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ON MACAULAYIFICATION OF LOCAL RINGS
—IN THE CASE OF $\dim \text{non-CM} \leq 2$

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ABSTRACT. Let $X$ be a Noetherian scheme. A birational proper morphism $Y \to X$ is said to be a Macaulayfication of $X$ if $Y$ is a Cohen-Macaulay scheme. In 1978 Faltings constructed a Macaulayfication of $X$ if the dimension of its non-Cohen-Macaulay locus $\text{non-CM}X$ is at most one. Recently the author constructed a Macaulayfication of $X$ in the case of $\text{non-CM}X = 2$. In the present article, we give another proof of them, which still work in general case except for only one lemma.

1. INTRODUCTION

Let $X$ be a Noetherian scheme. A Macaulayfication of $X$ is a birational proper morphism $Y \to X$ such that $Y$ is a Cohen-Macaulay scheme. If $X = \text{Spec } A$ is an affine scheme, then by abuse notation the Macaulayfication $Y \to \text{Spec } A$ is said to be the one of $A$. In 1978, Faltings [4] gave the notion of Macaulayfication and constructed a Macaulayfication of Noetherian local ring $A$ if it possesses a dualizing complex and $\dim \text{non-CM} A \leq 1$. Here $\text{non-CM } A = \{ \mathfrak{p} \in \text{Spec } A \mid A_{\mathfrak{p}} \text{ is not Cohen-Macaulay} \}$ is the non-Cohen-Macaulay locus of $A$, which is closed subset of $\text{Spec } A$ if $A$ possesses a dualizing complex. In the present article, we will construct a Macaulayfication of a Noetherian local ring $A$ in the case of $\dim \text{non-CM} A \leq 2$.

Theorem 1.1 ([9]). Let $A$ be a Noetherian local ring possessing a dualizing complex. If $\text{Ass } A = \text{Assh } A$ and $\dim \text{non-CM } A \leq 2$, then $A$ has a Macaulayfication.

Here $\text{Ass } A$ denotes the set of associated prime ideals of $A$ and $\text{Assh } A = \{ \mathfrak{p} \in \text{Ass } A \mid \dim A/\mathfrak{p} = \dim A \}$.

The notion of Macaulayfication is an analogue of the resolution of singularities. In 1964, Hironaka [8] gave a resolution of singularities of an algebraic variety over a field of characteristic zero. However the general resolution problem is still open even a variety over a field of positive characteristic. On the other hand, Faltings’

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method to construct a Macaulayfication is independent of the characteristic of $A$. In particular, it still works if $A$ is mixed characteristic. Of course, our method is also independent of the characteristic.

In the last section, we give an application of Macaulayfication. A dualizing complex is an important tool of Commutative Algebra and Algebraic Geometry, though we know what rings possesses it not well. It is well-known that a homomorphic image of a Gorenstein local ring possesses a dualizing complex. In 1979, Sharp asked whether its converse is true [14]. Aoyama and Goto [1] gave a partial answer to Sharp’s question by using Faltings’ Macaulayfication. They showed that Sharp’s question is true for a rings with dim non-CM $\leq 1$. Their argument still works in the case of dim non-CM $= 2$. We will show the following theorem.

**Theorem 1.2.** Let $A$ be a Noetherian local ring possessing a dualizing complex. If dim non-CM $A \leq 2$, then $A$ is a homomorphic image of a Gorenstein local ring.

Throughout this article, $A$ denotes a Noetherian local ring with maximal ideal $m$. Assume that $d = \dim A > 0$.

2. A system of parameters

In this section, we state on the p-standard system of parameters, which was introduced by Cuong [2]. First we recall the definition of u.s.d-sequences.

**Definition 2.1 ([7]).** Let $M$ be an $A$-module. A sequence $x_1, \ldots, x_u \in A$ is said to be a $d$-sequence on $M$ if

$$(x_1, \ldots, x_{i-1})M : x_ix_j = (x_1, \ldots, x_{i-1})M : x_j \text{ for any } 1 \leq i \leq j \leq u.$$ 

A sequence $x_1, \ldots, x_u$ is said to be a $u.s.d$-sequence on $M$ if $x_1^{n_1}, \ldots, x_u^{n_u}$ is a $d$-sequence on $M$ for any integers $n_1, \ldots, n_u > 0$ and in any order.

The following definition and lemmas are useful to find a u.s.d-sequence, which were given by Schenzel [12, 13].

**Definition 2.2.** For any finitely generated $A$-module $M$, let $a_i(M)$ be the annihilator of $H_i^a(M)$ and $a(M) = \prod_{i \neq \dim M} a_i(M)$.

**Lemma 2.3.** Let $M$ be a finitely generated $A$-module. If $A$ possesses a dualizing complex, then the following statements are true:

1. For all $i$, $\dim A/a_i(M) \leq i$. In particular, $\dim A/a(M) < \dim M$.
2. Let $p$ be a prime ideal of $A$ such that $\dim A/p = i$. Then $p \in \Ass M$ if and only if $p \in \Ass A/a_i(M)$. In particular, $A = \Ass M$ if and only if $\dim A/a_i(M) < i$ for all $i < \dim M$.
3. If $M$ is equidimensional, then non-CM $M = V(a(M))$. 


Lemma 2.4. Let $M$ be a finitely generated $A$-module and $x_1, \ldots, x_u$ a system of parameters for $M$. Then

$$(x_1, \ldots, x_{i-1})M : x_i \subseteq (x_1, \ldots, x_{i-1})M : a(M) \quad \text{for all } 1 \leq i \leq n.$$ 

The following definition is slightly different from Cuong's one.

Definition 2.5. Let $M$ be a finitely generated $A$-module and $x_1, \ldots, x_u$ is a system of parameters for $M$. We say that $x_1, \ldots, x_u$ is a p-standard system of parameters of type $s$ if

$$\begin{cases} 
  x_{i+1}, \ldots, x_u \in a(M) \\
  x_i \in a(M/(x_{i+1}, \ldots, x_u)M) 
\end{cases}$$

for $i \leq s$. If $A$ possesses a dualizing complex and $s \leq \dim A/a(M)$, then we can take a p-standard system of parameters of type $s$ for $M$ by using (1) of Lemma 2.3.

The following is the main theorem of this section, which was given by Cuong in his unpublished work.

Theorem 2.6. Let $M$ be a finitely generated $A$-module, $x_1, \ldots, x_u$ its p-standard system of parameters of type $s$ and $t \leq u$ a positive integer. Then $x_t^{n_t}, \ldots, x_u^{n_u}$ is a $d$-sequence on $M$ for any integers $n_t, \ldots, n_u > 0$.

Proof. We have to prove that

$$(x_t^{n_t}, \ldots, x_i^{n_i} - 1)M : x_j^{n_j} = (x_t^{n_t}, \ldots, x_i^{n_i} - 1)M : x_j^{n_j}$$

for any $t \leq i \leq j \leq u$. If $j \geq s + 1$, then the both side of (2.6.1) equal to $(x_t^{n_t}, \ldots, x_i^{n_i} - 1)M : a(M)$.

Assume that $j \leq s$ and take an element $a$ of the left hand side of (2.6.1). Then

$$a \in (x_t^{n_t}, \ldots, x_i^{n_i} - 1, x_{j+1}, \ldots, x_d)M : x_j^{n_j} = (x_t^{n_t}, \ldots, x_i^{n_i} - 1, x_{j+1}, \ldots, x_d)M : x_j^{n_j}.$$ 

Thus we have

$$x_j^{n_j} a \in (x_t^{n_t}, \ldots, x_i^{n_i} - 1, x_{j+1}, \ldots, x_d)M \cap (x_t^{n_t}, \ldots, x_i^{n_i} - 1)M : x_t^{n_t}.$$ 

The following lemma assures us that the right hand side of this equation is equal to $(x_t^{n_t}, \ldots, x_i^{n_i} - 1)M$. □

Lemma 2.7. In the same notation as Theorem 2.6,

$$(x_t^{n_t}, \ldots, x_i^{n_i} - 1, x_{j+1}, \ldots, x_u)M \cap (x_t^{n_t}, \ldots, x_i^{n_i} - 1)M : x_i^{n_i} = (x_t^{n_t}, \ldots, x_i^{n_i} - 1)M$$

for all $t \leq i \leq j \leq u$. 

Proof. We work by descending induction on \( j \). If \( j = u \), then there is nothing to prove. Assume that \( j < u \) and let \( a \) be an element of the left hand side of (2.7.1). Then \( a = b + x_{j+1}c \) with \( b \in (x_{r}^{n_{r}}, \ldots, x_{i-1}^{n_{i-1}}, x_{j+2}, \ldots, x_{u})M \) and \( c \in M \). By using Lemma 2.4, we have
\[
c \in (x_{t}^{ni}, \ldots, x_{i-1}^{ni-1}, x_{j+2}, \ldots, x_{u})M : x_{i}^{ni}x_{j+1}
\]
Hence
\[
a \in (x_{t}^{ni}, \ldots, x_{i-1}^{ni-1}, x_{j+2}, \ldots, x_{u})M : x_{i}^{ni} \cap (x_{t}^{ni}, \ldots, x_{i-1}^{ni-1}, x_{j+2}, \ldots, x_{u})M
\]
by induction hypothesis. □

3. The Proof of Theorem 1.1

The main theorem of this section is the following

**Theorem 3.1.** Assume that \( d \geq 2 \) and there is a subsystem of parameters \( x_{1}, \ldots, x_{d} \) for \( A \) satisfying the following two conditions for some integer \( s \geq t - 1 \):

(\#) \( x_{t}^{n_{t}}, \ldots, x_{i}^{n_{i}}, x_{i+1}^{(s+1)}, \ldots, x_{d}^{nd} \) is a \( d \)-sequence on \( A \) for any positive integers \( n_{t}, \ldots, n_{d} \) and for any permutation \( \sigma \) of \( s+1, \ldots, d \).

(\%) \( x_{1}, \ldots, x_{i} \) is a \( d \)-sequence on \( A/(x_{i+1}, \ldots, x_{d}) \) for all \( t \leq i \leq s + 1 \).

We put \( q_{i} = (x_{i}, \ldots, x_{d}) \), \( b_{i} = q_{i} \cdots q_{s+1} \) and \( X_{i} = \text{Proj } A[b_{i}T] \) for \( t \leq i \leq s + 1 \), where \( T \) is an indeterminate.

If \( s - 1 \leq t \leq s + 1 \), then \( \text{depth } O_{X_{t}, p} \geq d - t + 1 \) for all closed point \( p \in X_{t} \).

Theorem 1.1 immediately comes from Theorem 3.1. In fact, if \( d \leq 1 \), then \( A \) itself is Cohen-Macaulay. If \( d \geq 2 \), then \( s = \text{dim } \text{non-CM } A \leq d - 2 \) by (2) and (3) of Lemma 2.3. Let \( x_{1}, \ldots, x_{d} \) be a \( p \)-standard system of parameters of type \( s \) for \( A \). Theorem 2.6 says that \( x_{1}, \ldots, x_{d} \) satisfies (\#) and (\%). Hence \( X_{1} \) is a Cohen-Macaulay scheme.

The rest of this section is devoted to the proof of Theorem 3.1. From now on, we use the notation of Theorem 3.1. Of course, \( x_{i+1}, \ldots, x_{d} \) satisfy (\#) and (\%) as a system of parameters for \( A/x_{i}^{l}A \) for any positive integer \( l \leq s + 1 \). Furthermore, they satisfy (\#) and (\%) as a system of parameters for \( A \). For example, we get
\[
(x_{t+1}, \ldots, x_{i-1}): x_{i}x_{j} = \bigcap_{l}(x_{t}^{l}, x_{t+1}, \ldots, x_{i-1}): x_{i}x_{j}
\]
\[
= \bigcap_{l}(x_{t}^{l}, x_{t+1}, \ldots, x_{i-1}): x_{j}
\]
\[
= (x_{t+1}, \ldots, x_{i-1}): x_{j}
\]
by Krull's intersection theorem.
**Lemma 3.2.** Let $y_0, \ldots, y_u \in A$. If $y_1, \ldots, y_u$ is a $d$-sequence on $A/y_0A$, then

$$ (y_1, \ldots, y_k)(y_1, \ldots, y_u)^n : y_0 = (y_1, \ldots, y_k)[(y_1, \ldots, y_u)^n : y_0] + 0 : y_0 $$

for all $n > 0$ and $1 \leq k \leq u$.

**Proof.** We work by induction on $k$. Let $k = 1$ and $a$ an element of the left hand side of (3.2.1). Then $y_0a = y_1b$ with $b \in (y_1, \ldots, y_u)^n$. By using Theorem 1.3 of [7], $b \in (y_0) : y_1 \cap (y_1, \ldots, y_u)^n \subseteq (y_0)$. If we put $b = y_0a'$, then $a' \in (y_1, \ldots, y_u)^n : y_0$ and $a - y_1a' \in 0 : y_0$. Thus $a$ belongs to the right hand side of (3.2.1).

Assume that $k \geq 2$ and let $a$ be an element of the left hand side of (3.2.1). We put $y_0a = y_kb + b'$ with $b \in (y_1, \ldots, y_u)^n$ and $b' \in (y_1, \ldots, y_{k-1})(y_1, \ldots, y_u)^n$. Then we have

$$ c \in (y_0, y_1, \ldots, y_{k-1}) : y_k \cap [(y_0) + (y_1, \ldots, y_u)^n] $$

$$ = (y_0) + (y_1, \ldots, y_{k-1})(y_1, \ldots, y_u)^{n-1} $$

by using Theorem 1.3 of [7] again. Let

$$ b = y_0a' + c $$

with $c \in (y_1, \ldots, y_{k-1})(y_1, \ldots, y_u)^{n-1}$. Then $a' \in (y_1, \ldots, y_u)^n : y_0$ and

$$ a - y_ka' \in (y_1, \ldots, y_{k-1})(y_1, \ldots, y_u)^n : y_0 $$

$$ = (y_1, \ldots, y_{k-1})[(y_1, \ldots, y_u)^n : y_0] + 0 : y_0 $$

by induction hypothesis. The proof is completed. \(\square\)

The following is a bottle neck of the general Macaulayfication problem.

**Proposition 3.3.** If $i = s$ or $s + 1$, then

$$ q_{i-1}[b_i^n : x_{i-1}^l] \subseteq b_i^n $$

for all $n > 0$ and $l > 0$.

**Proof.** Assume that $i = s + 1$. Then Lemma 3.2 says that

$$ q_{s+1}^n : x_s^l = q_{s+1}^{n-1}[q_{s+1} : x_s^l] + 0 : x_s^l $$

$$ = q_{s+1}^{n-1}[q_{s+1} : x_s^l] + 0 : x_s^l $$

Thus we have the assertion.

Next assume that $i = s$. We prove

$$ b_s^n : x_{s-1}^l = b_{s-1}^{n-1}q_{s+1}[q_s : x_{s-1}] + x_s^n q_{s+1}^{n-1}[q_{s+1} : x_{s-1}] + 0 : x_{s-1} $$
for all $n > 0$ and $l > 0$. Let $a$ be an element of the left hand side of (3.3.1). Then by Lemma 3.2, we have

\[ a \in q_{s}^{2n} : x_{s-1}^{l} \]
\[ = q_{s}^{2n-1}[q_{s} : x_{s-1}] + 0 : x_{s-1} \]
\[ = q_{s}^{n-1}q_{s+1}^{n}[q_{s} : x_{s-1}] + 0 : x_{s-1} + x_{s}^{n}q_{s}^{n-1}[q_{s} : x_{s-1}] . \]

Hence we may assume that $a = x_{s}^{n}a'$ with $a' \in q_{s}^{n-1}[q_{s} : x_{s-1}]$. Then

\[ x_{s-1}^{n}x_{s}^{n}a' \in q_{s+1}^{2n} + \cdots + x_{s}^{n}q_{s+1}^{n+1} + x_{s}^{n}q_{s+1} . \]

We put $x_{s-1}^{l}x_{s}^{n}a' = b + x_{s}^{n}b'$ with $b \in q_{s+1}^{n+1}$ and $b' \in q_{s+1}^{n}$. Since

\[ x_{s}^{l}a' - b' \in q_{s+1}^{n+1} : x_{s}^{n} \cap q_{s} \]
\[ = q_{s+1}^{n}[q_{s+1} : x_{s}] + 0 : x_{s} \cap q_{s} \]
\[ \subset q_{s+1}^{n} . \]

Therefore

\[ a' \in q_{s+1}^{n} : x_{s-1}^{l} = q_{s+1}^{n-1}[q_{s+1} : x_{s-1}] + 0 : x_{s-1} \]

by Lemma 3.2. Thus (3.3.1) is proved and the assertion comes from it.

Next we consider affine charts of $X_i$. We put

\[ c_{i} = \left( x_{s+i+2}^{s+i+2}, \ldots, x_{i}^{s+i+2} \right) \]
\[ + \left( x_{1}^{\alpha_{1}}, x_{2}^{\alpha_{2}-\alpha_{1}}, \ldots, x_{k}^{\alpha_{k}-\alpha_{k-1}}, x_{s}^{s-\alpha_{k}} \right) \mid i \leq \alpha_{1} < \cdots < \alpha_{k-1} \leq s < \alpha_{k} \]

for all $i \leq s + 1$.

Lemma 3.4. The ideal $c_{i}$ is a reduction of $b_{i}$, that is, $b_{i}^{n} = c_{i}b_{i}^{n-1}$ for a sufficiently large $n$.

Proof. We work by descending induction on $i$. If $i = s + 1$, then $b_{s+1} = c_{s+1} = q_{s+1}$. There is nothing to prove.

Assume that $i \leq s$ and $b_{j}^{n} = c_{j}b_{j}^{n-1}$ for all $i < j \leq s + 1$ and for a sufficiently large $n$. Let $k$ be an integer such that $0 \leq k \leq s - i$. Then, since $x_{i+k}^{k+1}c_{i+k+1} \subset c_{i}$, we have

\[ b_{i+k+1}^{k+1} \cdot q_{i+k+1}^{k} - q_{i+k-1}^{k-1} q_{i+k}^{k-1} \cdot q_{i+k+1}^{k} \]
\[ = c_{i+k+1}^{k+1} \cdot q_{i+k+1}^{k} - q_{i+k-1}^{k-1} q_{i+k}^{k-1} \]
\[ + q_{i+k}^{k+1} \cdot q_{i+k-1}^{k-1} q_{i+k}^{k-1} \]
\[ \subset c_{i} b_{i+k+1}^{n-1} + q_{i+k}^{k+1} \cdot q_{i+k+1}^{k} + c_{i} b_{i+k}^{n-1} . \]
Hence
\[
\begin{align*}
\mathbf{b}_i^n &= \mathbf{q}_i^n \mathbf{b}_{i+1}^n \\
&\subseteq \mathbf{q}_i^n \mathbf{b}_{i+1}^n + c_i \mathbf{b}_i^{n-1} \\
&\subseteq \mathbf{q}_{i+1} \mathbf{q}_i^{n-1} \mathbf{b}_{i+2}^n + c_i \mathbf{b}_i^{n-1} \\
&\ldots \\
&\subseteq \mathbf{q}_{i+1} \mathbf{q}_i^{s-i+1} \mathbf{b}_{s+1}^{n-(s-i+1)} + c_i \mathbf{b}_i^{n-1} \\
&= c_i \mathbf{b}_i^{n-1}
\end{align*}
\]

because \((x_{s+1}^{s-i+2}, \ldots, x_d^{s-i+2}) \subseteq c_i\) is a reduction of \(\mathbf{q}_{s+1}\). \(\square\)

Thus \(X_i\) is covered by spectrum of such rings as
\[
A[\mathbf{b}_i/x_{\alpha}^{s-i+2}] = A \left[ \frac{x_{i}}{x_{\alpha}}, \ldots, \frac{x_{d}}{x_{\alpha}} \right]
\]
with \(s+1 \leq \alpha \leq d\) and
\[
A[\mathbf{b}_i/x_{\alpha_1}^{s-i+1} x_{\alpha_2}^{s-s_\alpha_1} \ldots x_{\alpha_k}^{s-s_\alpha_{k-1}+1}]
= A \left[ \frac{x_{i}}{x_{\alpha_1}}, \frac{x_{i}}{x_{\alpha_2}}, \ldots, \frac{x_{i}}{x_{\alpha_k}}, \ldots, \frac{x_{d}}{x_{\alpha_1}}, \ldots, \frac{x_{d}}{x_{\alpha_k}} \right]
\]
with \(i \leq \alpha_1 < \cdots < \alpha_{k-1} \leq s < \alpha_k \leq d\). Assume that \(i > t\). Then it is easy to verify that
\[
\begin{align*}
A[\mathbf{b}_{i-1}/x_{\alpha}^{s-i+3}] &= A[\mathbf{b}_i/x_{\alpha}^{s-i+2}][x_{i-1}/x_{\alpha}], \\
A[\mathbf{b}_{i-1}/x_{i-1} x_{\alpha}^{s-i+2}] &= A[\mathbf{b}_i/x_{\alpha}^{s-i+2}][x_{\alpha}/x_{i-1}], \\
A[\mathbf{b}_{i-1}/x_{\alpha_1}^{s-i} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}] &= A[\mathbf{b}_i/x_{\alpha_1}^{s-i+1} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}][x_{i-1}/x_{\alpha_1}], \\
A[\mathbf{b}_{i-1}/x_{i-1} x_{\alpha_1}^{s-i-1} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}] &= A[\mathbf{b}_i/x_{\alpha_1}^{s-i+1} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}][x_{\alpha_i}/x_{i-1}], \\
\mathbf{q}_i A[\mathbf{b}_i/x_{\alpha}^{s-i+2}] &= x_{\alpha} A[\mathbf{b}_i/x_{\alpha}^{s-i+2}]
\end{align*}
\]
and
\[
\mathbf{q}_i A[\mathbf{b}_i/x_{\alpha_1}^{s-i+1} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}] = x_{\alpha_1} A[\mathbf{b}_i/x_{\alpha_1}^{s-i+1} \ldots x_{\alpha_k}^{s-\alpha_{k-1}+1}].
\]

Therefore

**Corollary 3.5.** The sheaf \(\mathbf{q}_{i-1} \mathcal{O}_{X_i}\) of ideals is locally generated by two elements and \(X_{i-1}\) is the blowing-up of \(X_i\) with respect to \(\mathbf{q}_{i-1} \mathcal{O}_{X_i}\) for all \(t < i \leq s + 1\).
Now we prove Theorem 3.1 by induction on $t$. We may assume that $A/\mathfrak{m}$ is algebraically closed without loss of generality: see the proof of [6, Proposition 3.5].

If $t = s + 1$, then $x_{s+1}, \ldots, x_{d}$ is a u.s.d-sequence on $A$. Let $R = A[\mathfrak{m}T]$ and $\mathfrak{M} = mR + R_{+}$. Then $H_{\mathfrak{M}}^{i}(R)$ is finitely graded for all $i \leq d - s$, that is, the homogeneous component $[H_{\mathfrak{M}}^{i}(R)]_{n}$ is zero for all but finitely many $n$. By using [3, Satz 1], we have depth $\mathcal{O}_{X_{s+1}} \geq d - s$ for all closed point $p \in X_{s+1}$.

Next we assume that $t \leq s$ and let $p$ be a closed point of $X_{t}$. Since the blowing-up $X_{t} \to \text{Spec } A$ is a closed map, we have an expression:

\[ \mathcal{O}_{X_{t},p} = A \left[ \frac{x_{t}}{x_{\alpha_{1}}}, \frac{x_{t+1}}{x_{\alpha_{1}}}, \ldots, \frac{x_{d}}{x_{\alpha_{1}}} \right] \]

(or $\mathcal{O}_{X_{t},p} = A[b_{t}/x_{\alpha_{1}}^{-(t+2)}]$ where $b_{t}$ is a non-zero element of $A$). Assume that $\alpha_{1} > t$ and let $l$ be a positive integer. Let

\[ B = A \left[ \frac{x_{t+1}}{x_{\alpha_{1}}}, \ldots, \frac{x_{d}}{x_{\alpha_{1}}} \right] \]

and $n$ be the maximal ideal of $B$. Since $x_{t+1}, \ldots, x_{d}$ satisfies (\#) and (\%) as a subsystem of parameters for $A$ and for $A/x_{t}A$, the induction hypothesis says that depth $B$, depth $B^{(l)} \geq d - t$.

We compute $H_{n}^{i}B$. Since $q_{l}B$ is generated by $x_{t}$ and $x_{\alpha_{1}}$, which are non-zero divisors on $B$, we have $H_{n}^{2}B = 0$ for $q \neq 1, 2$. Taking direct limit, local cohomology with respect to $x_{\alpha_{1}}$ and localization of a short exact sequence

\[(3.5.1) \quad 0 \to \bigoplus_{n>0} \frac{b_{t+1}^{n} + 0}{b_{t+1}^{n} + 0 : x_{t}} \to \bigoplus_{n>0} \frac{b_{t+1}^{n} + (x_{t})}{b_{t+1}^{n} + (x_{t})} \to 0,
\]

we obtain $H_{n}^{i}A_{x_{t}}H_{x_{\alpha_{1}}}^{p-1}(-) \to H_{n}^{i}A_{x_{t}}H_{x_{\alpha_{1}}}^{p}(-) \to H_{n}^{i}A_{x_{t}}H_{x_{\alpha_{1}}}^{p}(-) \to 0$.

Hence $H_{n}^{2}A_{x_{t}} = \lim_{l,m} B^{(l)}/x_{\alpha_{1}m}B^{(l)}$ because the left term of (3.5.1) is annihilated by $x_{\alpha_{1}}$: see Proposition 3.3. The spectral sequence $E_{2}^{pq} = H_{x_{t}}^{p}H_{x_{\alpha_{1}}}^{q}(-) \Rightarrow H_{n}^{p}H_{x_{t}}^{q}(-)$ induces a short exact sequence

\[ 0 \to H_{x_{t}}^{1}H_{x_{\alpha_{1}}}^{p}(-) \to H_{x_{t}}^{p}H_{x_{\alpha_{1}}}^{q}(-) \to H_{x_{t}}^{0}H_{x_{\alpha_{1}}}^{p}(-) \to 0.
\]

Furthermore, we get

\[ H_{n}^{1}A[b_{t}]_{+}^{1}H_{x_{t}}^{1}A[b_{t}]_{+}^{1} = \bigoplus_{n>0} \frac{b_{t+1}^{n} + 0}{b_{t+1}^{n} + 0 : x_{t}},
\]
from (3.5.1). In fact, $x_{\alpha_{1}}$ is a non-zero divisor on the right term of (3.5.1) because $(x_{1}^{l}:x_{\alpha_{1}} \cap [(x_{1}^{l}+q_{\alpha_{1}})] = (x_{1}^{l})$. Therefore $q_{t}H_{\alpha_{1}}^{1}(B) = 0$.

Consider the spectral sequence $E_{2}^{pq} = H_{n}^{p}H_{\alpha_{1}}^{q}(\cdot) \Rightarrow H_{n}^{p}(\cdot)$. Since depth $B \geq d-t$, $E_{2}^{pq} = H_{n}^{p+1}(B) = 0$ for $p < d-t-1$. Thus

$$H_{n}^{p}H_{\alpha_{1}}^{q}(B) = 0 \quad \text{if } q \neq 1, 2 \text{ or } p < d-t-1$$

and

$$q_{t}H_{\alpha_{1}}^{1}(B) = 0.$$ 

By using this, we compute the depth of

$O_{X_{t},p} = B[x_{t}/x_{\alpha_{1}}](n_{x_{t}/x_{\alpha_{1}}-n_{t}}) \cong \left( \frac{B[U]}{\bigcup_{l>0}(x_{\alpha_{1}}U-x_{t})} \right)_{(n_{U-n_{t}})}$(n_{U-n_{t}}),

where $U$ denotes an indeterminate. Taking local cohomology with respect to $(x_{t}, x_{\alpha_{1}})$ of a short exact sequence

$$0 \rightarrow B[U] \xrightarrow{x_{\alpha_{1}}U-x_{t}} B[U] \rightarrow B[U]/(x_{\alpha_{1}}U-x_{t}) \rightarrow 0,$$

we obtain an exact sequence

$$0 \rightarrow H_{q_{t}}^{1}(B[U]) \rightarrow H_{q_{t}}^{1}(B[U]/(x_{\alpha_{1}}U-x_{t})) \rightarrow H_{q_{t}}^{2}(B[U]) \rightarrow H_{q_{t}}^{2}(B[U]) \rightarrow 0.$$ 

By using an exact sequence

$$0 \rightarrow H_{U-a_{t}}^{1}(B[U]) \rightarrow H_{U-a_{t}}^{1}(B[U]/(x_{\alpha_{1}}U-x_{t})) \rightarrow H_{U-a_{t}}^{2}(B[U]) \rightarrow H_{U-a_{t}}^{2}(B[U]) \rightarrow 0,$$

we get $H_{U-a_{t}}^{p}(B[U]) = 0$ if $q \neq 1, 2$ or $p < d-t$. Hence we obtain

$$H_{U-a_{t}}^{p}(B[U]/(x_{\alpha_{1}}U-x_{t})) = 0 \quad \text{for } p < d-t.$$ 

Taking local cohomology of a short exact sequence

$$0 \rightarrow \bigcup_{l>0}(x_{\alpha_{1}}U-x_{t}) \xrightarrow{x_{\alpha_{1}}U-x_{t}} B[U] \xrightarrow{x_{\alpha_{1}}U-x_{t}} B[x_{t}/x_{\alpha_{1}}] \rightarrow 0,$$

we have

$$H_{q_{t}}^{1}(B[x_{t}/x_{\alpha_{1}}]) \cong H_{q_{t}}^{1}(B[U]/(x_{\alpha_{1}}U-x_{t}))$$

that is,

$$H_{q_{t}}^{p}(B[x_{t}/x_{\alpha_{1}}]) = H_{q_{t}}^{1}(B[U]/(x_{\alpha_{1}}U-x_{t}))$$

for $p < d-t$.

Of course, $H_{q_{t}}^{1}(B[x_{t}/x_{\alpha_{1}}]) = 0$ if $q \neq 1$. The spectral sequence

$$E_{2}^{pq} = H_{n}^{p}(B[x_{t}/x_{\alpha_{1}}]) \Rightarrow H_{n}^{p}(B[x_{t}/x_{\alpha_{1}}])$$

says that depth $O_{X_{t},p} \geq d-t+1$.

If $\alpha_{1} = t$ or $O_{X_{t},p} = A[x_{t}/x_{\alpha_{1}}^{s-t+2}](m_{x_{t}/x_{\alpha_{1}}-n_{t}})$, then we can also show depth $O_{X_{t},p} \geq d-t+1$ in the same way as above. Thus Theorem 3.1 is proved.
4. THE PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 in the same way as [1]. Let A be a Noetherian local ring possessing a dualizing complex and \( s = \dim \text{non-CM} A \).

First we assume Ass A = Assh A. We work by induction on s. If \( s < 0 \), that is, A is Cohen-Macaulay, then the idealization \( A \ltimes K_A \) of the canonical module \( K_A \), which exists because A possesses a dualizing complex, is a Gorenstein ring [11] and A is its homomorphic image.

When \( 0 \leq s \leq 2 \), let \( x_1, \ldots, x_d \) be a p-standard system of parameters of type s for A, \( q_i = (x_{i1}, \ldots, x_{id}) \) and \( b_i = q_i \cdot \ldots \cdot q_{i+1} \) for \( i \leq s + 1 \). We consider \( R = A[b_1^{d-1}T] \) and \( M = m + R_+ \). If \( s = 0 \), then \( R_{\mathfrak{M}} \) is Cohen-Macaulay [7, Theorem 7.11] and A is its homomorphic image. Since \( R_{\mathfrak{M}} \) also possesses a dualizing complex, A is a homomorphic image of a Gorenstein ring.

Assume that \( s > 0 \) and let \( \mathfrak{P} \subset R \) be a prime ideal such that \( \dim R/\mathfrak{P} \geq s \). We show that \( R_{\mathfrak{P}} \) is Cohen-Macaulay, hence \( \dim \text{non-CM} R_{\mathfrak{M}} < s \). Without loss of generality, we may assume that \( \mathfrak{P} \) is homogeneous. If \( \mathfrak{P} \supset R_+ \), then \( R_{\mathfrak{P}} \) is Cohen-Macaulay by Theorem 1.1. If \( \mathfrak{P} \supset R_+ \), then we put \( \mathfrak{P} = \mathfrak{p}R + R_+ \) with \( \mathfrak{p} \in \text{Spec} A \). If \( \mathfrak{p} \supset q_{s+1} \), then \( R_{\mathfrak{p}} = A_{\mathfrak{p}}[T] \) is Cohen-Macaulay. If \( \mathfrak{p} \subset q_{s+1} \), then \( x_{s+1}, \ldots, x_d \) is a system of parameters for \( A_{\mathfrak{p}} \) which forms a u.s.d-sequence on \( A_{\mathfrak{p}} \) because \( \dim A/\mathfrak{p} = \dim R/\mathfrak{P} \geq s \). Hence \( R_{\mathfrak{p}} = A_{\mathfrak{p}}[q_{s+1}q_{s+1}A_{\mathfrak{p}}T] \) is Cohen-Macaulay. By induction hypothesis, we find that \( R_{\mathfrak{M}} \) is a homomorphic image of a Gorenstein ring and A is also.

Next we consider the general case, we work by induction on \( d = \dim A \). If \( d = 0 \), then there is nothing to prove. Assume that \( d > 0 \). Let \( (0) = \mathfrak{t}_1 \cap \cdots \cap \mathfrak{t}_n \) be a primary decomposition of \((0)\) in A. By renumbering \( \mathfrak{t}_i \), we may assume that there is an integer \( l \leq n \) such that \( \dim A/\mathfrak{t}_i = d \) if and only if \( i \leq l \). Let \( \mathfrak{f} = \mathfrak{t}_1 \cap \cdots \cap \mathfrak{t}_l \) and \( \mathfrak{f}' = \mathfrak{t}_{l+1} \cap \cdots \cap \mathfrak{t}_n \).

Let \( \mathfrak{p} \) such that \( \dim A/\mathfrak{p} \geq s \). Then \( A_{\mathfrak{p}} \) is Cohen-Macaulay, hence equidimensional. Therefore \( \mathfrak{p} \supset \mathfrak{f} \) if and only if \( \mathfrak{p} \not\supset \mathfrak{f}' \). This implies that \( \dim \text{non-CM} A/\mathfrak{f}, \dim \text{non-CM} A/\mathfrak{f}' \leq s \). By induction hypothesis and the case of Ass A = Assh A, there are Gorenstein local rings B and \( B' \) such that \( A/\mathfrak{f} \) and \( A/\mathfrak{f}' \) are their homomorphic image, respectively. We may assume that \( \dim B = \dim B' = d \).

Consider A as a subring of \( A/\mathfrak{f} \oplus A/\mathfrak{f}' \). Let C be the inverse image of A by \( B \oplus B' \rightarrow A/\mathfrak{f} \oplus A/\mathfrak{f}' \). Then there exists a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C & \longrightarrow & B \oplus B' & \longrightarrow & B \oplus B'/C & \longrightarrow & 0 \\
& & g & & f & & & & \\
0 & \longrightarrow & A & \longrightarrow & A/\mathfrak{f} \oplus A/\mathfrak{f}' & \longrightarrow & A/\mathfrak{f} + \mathfrak{f}' & \longrightarrow & 0
\end{array}
\]

with exact rows and epimorphisms f and g.
Since $A/f + f'$ is finitely generated over $A$, $B \oplus B'/C$ and $B \oplus B'$ are finitely generated over $C$. Therefore $C$ is a Noetherian local ring by Eakin-Nagata theorem. Since $B \oplus B'/C$ and $B \oplus B'$ are finitely generated over $C$. Therefore $C$ is a Noetherian local ring by Eakin-Nagata theorem. Since $Carrow B \downarrow B' \downarrow Barrow B \oplus B'/C$ is a fiber product, $B$ possesses a dualizing complex: see [5, Lemma 3 and 5] or [10, Corollary 3.7]. Furthermore, dim non-CM $C \leq s$ and $Ass C = Ash C$ because $B \oplus B'$ is a Cohen-Macaulay $C$-module and $\dim A/f + f' \leq s$. Thus $C$ is a homomorphic image of a Gorenstein local ring and $A$ is also.

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