1} Y_{k-1} \quad (1 \leq f \leq n)$$

By comparing $LIS(w, \beta)$ with $LIS(v, \beta')$, one can see easily that $LIS(v, \beta')$ have the following integer solution.

(i) $Y_j = r_j \quad (1 \leq j < g, \, j \neq f)$

(ii) $Y_f = r_f + r_g$

(iii) $Y_j = r_{j+1} \quad (g \leq j \leq k-1)$
Since $|v| < |w|$, this is a contradiction to the inductive hypothesis. Thus $LIS(w, \beta)$ have no integer solution. Hence (3) holds.

By Theorem 4.3, the inequity problem of union F-automata can be reduced to the problem of solving simultaneous linear inequities. The latter problem is decidable [3]. Thus the following theorem holds.

**Theorem 4.4** The inequity problem of union F-automata is decidable.

### 4.2.3 Decidability of the equivalence problem of union F-automata

**Definition 4.11** The following problem is called the equivalence problem of union F-automata.

**Problem**

**Input:** $m(m \geq 1)$ deterministic F-automata

$A_i = \langle \Sigma, Q_i, \delta_i, \{s_i\}, F_i, f_i \rangle (1 \leq i \leq m)$.

$n(n \geq 1)$ deterministic F-automata

$A_{m+j} = \langle \Sigma, Q_{m+j}, \delta_{m+j}, \{s_{m+j}\}, F_{m+j}, f_{m+j} \rangle (1 \leq j \leq n)$.

**Output:**

- if $A_1 \cup \cdots \cup A_m \equiv A_{m+1} \cup \cdots \cup A_{m+n}$, then "yes".
- if $A_1 \cup \cdots \cup A_m \not\equiv A_{m+1} \cup \cdots \cup A_{m+n}$, then "no".

**Theorem 4.5** The equivalence problem of union F-automata is decidable.

[Proof] From Proposition 4.2 and Theorem 4.4, the assertion follows immediately.

## 5 Decidability of the equivalence problem of finitely ambiguous F-automata

In this section, we shall prove decidability of the equivalence problem of finitely ambiguous F-automata. To do this, we shall present an algorithm for decomposing a finitely ambiguous F-automaton $A$ to a finite set of deterministic F-automata whose union is equivalent to $A$.

### 5.1 Finitely ambiguous F-automata

In this subsection, we prove the problem of decidibility whether a given F-automaton is finitely ambiguous is decidable. This result is already known, but we shall present the proof since the proof seems new.

**Definition 5.1** For a nondeterministic F-automaton $A = \langle \Sigma, Q, \delta, S, F, f \rangle$ and a word $w = a_1a_2\cdots a_m \in L(A) (m \geq 1, a_i \in \Sigma)$, a successful path of $w$ is a path spelling $w$ from $S$ to $F$, i.e.,

path $(p_0, a_1, p_1, a_2, p_2, \cdots, p_{m-1}, a_m, p_m)$

Here $p_i \in Q(0 \leq i \leq m)$, $p_j \in \delta (p_{j-1}, a_j)(1 \leq j \leq m)$, $p_0 \in S$, $p_m \in F$.

**Definition 5.2** A nondeterministic F-automaton $A = \langle \Sigma, Q, \delta, S, F, f \rangle$ is $k$-ambiguous if there exists a positive integer $k$ such that for any $w \in L(A)$, the number of successful paths of $w$ is at most $k$. $A$ is finitely ambiguous if for some positive integer $k$, it is $k$-ambiguous.
In the following, we shall present a necessary and sufficient condition for a nondeterministic $F$-automaton $A = \langle \Sigma, Q, \delta, S, F, f \rangle$ not to be finitely ambiguous.

**Definition 5.3** Let $A = \langle \Sigma, Q, \delta, S, F, f \rangle$ be a nondeterministic $F$-automaton. For any $w \in \Sigma^+ \cap L(A)$ and a decomposition of $w$, $w = uw(v, v \in \Sigma^*)$, $SP(A, w, u)$ denotes the number of subpaths appearing at the part from $S$ to $\delta(S, u)$ of all successful paths of $w$. We denote by $Q(A, w, u)$ the set $\{ q \in Q \mid q \in \delta(S, u) \}$.

**Theorem 5.1** For any nondeterministic $F$-automaton $A = \langle \Sigma, Q, \delta, S, F, f \rangle$, the following three conditions are equivalent.

1. $A$ is not finitely ambiguous.

2. There exist $w \in \Sigma^+ \cap L(A)$ with length at most $2|Q| + 2$ and a decomposition of $w$, $w = xyz$, such that it holds $x, z \in \Sigma^*$, $y \in \Sigma^+$, $Q(A, w, x) = Q(A, w, xy)$ and $SP(A, w, x) < SP(A, w, xy)$.

3. There exists $w \in \Sigma^+ \cap L(A)$ such that the number of successful paths of $w$ is at least $|Q|^{2|Q|}$.

**Proof** Put $m = |Q|$.

(1) $\Rightarrow$ (2). Assume (2) does not hold. Put $n = \max\{ SP(A, w, w) \mid w \in \Sigma^+ \cap L(A) \text{ and } |w| \leq 2^{m+2} \}$. For $w \in \Sigma^+ \cap L(A)$, we shall prove by induction on $|w|$ that it holds $SP(A, w, w) \leq n$, and for a decomposition of $w = xyz$ as (2), it holds $SP(A, w, x) = SP(A, w, xy)$.

When $|w| \leq 2^{m+2}$, the assertion is clear.

Let $|w| > 2^{m+2}$, and assume for any word with length less than $|w|$, the assertion holds. Consider a decomposition of $w$ as (2), $w = xyz$, $x, z \in \Sigma^*$, $y \in \Sigma^+$, $Q(A, w, x) = Q(A, w, xy)$.

Since $|w| > 2^{m+2}$, it holds $|x| \geq 2^m$, $|y| \geq 2^{m+1}$ or $|z| \geq 2^m$. We shall present the proof in the case $|x| \geq 2^m$. Other cases can be handled similarly.

Since $|x| \geq 2^m$, there exists a decomposition of $x$, $x = x_0y_0x_1$, such that $Q(A, w, x_0) = Q(A, w, x_0y_0)$, $x_0, x_1 \in \Sigma^*$, $y_0 \in \Sigma^+$. Put $w_0 = x_0y_0x_1^2$ and $w_1 = x_0x_1y_2$. By induction, $SP(A, w_0, x_0) = SP(A, w_0, x_0y_0)$, $SP(A, w_1, x_0x_1) = SP(A, w_1, x_0x_1y_2)$ and $SP(A, w_0, w_0), SP(A, w_1, w_1) \leq n$.

Then on can see the following holds.

$SP(A, w, x) = SP(A, w, xy)$ and $SP(A, w, w) \leq n$

Hence $A$ is finitely ambiguous.

(2) $\Rightarrow$ (1). Assume there exist a word $w$ and a decomposition of $w$, $w = xyz$, satisfying (2). Since $Q(A, w, x) = Q(A, w, xy)$, for any $i \geq 0$, $Q(A, w, xy^i) = Q(A, w, x)$. Let $Q(A, w, x) = \{ p_1, p_2, \cdots, p_r \}$. Since $w \in L(A)$, $r \geq 1$ and $SP(A, w, x) < SP(A, w, xy)$, there exists $p_j \in Q(A, w, x)$ such that $|\delta(p_j, y) \cap Q(A, w, x)| \geq 2$. Then for any $i \geq 1$, by comparing $SP(A, w, xy^{i-1})$ and $SP(A, w, xy^i)$, one can see that for each path $P$ of $xy^{i-1}$ form $S$ counted in $SP(A, w, xy^{i-1})$, when $y$ follows $xy^{i-1}$, a subpath of $y$ after $P$ continues to $F$. Thus $SP(A, w, xy^i) \geq SP(A, w, xy^{i-1})$.

Moreover in the subpath corresponding to $p_j$, when $y$ follows $xy^{i-1}$, the number of subpaths increases. Thus $SP(A, w, xy^i) > SP(A, w, xy^{i-1})$. By induction on $i$, it holds
Hence $A$ is not finitely ambiguous.

$(1) \Rightarrow (3)$ is clear.

$(3) \Rightarrow (1)$. If $A$ is finitely ambiguous, then in the proof of $(1) \Rightarrow (2)$, for any $w \in \Sigma^+ \cap L(A)$, there exists $v \in \Sigma^+ \cap L(A)$ with length at most $2^{|Q|} - 1$ such that the number of successful paths of $w$ is equal to that of $v$. Thus if $v = a_1 \cdots a_m (a_i \in \Sigma)$, then for each $a_i$, the number of successful subpaths corresponding to the part of $a_i$ increases at most by factor $|Q|$. Hence (the number of successful paths of $w$) = (the number of successful paths of $v$) is at most $|Q|^{|Q|} - 1$.

The following theorem holds.

**Theorem 5.2** It is solvable to decide whether any given nondeterministic $F$-automaton is finitely ambiguous.

### 5.2 Decomposition of finitely ambiguous $F$-automata

We consider the following problem.

**Problem** Find an algorithm for decomposing any given finitely ambiguous $F$-automaton $A$ to a union $F$-automaton equivalent to $A$, that is, finding a set of deterministic $F$-automata $A_i$ ($1 \leq i \leq m$, $m \geq 1$) satisfying $A \equiv A_1 \cup \cdots \cup A_m$.

From the following Definition 5.4 to Theorem 5.3, let $A = \langle \Sigma, Q, \delta, S, F, f \rangle$ be a finitely ambiguous $F$-automaton.

**Definition 5.4** For any $w \in \Sigma^+$ with length at most $|Q|$ and a path $P = (p_0, a_1, p_1, \cdots, p_{m-1}, a_m, p_m)$, define a deterministic $F$-automaton $A(P) = \langle \Sigma, Q_P, \delta_P, S_P, F_P, f_P \rangle$ as follows.

1. Let $X = \{q_0, \cdots, q_m\}$ be a new set with $m + 1$ elements. Then put $Q_P = X$.
2. For each $0 \leq i \leq m$, define the following.
   - $\delta_P(q_i, a_i) = q_{i+1}$, $f_P(q_i, a_i, q_{i+1}) = f(p_i, a_i, p_{i+1})$
   - $S_P = \{q_0\}$, $F_P = \{q_m\}$ if $p_m \in F$ and $F_P = \phi$ otherwise.

**Definition 5.5** For any $w \in \Sigma^+$ with length greater than $|Q|$ and a path $w$, $P = (p_0, a_1, p_1, a_2, p_2, \cdots, p_{m-1}, a_m, p_m)$, define a deterministic $F$-automaton $A(P) = \langle \Sigma, Q_P, \delta_P, S_P, F_P, f_P \rangle$ as follows.

1. Let $X = \{q_0, \cdots, q_m\}$ be a new set of $m + 1$ elements.
2. For each $i \geq 0$, define an equivalence relation $\equiv_i$ over $X$ inductively as follows. where $equiv_{i+1}$ is a refinement of $equiv_i$ for $i \geq 0$.

   1. For any $q_j, q_k \in X$, $q_j \equiv_0 q_k$ iff $p_j = p_k$.
   2. For $0 \leq r \leq m$, let $B = \{q \in X \mid q \equiv_0 q_r\} = \{q_{j_1}, \cdots, q_{j_k}\}$, $0 \leq j_1 < \cdots < j_k \leq m$. Then $\equiv_1$ over $B$ is the maximal equivalence relation such that for any $1 \leq s < t \leq k$, $q_s \not\equiv_1 q_t$ iff $a_{j_s+1} = a_{j_t+1}$ and $p_{j_s+1} \neq p_{j_t+1}$.
   3. For any $i \geq 1$ and $q_j, q_k \in X$, $q_j \equiv_i q_k$ iff (i) $q_j \equiv_{i-1} q_k$ and $q_{j+1} \equiv_{i-1} q_{k+1}$ if $k + 1 \neq m$, and (ii) $q_j \equiv_{i-1} q_k$ if $k + 1 = m$. 

$$SP(A, w, xy') \geq r + i$$
(3) Let $\equiv'$ be the equivalence relation over $X$ such that for some $i \geq 0$, $\equiv' = \equiv_{i} = \equiv_{i+1}$. Define the equivalence relation $\equiv$ by: $\forall 0 \leq j < k \leq m-1$, $q_j \equiv q_k$ iff $q_j \equiv' q_k$ and if for $\exists 0 \leq j \leq m-1$, $q_j \equiv' q_{m-1}$, then $q_m \equiv q_{i+1}$. For each $q \in X$, let $[q]$ denote the equivalence class of $q$ under $\equiv$.

(4) Define $Q_F = \{[q] \mid q \in X\}$ and for each $0 \leq i < m$ and $a_i \in \Sigma$, define the following.

(4.1) $\delta_P([q_i], a_i) = [q_{i+1}]$

(4.2) $f_P([q_i], a_i, [q_{i+1}]) = f(p_i, a_i, p_{i+1})$

(5) $S = \{[q_0]\}$, $F = \{[q_m]\}$ if $p_m \in F$, and $F = \phi$ otherwise.

The following proposition is clear.

**Proposition 5.1** For any $w \in \Sigma^+$ and a path $P$ of $w$, $A(P)$ is deterministic.

The following proposition is clear since $A$ is finitely ambiguous.

**Proposition 5.2** $S(A)$ is a finite set.

The following theorem holds.

**Theorem 5.3** For any given finitely ambiguous $F$-automaton $A = \langle \Sigma, Q, \delta, S, F, f \rangle$, one can construct a finite set of deterministic $F$-automata $\{A_1, \cdots, A_m\}$ ($m \geq 1$) such that $A \equiv A_1 \cup \cdots \cup A_m$.

[Proof] For each $i \geq 1$, construct the set of deterministic $F$-automata $Y_i$ by $Y_i = \{A(P) \mid$ for some $w \in \Sigma^\ell, P$ is a path of $w\}$ until it holds $Y_k = Y_{k+1}$ for some $k \geq 1$. Such $k$ exists since $S(A)$ is a finite set. Then it holds $S(A) = Y_k$. Let $Y_k = \{A_1, \cdots, A_m\}$. Then it holds $A \equiv A_1 \cup \cdots \cup A_m$. $\square$

From Theorems 4.5 and 5.3, we have the following theorem.

**Theorem 5.4** The equivalence problem of finitely ambiguous $F$-automata is decidable.

**References**


