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Some Homotopy Equivalences for Sporadic Groups

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Abstract

This is a report of my recent joint work with Stephen D. Smith, in which some sporadic geometries are shown to be homotopy equivalent to the nontrivial p-subgroup complexes.

1. Motivation.

The last few years have seen particularly vigorous development of mod -p cohomology of sporadic simple groups. As far as I know, structures of mod -p cohomology rings are (almost) determined for the following groups:

For $p=2$, M_{11}, J_1 [AM94a], M_{22} [AM95], M_{23} [Mil93], M_{24} [Mil95], McL [AM94b], $O'N$ [AM94c], $Co3$ [Ben94], M_{12} [BW95], M_{12}, J_2, Ru [Mag95];
For p odd and the sporadic groups with a Sylow p-subgroup isomorphic to the extraspecial group of order p^3 [TY95].

Before determining the ring structure of group cohomology, it is often required to find its additive structure as a graded module. The *alternating-sum formula* provides many informations to obtain the additional structure from those for smaller subgroups. Here is a version of the theorem on the alternating-sum formula by [Web87, Thm A]

Theorem 1 *[Web87, Thm A] Let G denote a finite group acting on a simplicial complex Δ admissibly (i.e., if $g \in G$ fixes a simplex $\sigma \in \Delta$ then g fixes each vertex in σ), and let p be a prime dividing the order $|G|$. Assume that*

$(*)$: For each z of order p in G, the fixed subcomplex $\Delta^z := \{\sigma \in \Delta | \sigma^z = \sigma\}$ is contractible.

Then we have the following expression of the mod-p cohomology of G as an alternating sum over the orbit complex Δ/G of the cohomology of the stabilizers:

$$H^*(G)_p = \sum_{\sigma \in \Delta/G} H^*(G_\sigma)_p.$$

The contractibility condition $(*)$ above was investigated for the sporadic geometries (certain simplicial complexes admitting sporadic groups) in my eariler joint paper [RSY90] with Alex Ryba and Stephen D. Smith, although the expected applications of [RSY90] were in modular representation theory, as the title may suggest: we obtained projective modules via Webb's result [Web87, Thm A'] by verifying $(*)$. But the results of [RSY90] can also be applied to obtain the alternating-sum decomposition above, and this was in fact done in a number of cases—notably the work of Adem and Milgram on M_{22} [AM95, end of Intro.] [AM94a, p. 269] and on McL [AM94b, 1.6].

Indeed it seems that alternating sums over p-local geometries (typically smaller than the standard complex $\mathcal{A}_p(G)$ of all elementary p-groups) arise one way or another in virtually all of the recent work on sporadic cohomology; sometimes under hypotheses different from Webb's, notably in the papers of of Benson-Wilkerson on M_{12} [BW95, 3.1], Benson on $Co3$ [Ben94, 3.3ff], and Maginnis on M_{12}, J_2, Ru [Mag95, Thms 2,3; Ex 2,3].

Motivated mainly by these observations Smith and I started the last year to work with a somewhat unexpected continuation of [RSY90] and had a manuscript forcussing on the alternating-sum decompositions [SY1]. While completing it, we realzed similarity of our arguments to those of Quillen [Qui78, Secs 2,4]. This was furthermore investigated, and we finially realized that in many cases we actually proved homotopy equivalences which are much stronger than just verifying the contractibility condition $(*)$.

To explain this more precisely, let me recall some important results established in [Qui78].

Lemma 2 *([Qui78]) For a finite group G and a prime p dividing $|G|$, the simplicial complex $|\mathcal{A}_p(G)|$ for the poset $\mathcal{A}_p(G)$ of non-trivial elementary abelian p-subgroups of G satisfies the condition $(*)$.*

With Webb's result above, this implies that if we found a small simplicial complex Δ admitting an admissible action of a finite group G, and (G-)homotopic to $\mathcal{A}_p(G)$, then we can easily obtained the additional structure of the mod p cohomology of G. Typical example of such a nice simplicial complex Δ is given by a building for a group G of Lie type:

Lemma 3 *([Qui78]) If G is a finite group of Lie type defined over a field of characteristic p, the simplicial complex $|\mathcal{A}_p(G)|$ is homotopy equivalenet to the building Δ associated with G.*

Three questions naturally arise: For which triples (G, Δ, p) of sporadic groups G, simplicial complexes Δ admitting G (known as *sporadic geometries*) and a prime p dividing $|G|$ we have

(0) the alternating-sum decomposition of cohomology as in Webb's theorem,

(1) the contractibility consdition $(*)$ holds, or

(2) the homotopy equivalence of Δ with $\mathcal{A}_p(G)$ holds.

Clearly the affirmative answer for Question (2) implies those for (1) and then those for (0).

As I already mentioned, Qestion (0) has an affirmative answer for many sporadic geometries. Indeed there are lots of activities concerning this question–notably the current works by Dwyer [Dw96], but I will not discuss on that in this report. Question (1) was analyzed for many sporadic geometries in [RSY90] and [SY1].

In this report I mainly introduce the results on Question (2). The key notion to establish this stronger results is a seemingly-new "closed set" (see Section 3), for use in a standard equivalence method of Quillen, which in most cases we can use to demonstrate the expected homotopy equivalence. The examples of applications of this notion are given both in [SY2] and [Y], and so in this report I did not attempt to repeat them again but just tried to describe the stream of our thoughts.

2. Some (Older) Obsevations.

This section is not related to the latter sections so you may skip to Section 3. Here I just quote some part of the introduction of [SY1] for the readers who we are interested in what we observed at the time we wrote that paper.

The "unexpected" aspect has to do with the precise notion of p-local geometry to be analyzed. The original work of [RSY90] aimed at being comprehensive, in the sense of either proving or disproving Webb's hypothesis for all the then-known p-local geometries of group-theoretic interest. But the recent cohomological applications suggest it may be more natural to expand slightly the original notion of p-local geometry, since this leads to further examples seemingly of cohomological interest. For example in case $p = 2$, the sporadic examples in [RSY90] satisfying Webb's hypothesis were:

$$M_{22}, M_{24}, McL, J_3.$$

But this time, we will also allow certain geometries where one stabilizer might not be a local subgroup; and certain rank-2 geometries (which in general were too numerous to consider before); and we get the following further sporadic examples of Webb's hypothesis:

$$M_{11}, M_{23}, J_1, J_4, Co2, Th.$$

(The cohomology of the first three of these was in fact already known).

For odd p, we also regard it as likely that future work on sporadic-group cohomology will lead to further examples of geometries with Webb's hypothesis, beyond those in [RSY90]; but we will not attempt any "comprehensive" analysis this time round. However, we will at least indicate in this paper how the alternating-sum method could be applied as an alternative approach to some of the groups with extra-special Sylow group of order p^3 considered recently by Tezuka and Yagita [TY95].

By way of additional motivation, we mention here certain coincidences that have emerged from this new work, which suggest that in place of our case-by-case considerations there may be a more uniform geometric approach to sporadic cohomology than is known so far:

Observation 1a. The list for $p = 2$ as extended above agrees almost exactly with the list of sporadic groups satisfying the local-group-theoretic property of *characteristic-2 type* (involution centralizers are 2-constrained, i.e. have no normal odd-order or quasisimple normal subgroups). The actual Lie-type groups in characteristic 2 an alternating-sum decomposition over the Tits building in view of Quillen's standard result [Qui78, Thm 3.1]; and the groups $U_4(3)$ and $G_2(3)$ in [RSY90] also have characteristic 2-type. So the form of the conclusion with our extended notion of geometries suggests that there might be a common approach to these results which uses characteristic-2 type as its hypothesis.

Observation 1b. The reader may detect, as the authors have concluded, that the contractibility *proofs* also seem to follow a common outline. The arguments are to some extent reminiscent of those in the classical work of Quillen [Qui78, Secs 2,4] on the full elementary poset $\mathcal{A}_p(G)$; however in the generally-smaller p-local geometries considered here, contractibility proofs usually seem to require more than the two steps used in conical-contractibility (e.g. [Qui78, 4.4]) arguments. In fact there seems to be a relation with the previous Observation 1a about characteristic-2 type: the geometries appear to satisfy a strong analogue of the Borel-Tits theorem for Lie-type groups—so that the contractibility arguments have some of the flavor of the well-known proof (first due to Bouc?) that in Lie-type groups, the unipotent radicals are precisely the p-group which are the largest normal p-subgroups of their normalizers.

Observation 2a. In a number of cases we deal with subgroups $G_1 \subseteq G_2$ where the corresponding p-local geometries Δ_1, Δ_2 appear to be related, but rather weakly—in particular, not embedded and definitely not homomopy equivalent; however we find they have the same reduced Lefschetz module: $\tilde{L}(\Delta_1) = \tilde{L}(\Delta_2)$. That is, any differences in effect cancel out in the alternating sum. It seems desirable to understand these coincidences as instances of some more general result.

Observation 2b. A different interrelation of geometries can arise, now with respect to a single group G, from one way in which we are now extending the viewpoint of [RSY90]. In that earlier work, stabilizer G_v of vertices v were ordinarily not just maximal as subgroups of G, but also maximal as p-local subgroups. In the present work we find cases (again see discussion of M_{11}) where we can define one geometry Δ with a vertex stabilizer G_v which is a maximal p-local but not a maximal subgroup; and another geometry Δ' where the corresponding vertex stabilizer is an actual maximal subgroup $G_{v'}$ above G_v, but which is of course no longer p-local. Here we typically find that the geometries and even their reduced Lefschetz modules are different—but nonetheless we may find that both $\tilde{L}(\Delta)$ and $\tilde{L}(\Delta')$ are projective; so that for the purpose of alternating sums, we could work over either. Again, it would be good to know a general explanation of insensitivity to this distinction.

3. The Results.

Here I give the homotopy equivalences we verified as well as some observations (compare with those in §2), by quting [SY2, §2].

The results we obtained can be summarized as the table in the next page, wherewe will continue certain notational conventions from [RSY90]. For each row, G will denote a finite group, acting on a geometry (simplicial complex) Δ ([SY2] for the details of each geometry). A particular prime p is also indicated: we take coefficients in the p-adic integers $\mathbf{Z}_p$, and we establish the projectivity of the p-modular representation given by the reduced Lefschetz module $\tilde{L}(\Delta)$ of Δ (namely, the alternating sum of the chain groups). The Table indicates only the corresponding dimension, given by the reduced Euler characteristic $\tilde{\chi}(\Delta)$; it is standard that projectivity forces the p-part $|G|_p$ of the group order to divide that dimension. In the fourth column, we indicate that in most cases we are able to verify the stronger result of homotopy equivalence of Δ with the Quillen elementary complex $\mathcal{A}_p(G)$. In contrast to [RSY90], we do not attempt to decompose the new modules in projective covers of individual irreducibles; but in some cases indicate other relevant remarks. (A + in the first column indicates a new geometry beyond [RSY90]; in the cases for old geometries, all equivalence proofs are new).

We mention the intersection of these results with other work known to us: For the odd-p cases Ru, J_4, Th and $ON (p = 7)$, the group cohomology has been described in Tezuka-Yagita [TY95, 4.1]—though those authors did not require the use of projectivity. The homotopy equivalence for M_{24} was first established in unpublished work of Ronan (mid-1980s).

We conclude with a striking feature of the above Table: First recall Quillen's result [Qui78, Thm 3.1] that the Tits building of a Lie-type group G in characteristic p is homotopy equivalent to $\mathcal{A}_p(G)$; so the corresponding reduced Lefschetz module is projective by [Qui78, Cor 4.3]. Thus it is natural to adjoin the Lie-type groups in characteristic 2 to the $p = 2$ sublist of the above Table of groups with projective modules; the result then agrees almost exactly with the list of simple groups satisfying the local group-theoretic property of *characteristic-2 type* (that is, Involution centralizers are 2-constrained, i.e. have no odd-order or quasisimple normal subgroups). So we wonder if there might be a common approach to these results which actually uses characteristic-2 type as the basic hypothesis.

new?	G	p	$\sim \mathcal{A}_p$?	$\tilde{\chi}(\Delta)$	Remarks
	A_7	2	No	$2^3 7$	
	A_7	2	No	2^6	
	$U_4(3)$	2	$\sim$	$2^7 31$	
	$G_2(3)$	2	No	$2^7 85$	
	$G_2(3)$	2	$\sim$	$2^6 181$	
+	M_{11}	2	=	$2^4 31$	= restriction of $\tilde{L}$ from M_{12}
+	M_{11}	2	No	$2^6 5$	
	M_{22}	2	$\sim$	$2^8 7$	
+	M_{23}	2	$\sim$	$2^7 421$	
+	M_{23}	2	No	$2^{10} 21$	= restriction of $\tilde{L}$ from M_{24}
	M_{24}	2	$\sim$	$2^{10} 21$	
	McL	2	No	$2^8 521$	
	McL	2	$\sim$	$2^7 7483$	
+	J_1	2	$\sim$	$2^3 601$	abelian Sylow-2
	J_3	2	$\sim$	$2^8 511$	$= \mathcal{Z}_2(G)$
+	J_4	2	?Yes?	$2^{21} 2520315$	
+	$Co2$	2	?Yes?	$2^{19} 77$	
+	Th	2	$\sim$	$2^{18} 28729$	
	$L_3(4)$	3	No	$3^2 25$	
	$U_5(2)$	3	$\sim$	$3^6 7$	
	M_{11}	3	No	$3^2 5$	
+	Ru	3	=	$3^3 36281363$	
+	J_4	3	=	$3^3 27892486000017427$	
+	ON	3	$\sim$	$3^4 1755889$	$= \mathcal{B}_3(G)$
	McL	3	No	$3^6 106$	
	McL	3	$\sim$	$3^6 169$	
+	Ly	3	$\sim$	$3^8 80967584$	
	Ly	5	No	$5^6 7065863$	
	Ly	5	$\sim$	$5^6 1769293$	
+	Th	5	=	$5^3 241989183701$	
	ON	7	No	$7^3 162487$	

Projective modules $\tilde{L}(\Delta)$, extended from Table I of [RSY90]

4. Main Methods.

In this section, I quote [SY2, §3], where we provide the details for our homotopy equivalence method, with further discussion of the development from the original projective-module methods of [RSY90]. For the readers who need more accounts or explicit examples, I also refer to my article [Y] written in Japanese.

As above, we always consider a finite group G, acting on geometry (i.e. simplicial complex) Δ; and consider the reduced Lefschetz module $\tilde{L}(\Delta)$ defined by the alternating sum of the chain spaces of Δ—with coefficients taken in the p-adic integers $\mathbf{Z}_p$ for some fixed prime p.

To establish in most cases the homology equivalence, we will recall Quillen's technique of "closed sets in products", as specialized to our present notation. We consider the Cartesian product $\mathcal{A}_p(G) \times \Delta$ of posets, and say a subset $\mathcal{R}$ is *closed* (or an *order-ideal*, in the combinatorial literature) if: whenever $(P, \sigma) \in \mathcal{R}$ with $Q \subseteq P$ and $\tau \subseteq \sigma$, we must also have $(Q, \tau) \in \mathcal{R}$. We also define the fibers of the two projections, namely

$$\mathcal{R}_P = \{\sigma \in \Delta : (P, \sigma) \in \mathcal{R}\}$$
$$\mathcal{R}_\sigma = \{P \in \mathcal{A}_p(G) : (P, \sigma) \in \mathcal{R}\}$$

Then our special case of the result takes the form:

Theorem 4 (Cor 1.8 in Quillen [Qui78])
Suppose $\mathcal{R}$ is closed, and all $\mathcal{R}_P$ and $\mathcal{R}_\sigma$ are contractible.
Then $\mathcal{A}_p(G)$ and Δ are homotopy equivalent.

In fact, it seems that in most applications of this result in the literature, the subset $\mathcal{R}$ has the specific structure of "stabilizing pairs": define $\mathcal{S} = \{(P, \sigma) : P \subseteq G_\sigma\}$. This definition guarantees the property of closure, since in the condition above we see that $Q \subseteq P \subseteq G_\sigma \subseteq G_\tau$, so that also $(Q, \tau) \in \mathcal{S}$. (The final containment assumes that action of G on Δ is *admissible*, namely that G_σ stabilizes all faces of σ. It is standard that we can always obtain this by passing to a barycentric subdivision.) Notice furthermore that for $\mathcal{S}$ the fibers have a very natural interpretation: namely, $\mathcal{S}_P$ is the fixed subcomplex Δ^P, and $\mathcal{S}_\sigma$ is the poset $\mathcal{A}_p(G_\sigma)$. At this point, we observe that the bulk of the geometries in our list are *fully* p-local, in the sense that for all simplices σ we have a non-trivial normal p-subgroup: $O_p(G_\sigma) \neq 1$; so in these cases we get contractibility of $\mathcal{S}_\sigma = \mathcal{A}_p(G_\sigma)$ by Quillen's standard result [Qui78, Prop 2.4]. On the other hand, our arguments verifying $(*)$ only check contractibility of $\mathcal{S}_P = \Delta^P$ for those P which have order exactly P. Now any larger-order $Q \in \mathcal{A}_p(G)$ certainly contains such a P, and contractibility has the consequence that Δ^P is mod-p acyclic; so a standard application of the P. A. Smith theorem (just as in Webb [Web87, p.148]) guarantees that Δ^Q is also mod-p acyclic. So for our fully p-local cases, if we replace "contractible" by "mod-p acyclic" we get the hypotheses of the natural analogue of Quillen's result: where the conlusion "homotopy equivalent" is replaced by "homology equivalent". So in these cases, we know at least that Δ has the same mod-p homology as $\mathcal{A}_p(G)$. And this can be regarded as evidence that the full homotopy equivalence should very probably hold.

Subsequently we realized that the stronger result of homotopy equivalence could be established via Quillen's technique 4, using a (seemingly new) closed set $\mathcal{I}$—which had been implicit in our original contractibility proofs for various Δ^P. Those proofs almost always took the form of a series of applications of a standard homological lemma, which we had stated as [RSY90, Lemma 2.1]: if the link (or *residue*) of a vertex is contractible, then removal of that vertex is a homotopy equivalence. This allowed us to reduce the original Δ^P to the (full) subcomplex on vertices which we called "Res-fixed" in [RSY90]: namely those vertices v for which we have not just $P \subseteq G_v$, but in fact $P \subseteq K_v$—where K_v is the kernel of the action of G_v on the residue of v. Motivated by this observation, we now go on to define for each simplex $\sigma \in \Delta$ the intersection of vertex kernels by:

$$I_\sigma = \cap K_v \text{ over vertices } v \in \sigma ,$$

and a corresponding subset of the Cartesian product by:

$$\mathcal{I} = \{(P, \sigma) : P \subseteq I_\sigma\} .$$

Notice that $\mathcal{I}$ is automatically closed, but for a different reason that $\mathcal{S}$ was: we have $Q \subseteq P \subseteq I_\sigma \subseteq I_\tau$—since the latter intersection is over a subset of the vertices of σ; giving (Q, τ) also in $\mathcal{I}$. And again the fibers of the projections take on appropriate meanings: we have $\mathcal{I}_\sigma = \mathcal{A}_p(I_\sigma)$, while $\mathcal{I}_P$ is just the full subcomplex on vertices Res-fixed by P, which had been prominent in our earlier proofs. As in our earlier discussion of the usual closed set $\mathcal{S}$, the p-local nature of the geometries will typically give $O_p(I_\sigma) > 1$, hence contractibility by [Qui78, Prop 2.4]. So via 4 we obtain a sufficient condition for equivalence:

Proposition 5 *If all $O_p(I_\sigma) > 1$, and all $\mathcal{I}_P$ are contractible, then $\mathcal{A}_p(G)$ and Δ are homotopy equivalent.*

We will use this technique in preference to the methods of [RSY90], whenever it applies (i.e., most of the time).

We make a few general remarks about the application of 5. We can *start* our proofs at the Res-fixed subcomplex $\mathcal{I}_P$, in contrast to [RSY90]—where we had to reduce from Δ^P down to it. This represents a very considerable saving of casework. Furthermore our earlier proofs then included the contractibility of this subcomplex, at least for those P of order exactly p. Partly offsetting the above saving, we do now have the requirement of considering arbitrarily large elementary Q in place of P. However, any such Q contains such a P, and we know immediately that $\mathcal{I}_Q$ is contained in $\mathcal{I}_P$—of known, contractible structure; and typically it is straightforward to get contractibility of the subcomplex $\mathcal{I}_Q$. (Unfortunately it need not be exactly the fixed subcomplex $(\mathcal{I}_P)^Q$, so that P. A. Smith-type approaches are not available).

Finally some caveats: $\mathcal{I}$ as defined seems to be very effective for many sporadic groups; however there is some flexibility in the method—and occasionally it will turn out to be natural to vary the method somewhat, for example by using subgroups still smaller than the kernels K_v. Also note that every elementary Q should be contained in some K_v, otherwise we could get $\mathcal{I}_Q$ empty, hence not contractible. Ordinarily we will check this containment at the start of our proofs; or if it fails, possibly replace $\mathcal{A}_p(G)$ by some larger equivalent poset for which we can check the necessary containment.

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