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<th>Hata-Yamaguti's result on Takagi function and its applications to digital sum problems (Analytic Number Theory)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1996), 961: 73-80</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1996-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60526">http://hdl.handle.net/2433/60526</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Hata-Yamaguti's result on Takagi function
and its applications to digital sum problems

1 Introduction

Let \( n \in \mathbb{N} \) and denote its binary expansion by \( n = \sum_{k \geq 0} \alpha_k(n) 2^k \) with \( \alpha_k(n) \in \{0, 1\} \). We define

\[
s(n) = \sum_{k \geq 0} \alpha_k(n) \quad \text{(the binary digital sum)},
\]

\[
F(\xi, N) = \sum_{n=0}^{N-1} e^{\xi s(n)} \quad \text{(the exponential sum)},
\]

\[
S_p(N) = \sum_{n=0}^{N-1} s(n)^p \quad \text{(the power sum)}
\]

for \( N \in \mathbb{N} \) and \( p, \xi \in \mathbb{R} \). We first review some fundamental results on these sums. If \( N \) is a power of 2, we immediately have \( F(\xi, N) = N^{\log_2(1+e^\xi)} \) and \( S_1(N) = N \log_2 N/2 \). However it is not so easy to obtain explicit formulas for arbitrary \( N \in \mathbb{N} \). In early times the asymptotic behavior of \( S_1(N) \) was studied:

\[
S_1(N) \sim N \frac{\log_2 N}{2} \quad (N \to \infty) \quad \text{(Bush [1])},
\]

\[
S_1(N) = N \frac{\log_2 N}{2} + O(N) \quad (N \to \infty) \quad \text{(Mirsky [8])}.
\]

Finally, Trollope [18] obtained a precise formula for \( S_1(N) \) and Delange [4] gave its elegant proof. Let \( F \) be a nowhere differentiable continuous periodic function of period 1 given by

\[
F(x) = 1 - x - 2^{1-x} T\left(\frac{1}{2^{1-x}}\right), \quad 0 \leq x \leq 1
\]

with \( T \) the Takagi function. Then

\[
S_1(N) = N \frac{\log_2 N}{2} + N \frac{1}{2} F(\log_2 N) \quad \text{(Trollope, Delange)}.
\]

Coquet [3] obtained a precise formula for positive integer powers.

**Theorem 1.1** (Coquet [3]) There are periodic functions \( G_{p,\tau} : \mathbb{R} \to \mathbb{R}, \ 0 \leq \tau \leq p, \) of period 1, such that

\[
S_p(N) = N \left(\frac{\log_2 N}{2}\right)^p + N \sum_{0 \leq \tau < p} (\log_2 N)^\tau G_{p,\tau}(\log_2 N).
\]
for every integer $p \geq 1$. Furthermore $G_{p,\tau}$ verify
\begin{equation}
2^{-\tau} \binom{d}{\tau} + \sum_{\tau < p < d} \binom{d}{p} G_{p,\tau} = 2^{1-d} \binom{d}{\tau} + \sum_{\tau < q < d} 2 \binom{q}{\tau} G_{d,q}
\end{equation}
for $d \geq 2$ and $\tau \leq d-2$.

**Theorem 1.2** (Coquet [3])

\[ S_2(N) = N \left( \frac{\log_2 N}{2} \right)^2 + N \frac{\log_2 N}{2} \left\{ \frac{1}{2} + F(\log_2 N) \right\} + NG(\log_2 N), \]

where $G$ is a nowhere differentiable continuous periodic function of period 1.

An explicit form of the function $G$ is stated in Osbaldestin [13]. However, for $p \geq 3$, we cannot get such an explicit formula via induction formulae (1) and the continuity of $G_{p,\tau}$ is unknown.

Concerning $F(\xi, N)$, Stolarsky [17] proved that $F(\log 2, N)/N^{\log_2 3}$ is not well-behaved asymptotically. Harborth [6] obtained the following estimates:

\begin{equation}
\limsup_{N \to \infty} \frac{F(\log 2, N)}{N^{\log_2 3}} = 1,
\end{equation}

\begin{equation}
0.812556 < \liminf_{N \to \infty} \frac{F(\log 2, N)}{N^{\log_2 3}} < 0.812557.
\end{equation}

We now introduce a function $G_\xi$ by
\[ G_\xi(\log_2 N) = \frac{F(\xi, N)}{N^{\log_2 3(1+\varepsilon)}}. \]

Coquet [3] and Stein [16] investigated the properties of $G_\xi$. Stein proved that $G_\xi$ is a continuous periodic function of period 1 by giving a formula of $F$. However, it is unknown if $G_\xi$ is differentiable. In this note, we get a simple explicit formula of $F(\xi, N)$ by the use of the connection between $s(n)$ and the binomial measure $\mu_r$. And using the results obtained in Hata-Yamaguti [7] and Sekiguchi-Shiota [15], we derive explicit formulas of the power sum $S_p(N)$. We notice that the higher order derivatives of the distribution function of $\mu_r$ with respect to $r$ play an important rule in the explicit formula of $S_p(N)$. The results in this note can be extended to the sum of $q$-adic digits by the use of multinomial measures (see M-O-S-S [9]).

2 Hata-Yamaguti’s result

Let $I = I_{0,0} = [0,1]$ and
\[ I_{n,j} = \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right), \quad j = 0, 1, \ldots, 2^n - 2, \quad I_{n,2^n-1} = \left[ \frac{2^n-1}{2^n}, 1 \right] \]
for $n = 1, 2, 3, \ldots$. 

Define the binomial measure $\mu_r$ ($0 < r < 1$) by a probability measure on $I$ such that

$$\mu_r(I_{n+1,2j}) = r\mu_r(I_{n,j}), \quad \mu_r(I_{n+1,2j+1}) = (1-r)\mu_r(I_{n,j})$$

for $n = 0, 1, 2, \ldots, j = 0, 1, \ldots, 2^n - 1$.

We denote the distribution function of $\mu_r$ by $L$:

$$L(r, x) = \mu_r([0, x])$$

It is well-known that $L(r, \cdot)$ is a strictly increasing continuous and singular function except for $r = 1/2$ (see Salem [14]). It immediately follows that $L(r, \cdot)$ satisfies the system of infinitely many difference equations:

$$\begin{cases}
L(r, \frac{2j+1}{2^{n+1}}) - (1-r)L(r, \frac{j}{2^n}) - rL(r, \frac{2j+1}{2^n}) = 0, \\
L(r, 0) = 0, \quad L(r, 1) = 1, \\
n = 0, 1, 2, \ldots, j = 0, 1, \ldots, 2^n - 1.
\end{cases}$$

This system is equivalent to the following functional equation:

$$L(r, x) = \begin{cases}
rL(r, 2x), & 0 \leq x \leq \frac{1}{2}, \\
(1-r)L(r, 2x - 1) + r, & \frac{1}{2} \leq x \leq 1.
\end{cases}$$

Let

$$R(x) = 1_{I_{1,0}}(x) - 1_{I_{1,1}}(x), \quad 0 \leq x \leq 1,$$

$$\phi(x) = \begin{cases}
2x, & 0 \leq x < 1/2, \\
2x - 1, & 1/2 \leq x \leq 1,
\end{cases}$$

$$\psi(x) = \int_0^x 2R(t)dt, \quad 0 \leq x \leq 1.$$

The Takagi function $T$ is defined by

$$T(x) = \sum_{n=0}^{\infty} 1_{2^{n+1}}(\psi^n(x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi^n(x), \quad 0 \leq x \leq 1.$$  

It is well-known that $T$ is a nowhere differentiable continuous function. And $T$ satisfies the system of infinitely many difference equations:

$$\begin{cases}
T(\frac{2j+1}{2^{n+1}}) - \frac{1}{2}T(\frac{j}{2^n}) - \frac{1}{2}T(\frac{2j+1}{2^n}) = \frac{1}{2^{n+1}}, \\
T(0) = 0, \quad T(1) = 0, \\
n = 0, 1, 2, \ldots, j = 0, 1, \ldots, 2^n - 1.
\end{cases}$$

Hata-Yamaguti [7] have obtained the following formula which connects the Takagi function $T$ with the function $L$.

**Theorem 2.1** (Hata-Yamaguti [7]) We have

$$\frac{1}{2} \frac{\partial}{\partial r}L(r, x) \bigg|_{r=\frac{1}{2}} = T(x).$$
Remark 2.1 We have
\[
\begin{aligned}
\frac{\partial}{\partial r}L(r, \frac{2j+1}{2^{n+1}}) & - (1-r)\frac{\partial}{\partial r}L(r, \frac{1}{2^{n+1}}) = L(r, \frac{2j+1}{2^{n+1}}) - L(r, \frac{1}{2^{n+1}}), \\
\frac{\partial}{\partial r}L(r, 0) & = 0, \quad \frac{\partial}{\partial r}L(r, 1) = 0,
\end{aligned}
\]
\[n = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, 2^n - 1.\]

Above system has a unique continuous solution (Hata-Yamaguti [7], S-S [15]).

More generally we have the following:

Theorem 2.2 (S-S [15]) \(L(r, x)\) is a continuous function valued analytic function of \(r \in I\) and the equality
\[
\left. \frac{\partial^k L(r, x)}{\partial r^k} \right|_{r=2^{n+1}} = k! T_{\frac{1}{2},n}(x)
\]
holds for \(k = 1, 2, 3, \ldots\) Here
\[
T_{\frac{1}{2},1}(x) = 2T(x),
\]
\[
T_{\frac{1}{2},n}(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} R(2^n x) T_{\frac{1}{2},n-1}(2^n(2^n-1)x).
\]

Remark 2.2 It also follows that \(T_{r,k}\) satisfies the system of infinitely many difference equations:
\[
\begin{aligned}
T_{r,k}(\frac{2j+1}{2^{n+1}}) & - (1-r)T_{r,k}(\frac{1}{2^{n+1}}) = T_{r,k-1}(\frac{2j+1}{2^{n+1}}) - T_{r,k-1}(\frac{1}{2^{n+1}}), \\
T_{r,k}(0) & = 0, \quad T_{r,k}(1) = 0,
\end{aligned}
\]
\[n = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, 2^n - 1.\]

3 An explicit formula of exponential sums

We first give a lemma which suggests a close connection between the distribution function \(L\) and digital sums. Set \(t = \log_2 N\) for \(N \in \mathbb{N}\) and denote by \([t]\) its integer part and by \(\{t\}\) its decimal part.

Lemma 3.1 We have
\[
L(r, 2^{1-[t]}) = \sum_{n=0}^{N-1} r^{[t]+1-s(n)(1-r)^s(n)}.
\]

Taking \(r = \frac{1}{1+e^t}\) in (3), we immediately have next theorem.

Theorem 3.1 We have
\[
F(\xi, N) = N^{\log_2(1+e^t)} 2^{(1-[t])}\log_2(1+e^t) L\left( \frac{1}{1+e^t}, \frac{1}{2^{1-[t]}} \right)
\]
for \(\xi \in \mathbb{R}\).

Remark 3.1 By (4), we know that \(G_{\xi}\) is differentiable for almost every \(t \in \mathbb{R}_+\).
4  Asymptotic behavior of $F(\xi, N)/N^{\log_2(1+e^\xi)}$

Theorem 4.1  We have

$$\limsup_{N \to \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \max_{\frac{1}{2} \leq x \leq 1} x^{\log_2 L(r, x)},$$
$$\liminf_{N \to \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \min_{\frac{1}{2} \leq x \leq 1} x^{\log_2 L(r, x)}.$$

Hence, to obtain the precise values on the left-hand sides of these equations, it suffices to estimate the function $g(x) = x^{\log_2 L(r, x)}$. However, it is very hard to get the maximum and the minimum of the function $g$.

Proposition 4.1  we have

$$\limsup_{N \to \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for} \quad \xi > 0,$$
$$\liminf_{N \to \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for} \quad \xi < 0.$$

These estimations are obtained by Stein [16].

Proposition 4.2  For $k = 1, 2, \ldots, 2^n - 1, n = 1, 2, \ldots$, we have

$$g\left(\frac{2k+1}{2^n+1}\right) > \min\{g\left(\frac{4k+1}{2^n+2}\right), g\left(\frac{4k+3}{2^n+2}\right)\}, \quad \text{if} \quad 0 < r < \frac{1}{2},$$
$$g\left(\frac{2k+1}{2^n+1}\right) < \max\{g\left(\frac{4k+1}{2^n+2}\right), g\left(\frac{4k+3}{2^n+2}\right)\}, \quad \text{if} \quad \frac{1}{2} < r < 1.$$

Remark 4.1  The above inequalities are essential in Harborth’s algorithm concerned with the lower bound of the function $F(\log_2 N)/N^{\log_3}$. Harborth’s algorithm is that, by starting with $n_0 = 1$ and $n_{r+1} = 2n_r \pm 1$ where + or - is chosen so that $q_{r+1} = F(\log_2 n_{r+1})/n_{r+1}^{\log_3}$ becomes minimal. Then $\{q_r\}$ is strictly decreasing and $q = \lim_{n \to \infty} q_r < 0.812556 \ldots$ (c.f. (2)). The question whether $q = \lim_{n \to \infty} q_r$ gives a true lower bound is still unknown.

5  From exponential sums to power sums

We set

$$E(r, t) = 2^{1-t} L\left(r, \frac{1}{2^{1-t}}\right), \quad 0 < r < 1, t \in \mathbb{R}.$$

Evidently $E(r, 0) = 2r$, $E(r, 1-) = 1$, $E(\frac{1}{2}, t) = 1$, and $E$ is continuous except for $t \in \mathbb{Z}$ and periodic of period 1 as a function of $t$. Furthermore $E$ is analytic in $r \in (0, 1)$. By use of $E(r, t)$,

$$F(\xi, N) = (1 + e^\xi)^t \left(\frac{1 + e^\xi}{2}\right)^{1-t} E\left(\frac{1}{1 + e^\xi}, t\right).$$
On the other hand, evidently the equality

$$S_k(N) = \frac{\partial^k}{\partial \xi^k} F(\xi, N) \bigg|_{\xi=0}$$

holds for $k = 1, 2, 3, \ldots$ Hence we can directly derive explicit formulas of power sums of lower order from these equations. We set

$$E^{(k)}(\frac{1}{2}, t) = \frac{\partial^k}{\partial r^k} E(r, t) \bigg|_{r=\frac{1}{2}}.$$ 

Then we have

$$S_1(N) = N\left(\frac{t}{2} + \frac{1 - \{t\}}{2} - \frac{1}{4} H_{2,1}(t)\right)$$

(Trollope [18], Delange [4]),

$$S_2(N) = N\left(\left(\frac{t}{2}\right)^2 + H_{2,1}(t)\right)^{\frac{t}{2}} + H_{2,0}(t)$$

(Coquet [3], Osbaldestin [13])

where

$$H_{2,1} = \frac{1}{2} + 1 - \{t\} - \frac{1}{2} E^{(1)}(\frac{1}{2}, t),$$

$$H_{2,0} = \frac{2 - 3\{t\} + \{t\}^2}{4} - \frac{1 - \{t\}}{4} E^{(1)}(\frac{1}{2}, t) + \frac{1}{16} E^{(2)}(\frac{1}{2}, t),$$

$$S_3(N) = N\left(\left(\frac{t}{2}\right)^3 + H_{3,2}(t)\left(\frac{t}{2}\right)^2 + H_{3,1}(t)\right)^{\frac{t}{2}} + H_{3,0}(t)$$

(Grabner, Kirschenhofer, Prodinger and Tichy [5], O-S-S [10])

where

$$H_{3,2}(t) = -\frac{3}{4} E^{(1)}(\frac{1}{2}, t) - \frac{3\{t\} - 6}{2},$$

$$H_{3,1}(t) = \frac{3}{16} E^{(2)}(\frac{1}{2}, t) + \frac{6\{t\} - 9}{8} E^{(1)}(\frac{1}{2}, t) + \frac{3\{t\}^2 - 12\{t\} + 9}{4},$$

$$H_{3,0}(t) = -\frac{1}{64} E^{(3)}(\frac{1}{2}, t) - \frac{3\{t\} - 3}{32} E^{(2)}(\frac{1}{2}, t) - \frac{3\{t\}^2 - 9\{t\} + 4}{16} E^{(1)}(\frac{1}{2}, t) - \frac{\{t\}^3 - 6\{t\}^2 + 9\{t\} - 4}{8},$$

$$S_4(N) = N\left(\left(\frac{t}{2}\right)^4 + H_{4,3}(t)\left(\frac{t}{2}\right)^3 + H_{4,2}(t)\left(\frac{t}{2}\right)^2 + H_{4,1}(t)\right)^{\frac{t}{2}} + H_{4,0}(t)$$

(O-S-S [11])

where

$$H_{4,3}(t) = -E^{(1)}(\frac{1}{2}, t) - 2\{t\} + 5,$$

$$H_{4,2}(t) = \frac{3}{8} E^{(3)}(\frac{1}{2}, t) + \frac{3\{t\} - 6}{2} E^{(1)}(\frac{1}{2}, t) + \frac{6\{t\}^2 - 30\{t\} + 27}{4},$$

$$H_{4,1}(t) = -\frac{1}{16} E^{(3)}(\frac{1}{2}, t) - \frac{6\{t\} - 9}{16} E^{(2)}(\frac{1}{2}, t) - \frac{3\{t\}^2 - 12\{t\} + 7}{4} E^{(1)}(\frac{1}{2}, t) - \frac{2\{t\}^3 - 15\{t\}^2 + 27\{t\} - 13}{4},$$

$$H_{4,0}(t) = \frac{1}{256} E^{(4)}(\frac{1}{2}, t) + \frac{\{t\} - 1}{32} E^{(3)}(\frac{1}{2}, t) + \frac{3\{t\}^2 - 9\{t\} + 2}{32} E^{(2)}(\frac{1}{2}, t) + \frac{\{t\}^3 - 6\{t\}^2 + 7\{t\} - 2}{8} E^{(1)}(\frac{1}{2}, t) + \frac{\{t\}^4 - 10\{t\}^3 + 27\{t\}^2 - 26\{t\} + 8}{16}.$$
We now extend Theorem 1.1 and 1.2 and get a precise formula of $S_p$.

**Theorem 5.1** (O-S-S [10]) We have

$$S_k(N) = N \sum_{p=0}^{k} H_{k,p}(t) \left(\frac{t}{2}\right)^p, \quad k = 0, 1, 2, \ldots$$

Here $H_{k,p}(t)$ is a periodic continuous function of period 1, defined inductively as follows:

$$H_{0,0}(t) = E\left(\frac{1}{2}, t\right) = 1,$$

$$\frac{(-2)^k}{k!} H_{k,0}(t) = \frac{1}{2^k k!} E^{(k)}\left(\frac{1}{2}, t\right) - \sum_{j=0}^{k-1} a(k, j, 1-t) H_{j,0}(t),$$

$$\frac{(-2)^k}{k!} H_{k,p}(t) = -\sum_{j=0}^{k-1} \sum_{p \leq j} \frac{2^q}{q!} a^{(q)}(k, j, 1-t) H_{j,q}(t) - \sum_{0 \leq j < p} \sum_{p \leq j} \frac{2^q}{q!} a^{(q)}(k, j, 1-t) H_{j,q}(t)$$

for $p = 1, 2, \ldots, k$.

where $a(k, j, t)$ is defined by

$$\frac{2^{t-k}}{k!} \frac{\partial^k}{\partial r^k} (r^{t-s}(1-r)^s) \bigg|_{r=\frac{1}{2}} = \sum_j a(k, j, t) s^j,$$

$s \in \mathbb{R}, \ t \in \mathbb{R}, \ k, j \in \mathbb{Z}, \ k \geq 0$, and $a^{(p)}(k, j, t) = \partial^p a(k, j, t) / \partial t^p$.

Furthermore the functions $H_{k,p}(t)$ satisfies the induction formulas:

$$\sum_{j=p}^{k-1} \binom{k}{j} H_{j,p}(t) = 2^{p+1} \sum_{j=p+1}^{k} 2^{-j} \binom{j}{p} H_{k,j}(t)$$

for $k \geq 1, \ 0 \leq p \leq k - 1$.

**References**


