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On Certain Starlike Functions

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Abstract
Let \( f(z) \) be analytic in \( |z| < 1 \), \( f(0) = f'(0) - 1 = 0 \) and suppose that
\[
1 + \text{Re}(zf''(z)/f'(z)) < 3/2 \quad \text{in} \quad |z| < 1.
\]

Then, R. Singh and S. Singh [Colloquium Mathematicum, 47, 309-314 (1982)] proved that \( f(z) \) is starlike in \( |z| < 1 \).

The authors proved that if \( f(z) \) is analytic in \( |z| < 1 \), \( f(0) = f'(0) - 1 = 0 \) and suppose that
\[
1 + \text{Re}(zf''(z)/f'(z)) < 1 + (\alpha/2) \quad \text{in} \quad |z| < 1
\]
for \( 0 < \alpha \leq 1 \), then we have
\[
|\text{arg}(zf'(z)/f(z))| < (\pi\alpha)/2 \quad \text{in} \quad |z| < 1.
\]

1 Introduction.

Let \( A \) denote the class of functions \( f(z) \) analytic in the open unit disk \( U = \{z : |z| < 1\} \) and normalized so that \( f(0) = f'(0) - 1 = 0 \).

A function \( f(z) \in A \) is called starlike with respect to the origin if
\[
\text{Re}\frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad U.
\]

It is well known that every starlike function is univalent in \( U \).

Ozaki [2] proved that if \( f(z) \in A \) and
\[
1 + \text{Re}\frac{zf''(z)}{f'(z)} < \frac{3}{2} \quad \text{in} \quad U,
\]
then \( f(z) \) is univalent in \( U \).

R. Singh and S. Singh [4, Theorem 6] proved that if \( f(z) \in A \) and satisfies the condition (1), then \( f(z) \) is starlike in \( U \).

In this paper, we need the following lemma.
Lemma 1. Let $f(z) \in A$ and starlike with respect to the origin in $U$. Let $C(r, \theta) = \{f(te^{i\theta}) : 0 \leq t \leq r\}$ and let $T(r, \theta)$ be the total variation of $\arg f(te^{i\theta})$ on $C(r, \theta)$, so that

$$T(r, \theta) = \int_{0}^{r} |\frac{\partial}{\partial t} \arg f(te^{i\theta})| dt.$$ 

Then we have

$$T(r, \theta) < \pi.$$ 

We owe this lemma to Sheil-Small [5, Theorem 1].

2 Main result.

Main Theorem. Let $f(z) \in A$ and

$$1 + \Re \frac{zf''(z)}{f'(z)} < 1 + \frac{\alpha}{2} \quad \text{in} \quad U,$$

where $0 < \alpha \leq 1$.

Then we have

$$|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} \alpha \quad \text{in} \quad U$$

or $f(z)$ is starlike in $U$.

Proof. Let us put

$$\frac{2}{\alpha}(1 + \frac{\alpha}{2} - 1 - \frac{zf''(z)}{f'(z)}) = \frac{zg'(z)}{g(z)}$$

where $g(z) = z + \Sigma_{n=2}^{\infty} b_n z^n$.

From the assumption (2), we have that

$$\Re \frac{zg'(z)}{g(z)} > 0 \quad \text{in} \quad U.$$ 

This shows that $g(z)$ is starlike and univalent in $U$.

From (3) and by an easy calculation ( see e.g. [1] ), we have

$$f'(z) = (\frac{g(z)}{z})^{-\alpha/2}.$$ 

Since $g(z)$ is univalent in $U$, we have that

$$f'(z) \neq 0 \quad \text{in} \quad U.$$
Therefore, we have

\[ \frac{f(z)}{zf'(z)} = \int_{0}^{1} \frac{f'(tz)}{f'(z)} dt = \int_{0}^{1} t^{\alpha/2} \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right)^{-\alpha/2} dt \]

where \( z = re^{i\theta}, 0 \leq \theta < 2\pi \) and \( 0 < r < 1 \).

Since \( g(z) \) is starlike in \( U \), from Lemma 1, we have

\[ -\pi < \arg g(tre^{i\theta}) - \arg g(re^{i\theta}) < \pi \]

for \( 0 < t \leq r \).

Putting

\[ s = t^{\alpha/2} \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right)^{-\alpha/2}, \]

then we have

\[ \arg s = -\frac{\alpha}{2} \arg \left( \frac{g(tre^{i\theta})}{g(re^{i\theta})} \right). \]

From (5) and (6), \( s \) lies in the convex sector

\[ |\arg s| \leq \frac{\pi}{2} \alpha \]

and the same is true of its integral mean of (4), (see e.g. [3, Lemma 1]).

Therefore we have

\[ |\arg \frac{f(z)}{zf'(z)}| < \frac{\pi}{2} \alpha \quad \text{in} \quad U \]

or

\[ |\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} \alpha \quad \text{in} \quad U. \]

This shows that

\[ \text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad U. \]

This completes our proof and this is another proof of [4, Theorem 6].
References


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