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HARDY CLASS OF FUNCTIONS DEFINED BY
SALAGEAN OPERATOR

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ABSTRACT. The object of the present paper is to derive some properties for Hardy class of analytic functions defined by Salagean operator.

1. INTRODUCTION

Let $A$ be the class of functions $f(z)$ of the form

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

that are analytic in the open unit disk $U = \{ z : |z| < 1 \}$.

For $f(z) \in A$, the Salagean operator $D^n$ (cf. [6]) is defined by

\[(1.2) \quad D^0 f(z) = f(z),\]
\[(1.3) \quad D^1 f(z) = D f(z) = zf'(z),\]
\[(1.4) \quad D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots \}).\]

A function $f(z)$ belonging to $A$ is said to be starlike of order $\alpha$ if it satisfies

\[(1.5) \quad \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)\]

for some $\alpha (0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $A$ consisting of functions which are starlike of order $\alpha$ in $U$.

A function $f(z) \in A$ is said to be convex of order $\alpha$ if it satisfies

\[(1.6) \quad \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)\]

for some $\alpha (0 \leq \alpha < 1)$. Also we denote by $K(\alpha)$ the subclass of $A$ consisting of all such functions. Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$ for $0 \leq \alpha < 1$.

Let $H^p (0 < p \leq \infty)$ be the class of all analytic functions in $U$ such that

\[(1.7) \quad ||f||_p = \lim_{r \rightarrow 1-} \{ M_p(r, f) \} < \infty,\]

where

\[(1.8) \quad M_p(r, f) = \left\{ \begin{array}{ll}
\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} & (0 < p < \infty) \\
\max_{\|z\| \leq r} |f(z)| & (p = \infty) 
\end{array} \right. \quad (\text{cf. [1]}).\]

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2. SOME LEMMAS

To discuss our problems for Hardy class $H^p$ of functions, we need the following lemmas.

**Lemma 1 ([7]).** If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta)$, where

\[
\beta = \beta(\alpha) = \begin{cases} 
\frac{1 - 2\alpha}{2(21 - 2\alpha - 1)} & (\alpha \neq \frac{1}{2}) \\
\frac{1}{2\log 2} & (\alpha = \frac{1}{2})
\end{cases}
\]

(2.1)

This result is sharp.

**Lemma 2 ([2]).** If $f(z) \in S^*(\alpha)$ and is not of the form

\[
f(z) = \frac{z}{(1 - ze^{it})^{2(1-\alpha)}},
\]

then there exists $\delta = \delta(f) > 0$ such that $\frac{f(z)}{z} \in H^{\delta + \frac{1}{2(1-\alpha)}}$.

**Lemma 3 ([5]).** If $p(z)$ is analytic in $U$ with $p(0) = 1$ and

\[
\text{Re}(p(z) + zp'(z)) > \frac{1 - 2\log 2}{2(1 - \log 2)} \quad (z \in U),
\]

then $\text{Re}(p(z)) > 0 \ (z \in U)$.

**Remark.** We see that

\[
\frac{1 - 2\log 2}{2(1 - \log 2)} = -0.629\ldots
\]

**Lemma 4 ([1]).** Every analytic function $p(z)$ with positive real part in $U$ is in the class $H^p$ for all $0 < p < 1$.

**Lemma 5 ([4]).** If $f(z) \in A$ satisfies $z^rf(z) \in H^p \ (0 < p < \infty)$ for a real $r$, then $f(z) \in H^p \ (0 < p < \infty)$.

**Lemma 6 ([1]).** If $f'(z) \in H^p$ for some $p \ (0 < p < 1)$, then $f(z) \in H^q \ (q = p/(1 - p))$.

**Lemma 7 ([3]).** Let $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r \ (0 \leq r < 1)$ at a point $z_0$, then we can write

\[
z_0w'(z_0) = kw(z_0),
\]

where $k$ is real and $k \geq 1$. 
3. Hardy Class of Functions

Our first result for Hardy class is contained in

Theorem 1. Let \( f(z) \in A \) satisfy

\[
\text{Re} \left\{ \frac{D^{n+1}f(z)}{D^nf(z)} \right\} > \alpha_0 \quad (z \in U)
\]

for some \( \alpha_0 (0 \leq \alpha_0 < 1) \), and let

\[
\alpha_j = \begin{cases} 
\frac{1-2\alpha_{j-1}}{2(2^{1-2\alpha_{j-1}}-1)} & (\alpha_{j-1} \neq \frac{1}{2}) \\
1 & (\alpha_{j-1} = \frac{1}{2})
\end{cases}
\]

for \( j = 1, 2, \ldots, n \). If \( D^{n-j}f(z) \) is not of the form

\[
D^{n-j}f(z) = \frac{z}{(1-ze^{it})^{2(1-\alpha_j)}}
\]

then there exists \( \delta > 0 \) such that \( D^{n-j}f(z) \in H^{\delta + \frac{1}{2(1-\alpha_j)}} \).

Proof. Note that

\[
D^{n+1}f(z) = D(D^{n}f(z))
\]

\[
= z(D^{n}f(z))'
\]

\[
= z(D^{n-1}f(z))' + z^2(D^{n-1}f(z))''
\]

and

\[
D^{n}f(z) = z(D^{n-1}f(z))'.
\]

This implies that

\[
\text{Re} \left\{ \frac{D^{n+1}f(z)}{D^nf(z)} \right\} = \text{Re} \left\{ 1 + \frac{z(D^{n-1}f(z))''}{(D^{n-1}f(z))'} \right\} > \alpha_0,
\]

so that, \( D^{n-1}f(z) \in K(\alpha_0) \). Therefore, an application of Lemma 1 leads to

\[
D^{n-1}f(z) \in K(\alpha_0) \Rightarrow D^{n-1}f(z) \in S^*(\alpha_1)
\]

\[
\iff D^{n-2}f(z) \in K(\alpha_1)
\]

\[
\iff D^{n-2}f(z) \in S^*(\alpha_2)
\]

\[
\cdots
\]

\[
\iff D^{n-j}f(z) \in K(\alpha_{j-1})
\]

\[
\iff D^{n-j}f(z) \in S^*(\alpha_j).
\]

Further, by using Lemma 2 and Lemma 5, we know that there exists \( \delta > 0 \) such that \( D^{n-j}f(z) \in H^{\delta + \frac{1}{2(1-\alpha_j)}} \).

Taking \( j = n \) in Theorem 1, we have
Corollary 1. Let \( f(z) \in A \) satisfy (3.1) for some \( \alpha_0 (0 \leq \alpha_0 < 1) \), and let
\[
\alpha_n = \begin{cases} 
1 - \frac{2\alpha_{n-1}}{2(2\alpha_{n-1} - 1)} & (\alpha_{n-1} \neq \frac{1}{2}) \\
\frac{1}{2\log 2} & (\alpha_{n-1} = \frac{1}{2})
\end{cases}
\]
If \( f(z) \) is not of the form (3.3), then there exists \( \delta > 0 \) such that \( f(z) \in H^{\delta + \frac{1}{2(1-\alpha_n)}} \).

Next, we derive

Theorem 2. Let \( f(z) \in A \) satisfy
\[
(3.7) \quad \text{Re} \left\{ \frac{D^{n+1}f(z)}{z} \right\} > \frac{1 - 2\log 2}{2(1 - \log 2)} \quad (z \in U).
\]
Then there exists \( p_j (j = 1, 2, \ldots, n+1) \) such that \( D^{n-j+1}f(z) \in H^{p_j} \), where
\[
(3.8) \quad p_k < \frac{1}{j-k+1} \quad (k = 1, 2, \ldots, j).
\]

Proof. Define the function \( p(z) \) by
\[
(3.9) \quad p(z) = \frac{D^n f(z)}{z}.
\]
Then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Since
\[
(3.10) \quad \text{Re} \left\{ \frac{D^{n+1}f(z)}{z} \right\} = \text{Re}(p(z) + zp'(z)) > \frac{1 - 2\log 2}{2(1 - \log 2)},
\]
Lemma 3 gives that
\[
(3.11) \quad \text{Re}(p(z)) = \text{Re} \left\{ \frac{D^n f(z)}{z} \right\} > 0 \quad (z \in U).
\]
Noting that
\[
\frac{D^n f(z)}{z} = (D^{n-1} f(z))',
\]
an application of Lemma 4 implies that \( (D^{n-1} f(z))' \in H^{p_1} \), so by Lemma 6,
\[
D^{n-1} f(z) \in H^{p_2} \quad (p_2 = \frac{p_1}{1 - p_1}).
\]
Further, since \( D^{n-1} f(z) = z(D^{n-2} f(z))' \), using Lemma 5, we obtain \( (D^{n-2} f(z))' \in H^{p_2} \). Taking this process again and again, we conclude that \( D^{n-j+2} f(z) \in H^{p_{j-1}} \) and \( 0 < p_{j-1} < 1/2 \). Thus, finally we have \( D^{n-j+1} f(z) \in H^{p_j} \) \( (0 < p_j < 1) \). This completes the proof of Theorem 2.

Letting \( j = n+1 \) in Theorem 2, we have

Corollary 2. Let \( f(z) \in A \) satisfy (3.7). Then there exists \( p_{n+1} \) such that \( f(z) \in H^{p_{n+1}} \), where
\[
p_k < \frac{1}{n-k+2} \quad (k = 1, 2, \ldots, n+1).
\]
4. HARDY CLASS OF BOUNDED FUNCTIONS

Next our theorem for Hardy class of bounded functions is contained in

**Theorem 3.** Let \( f(z) \in A \) satisfy

\[
(4.1) \quad \left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \right| < \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0} \quad (z \in U)
\]

for some \( \alpha_0 (1/3 \leq \alpha_0 \leq 1/2) \), or

\[
(4.2) \quad \left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 \right| < \frac{\alpha_0 - 2\alpha_0^2 + 1}{2\alpha_0} \quad (z \in U)
\]

for some \( \alpha_0 (1/2 \leq \alpha_0 < 1) \). If \( D^{n-j}f(z) \) is not of the form \( (3.3) \), then there exists \( \delta > 0 \) such that \( D^{n-j}f(z) \in H^{\delta + \frac{1}{\underline{\mathrm{o}}(1-\alpha j)}}(j = 1, 2, \cdots, n)_{\mathrm{Z}} \) where \( \alpha_j \) is given by \( (3.2) \).

**Proof.** Define the function \( w(z) \) by

\[
(4.3) \quad \frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + (1 - 2\alpha_0)w(z)}{1 - w(z)} \quad (w(z) \neq 1).
\]

Then \( w(z) \) is analytic in \( U \) and \( w(0) = 0 \). It follows from \( (4.3) \) that

\[
(4.4) \quad \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 = \left( \frac{w(z)}{1 - w(z)} \right) \left( 2(1 - \alpha_0) + \frac{zw'(z)}{w(z)} + \frac{(1 - 2\alpha_0)(1 - w(z))}{1 + (1 - 2\alpha_0)w(z)} \left( \frac{zw'(z)}{w(z)} \right) \right).
\]

Suppose that there exists a point \( z_0 \in U \) such that

\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).
\]

Then Lemma 7 leads us to \( w(z_0) = e^{i\theta} \) and

\[
z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).
\]

Therefore, we have

\[
(4.5) \quad \left. \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1 \right| = \left| \frac{w(z_0)}{1 - w(z_0)} \right| \left| 2(1 - \alpha_0) + \frac{zw'(z_0)}{w(z_0)} + \frac{(1 - 2\alpha_0)(1 - w(z_0))}{1 + (1 - 2\alpha_0)w(z_0)} \left( \frac{zw'(z_0)}{w(z_0)} \right) \right|
\]

\[
= \left| \frac{e^{i\theta}}{1 - e^{i\theta}} \right| \left| 2(1 - \alpha_0) + k + k \frac{(1 - 2\alpha_0)(1 - e^{i\theta})}{1 + (1 - 2\alpha_0)e^{i\theta}} \right|
\]

\[
\geq \frac{2(1 - \alpha_0) + k}{|1 - e^{i\theta}|} - k \frac{|1 - 2\alpha_0|}{|1 + (1 - 2\alpha_0)e^{i\theta}|}
\]

\[
\geq \frac{2(1 - \alpha_0) + k}{2} - k \frac{|1 - 2\alpha_0|}{2\alpha_0}.
\]
For $1/3 \leq \alpha_0 \leq 1/2$, we have

\[(4.6) \quad \left| \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1 \right| \geq \frac{5\alpha_0 - 2\alpha_0^2 - 1}{2\alpha_0} \]

and for $1/2 \leq \alpha_0 < 1$, we have

\[(4.7) \quad \left| \frac{D^{n+2}f(z_0)}{D^{n+1}f(z_0)} - 1 \right| \geq \frac{\alpha_0 - 2\alpha_0 - 1}{2\alpha_0} \]

Since the above contradicts our conditions (4.1) and (4.2) of the theorem, we conclude that $|w(z)| < 1$ for all $z \in U$. This implies that

\[(4.8) \quad \text{Re} \left\{ \frac{D^{n+1}f(z)}{D^{n}f(z)} \right\} > \alpha_0 \quad (z \in U). \]

Noting that (4.8) is equivalent to $D^n f(z) \in S^*(\alpha_0)$. Using the same manner in the proof of Theorem 1, we conclude that $D^{n-j} f(z) \in S^*(\alpha_j)$. Thus, applying Lemma 2 and Lemma 5, we can prove Theorem 3.

If we put $j = n$ in Theorem 3, then we have

Corollary 3. Let $f(z) \in A$ satisfy the condition (4.1) for some $\alpha_0 (1/3 \leq \alpha_0 \leq 1/2)$ or (4.2) for some $\alpha_0 (1/2 \leq \alpha_0 < 1)$. If $f(z)$ is not of the form (3.3), then there exists $\delta > 0$ such that $f(z) \in H^{\delta + \frac{1}{2(1-\alpha_0)}}$, where $\alpha_n$ is given by (3.2).

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