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REPORT ON THE FUNDAMENTAL LEMMA FOR $GL(4)$ AND $GSp(2)$

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Introduction.

Langlands' principle of functoriality [B] conjectures that there is a parametrization of the set $\mathrm{Rep}_F(G)$ of admissible [BZ] or automorphic [BJ] representations of a reductive group G over a local or global field F, by admissible homomorphisms $\rho : W_F \to \hat{G} \rtimes W_F$. Here W_F is a form of the Weil group [T] of F, and $\hat{G}$ is the connected (complex) Langlands dual group [B] of G, on which W_F acts via the absolute galois group of F. If H is another reductive group over F and there is an admissible map $\hat{H} \rtimes W_F \to \hat{G} \rtimes W_F$, then composing with $\rho_H : W_F \to \hat{H} \rtimes W_F$ we get $\rho : W_F \to \hat{G} \rtimes W_F$, and by the functoriality conjecture we would expect a "lifting" map $\mathrm{Rep}_F(H) \to \mathrm{Rep}_F(G)$.

The trace formula has been used to establish the lifting in a few cases. For a test function $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}))$, the convolution operator $r(f)$ maps ϕ in $L^2(G(F)\backslash G(\mathbb{A}))$ to the function whose value at $h \in G(\mathbb{A})$ is $\int_{G(\mathbb{A})} f(g)\phi(hg)dg$. It is an integral operator with kernel $K_f(x,y)$ which has geometric expansion $\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$, and spectral expansion $\sum_\pi \sum_\Phi r(f)\phi(x)\overline{\phi}(y)$. Here π ranges over the set of the irreducible direct summands of L^2 as a module under the action of $G(\mathbb{A})$ by multiplication on the right, and ϕ ranges over an orthonormal basis of smooth vectors. Integrating over $x = y \in G(F)\backslash G(\mathbb{A})$ we obtain the trace formula $\sum_\pi \mathrm{tr}\,\pi(f) = \sum_{G/\sim} \Phi_f(\gamma)$. Here $G/\sim$ denotes the set of conjugacy classes in $G(F)$, and $\Phi_f(\gamma) = \int_{G(\mathbb{A})/Z(\gamma)} f(x\gamma x^{-1})dx$ is an orbital integral of f. In this outline we ignore all questions of convergence, which make the development of the trace formula such a formidable task.

To develop a theory of liftings of representations from the group H to G, one proves a trace formula for a test function f_H on $H(\mathbb{A})$, of the form $\sum_{\pi_H} \mathrm{tr}\,\pi_H(f_H) = \sum_{H/\sim} \Phi_{f_H}(\gamma_H)$. One then compares the geometric sides of the two trace formulae. For this one needs: (1) A notion of a norm map $N : \{G/\sim\} \to \{H/\sim\}$, sending a stable conjugacy class γ in $G(F)$ to γ_H in $H(F)$, locally and globally. In our context, this has been defined by Kottwitz-Shelstad [KS]. (2) A statement of transfer of orbital integrals, asserting that given a test function $f \in C_c^\infty(G(F))$, where F is a local field, there exists a test function f_H, and given f_H there is an f, with "matching orbital integrals", i.e. $\Phi_f(\gamma) = \Phi_{f_H}(N\gamma)$. The global test function f is a product of local functions which are almost all the unit element f^0 of the Hecke algebra of spherical (bi-invariant by a standard maximal compact subgroup K of the local group $G(F)$ (K is hyperspecial, [Ti, 3.9.1]) functions on $G(F)$. Hence one must have also the statement: (3) $\Phi_{f^0}(\gamma) = \Phi_{f_H^0}(N\gamma)$ for all (regular) γ. This statement is called the fundamental lemma. It is a necessary initial point for the comparison to exist.

Further, the admissible map $\hat{H} \rtimes W_F \to \hat{G} \rtimes W_F$ defines a lifting map for unramified representations from $H(F)$ to $G(F)$, and via the Satake transform a dual map from the Hecke algebra of G (locally) to the Hecke algebra of H, and one needs: (4) an extended

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fundamental lemma, relating the orbital integrals of the corresponding spherical functions. Once all this is accomplished, the spectral sides of the trace formulae are equal for sufficiently many corresponding test functions, which are used to isolate individual contributions to the formula, and thus derive the lifting of global and local representations.

The technique of comparison of trace formulae has been applied to lift representations of the multiplicative group of a central simple algebra of degree n, to $GL(n)$. Note that inner forms of G all have the same dual group $\hat{G}$. This is due to Jacquet-Langlands for $n = 2$, Deligne-Kazhdan for all n and local as well as automorphic representations with two supercuspidal components, and [FK2] with "one" rather than "two" such constraints (see [F1] for the special case of a division algebra). However, in this case the two groups under comparison are isomorphic for almost all completions of the global field F, and the fundamental lemma holds automatically.

The next case of such a comparison concerns endoscopy for $G = GL(n, F)$, where $H = GL(m, E)$, E/F is a cyclic field extension of degree n/m. Labesse-Langlands dealt with $n = 2$, Kazhdan [K] with all n and $m = 1$, and Waldspurger [W1] with the general case. The fundamental lemma in this endoscopic case implies the fundamental lemma needed to establish the metaplectic correspondence of [FK1], between $GL(n)$ and any central topological covering group of it. This lifting generalizes Shimura's in the case of $n = 2$. The extended fundamental lemma follows (as in [F2]) from the fundamental lemma of [W1] by means of the (simple) regular functions technique introduced in [FK1], or alternatively by using the spherical functions technique of Clozel.

For a cyclic extension E/F one has the base change lifting from $H(F)$ to $H(E)$. Viewing $H(E)$ as the group of F-points of the F-group $G = \mathrm{Res}_{E/F} H$ obtained by restricting scalars from E to F, the lifting is compatible with the diagonal map of $\hat{H} \rtimes W_F$ to $\hat{G} \rtimes W_F$. Here $\hat{G}$ is a product of $[E : F]$ copies of $\hat{H}$, on which W_F acts via its quotient $\mathrm{Gal}(E/F)$. H. Saito used (in the context of modular forms) the twisted (by a generator σ of the galois group $\mathrm{Gal}(E/F)$) trace formula $\sum \mathrm{tr}\, \pi(f\sigma) = \sum \Phi_f(\gamma\sigma)$, for the convolution operator $r(f\sigma)$. Here the twisted orbital integrals are $\int f(x^{-1}\gamma\sigma(x))dx$. For $n = 2$ the base change lifting for $GL(n)$ has been carried out by Saito, Shintani, Langlands, and for general n by Arthur-Clozel [AC]. The stable fundamental lemma, matching stable orbital integrals and stable twisted ones, has been proven by Kottwitz [Ko] for any G. Regular functions are used in [F3] to give a simple proof of the (unconditional) base change lifting for $GL(2)$, and in [F4] for cusp forms on $GL(n)$ with a supercuspidal component.

Naturally one can consider actions other than that of the Galois group. Twisting by the outer automorphism $\theta(g) = {}^t g^{-1}$ (t for "transpose") of $GL(n)$ would lead to liftings from symplectic and orthogonal groups to $GL(n)$. The first example in this line concerns the symmetric square lifting ([F6]) from $H = SL(2)$ to $G = PGL(3)$, which is associated with the dual group homomorphism embedding $\hat{H} = PGL(2, \mathbb{C}) = SO(3, \mathbb{C}) = \hat{G}^{\hat{\theta}}$ in $\hat{G} = SL(3, \mathbb{C})$. Here $\hat{H} = Z_{\hat{G}}(\hat{\theta})$ is a twisted endoscopic group. More generally, for $n \geq 3$, $\hat{G} = GL(n, \mathbb{C})$, $\theta(g) = J^t g^{-1} J^{-1}$ for some symmetric or anti-symmetric matrix J, since $\hat{H} = Sp(n/2, \mathbb{C})$ or $SO(n, \mathbb{C})$, one expects to obtain liftings from orthogonal or symplectic groups to the general linear group. The purpose of this lecture is to report on a proof of the fundamental lemma in the next case, of $GL(4)$, by means of a new technique, which also provides a more elementary proof in other (known) cases, and a hope for extension.

The orbital integral $\int_G f^0(x^{-1}\gamma x)dx$ is the number of cosets xK in G/K (G is a p-adic group and K denotes a hyperspecial maximal compact subgroup), which are fixed by the action of γ. Since G/K is the Bruhat-Tits building of G, Langlands interpreted the computation of the orbital integral as a problem of counting points on the building. This led to a satisfactory proof of the stable fundamental lemma for base change ([Ko]), and to a counting proof for the symmetric square lifting ([F5, §4]). Langlands and Shelstad then studied the asymptotic expansion of orbital integrals of general (C_c^∞) functions for a general G, and Hales [H] in the context of $Sp(2)$. A recent coherence result of Waldspurger [W2] for the unit element f^0 should lead to a computation of the orbital integral of f^0 too. Our – elementary – approach is entirely different. It involves neither buildings nor germs.

To start with, we note that a useful reduction of the computation of the orbital integral of f^0 at an element k of K is given by Kazhdan's decomposition [K] of k as a commuting product of an absolutely semi-simple element s, and a topologically unipotent element u. The integral is then reduced to that of u, where G and K are replaced by the centralizers of s in these groups. A twisted analogue of this result is developed in [F7], where – taking the group to be the semi direct product of $PGL(3, F)$ and the group generated by the twisting σ – the twisted orbital integrals of f^0 are reduced to orbital integrals on forms of $GL(2)$, which can be directly computed, and compared with the orbital integrals on the "lifted" groups ($SL(2)$ and $PGL(2)$). This reduction is carried out in the context of $GL(4)$ rather than $GL(3)$ in the work reported about below. It permits us to compare the resulting integrals on the group $Sp(2)$ of fixed points of $\sigma(g) = J^t g^{-1} J^{-1}$ on $GL(4)$, with the integrals of f^0 on $GSp(2)$ at the norm of the element u.

The basic idea for the computation of the non twisted orbital integrals comes ¿from the work of Weissauer [We]. Since the orbital integral is an integral over $T\backslash G/K$, where T is the centralizer of our regular element in G, it suffices to find a double coset decomposition for $H\backslash G/K$, for a subgroup H of G which contains T, and then the computation of the orbital integral is reduced to one on the subgroup H, which should be simpler than G. Weissauer [We] proved the fundamental lemma for $GSp(2)$ and its endoscopic group $SO(4)$. We report here on the proof of this lemma from $GL(4)$ to all of its twisted endoscopic groups, especially $GSp(2)$, using this approach. Of course here we consider all tori T of $GSp(2)$, not only those which transfer to its endoscopic group, and compute the norm map.

T. Oda pointed out at the end of my talk that results of Murase and Sugano [MS] on double coset decompositions of the form $H\backslash G/K$ existed for all classical quasi-split groups, and our direct and elementary approach might extend to deal with twisted $GL(n)$ for all n, namely with all symplectic and orthogonal groups. It is easy to obtain such a double coset decomposition in the context of $U(2) \times U(1)\backslash U(2,1)/K$, where U denote unitary groups of a quadratic field extension E/F. I have recently used this to prove the fundamental lemma for $U(2,1)$ and its endoscopic group $U(1,1) \times U(1)$, for a torus T split over E when it is a quadratic unramified extension of F, or over a biquadratic extension of F.

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Let us proceed to describe the fundamental lemma in our case, and steps in its proof.

We simply extract paragraphs from [F8], following its numbering.

Part I. Preparations. A. Statement of Theorem.

Let R denote the ring of integers in a local non archimedean field F. Let $\mathbf{G}$ be the F-group $\mathbf{G}_1 \times \mathbf{G}_m$, where $\mathbf{G}_1 = GL(4)$ and $\mathbf{G}_m = GL(1)$. Put ${}^t g_1$ for the transpose of $g_1 \in \mathbf{G}_1$. Define $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$, $\theta(g_1) = J{}^t g_1^{-1} J^{-1}$, and $\theta(g_1, e) = \big(\theta(g_1), e\|g_1\|\big)$ for $g = (g_1, e) \in \mathbf{G}$; $\|g_1\|$ denotes the determinant of g_1. Put $\mathbf{H} = GSp(2) = GSp(J)$ for the group $\{g_1 \in \mathbf{G}_1; \theta(g_1) = eg_1 \text{ for some } e = e(g_1) \in GL(1)\}$ of symplectic similitudes. We write $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$ for the groups of F-points, and $K = \mathbf{G}(R)$ and $K_H = \mathbf{H}(R)$ for the standard maximal compact subgroups. Similarly we have $G_1, K_1, \dots$.

We choose Haar measures $dg, dh, \dots$ on $G, H, \dots$, and denote by $1_K = 1_{K_G}$ the quotient by the volume $|K|$ of K of the characteristic function of $K = K_G$ in G, by 1_{K_H} the analogous object for K_H, 1_{K_1} for K_1 in G_1, etc. Then 1_K lies in the space $C_c^\infty(G)$ of locally constant compactly supported functions on G. We often omit the subscript of K, when it is clear from the context. Identify $C_c^\infty(G)$ with $C_c^\infty(G\theta)$ by $f(g) = f(g\theta)$, put $\mathrm{Int}(g)(t\theta) = gt\theta g^{-1} = gt\theta(g^{-1})\theta$, and introduce the *orbital integral*

$$\Phi_f^G(t\theta) = \Phi_f^G(t\theta; d_G/d_{Z_G(t\theta)}) = \int_{G/Z_G(t\theta)} f\Big(\big(\mathrm{Int}(g)\big)(t\theta)\Big) dg/d_{Z_G(t\theta)}$$

of $f \in C_c^\infty(G)$ at $t\theta, t \in G$ (it is also called the *θ-orbital integral* of f at t). Here

$$Z_G(t\theta) = \{g \in G; \mathrm{Int}(g)(t\theta) = t\theta\}$$

is the *θ-centralizer* of t in G, or the centralizer of $t\theta$ in G.

The elements t, t' of G are called *stably θ-conjugate* if $t'\theta = \mathrm{Int}(g)(t\theta)$ for some $g \in \mathbf{G}$ $\big(= \mathbf{G}(\overline{F}), \overline{F} = \text{algebraic closure of } F\big)$. There are finitely many θ-conjugacy classes $\big(\mathrm{Int}(g)(t\theta), g \in G\big)$ in a stable θ-conjugacy class, and we define the stable orbital integral $\Phi_f^{G,st}(t\theta)$ of f at $t\theta$ to be the sum $\sum \Phi_f^G(t'\theta)$ over a set of representatives t' for the θ-conjugacy classes within the stable θ-conjugacy class of t (in G). Note that $Z_{\mathbf{G}}(t\theta)$ and $Z_{\mathbf{G}}(t'\theta)$ are isomorphic when t, t' are stably θ-conjugate, this isomorphism is used to relate the measures on these groups. Similarly we have the stable orbital integral $\Phi_f^{H,st}(h; d_H/d_{Z_H(h)})$ of $f \in C_c^\infty(H)$ at $h \in H$.

The purpose of this lecture is to outline steps – mainly involving listing tori, conjugacy classes within stable ones, endoscopic groups, decompositions, norms, but not the computations themselves – in the proof of the following.

Theorem. *For any strongly θ-regular $t \in G$ we have*

$$\Phi_{1_K}^{G,st}(t\theta; d_G/d_{T^\theta}) = \Phi_{1_{K_H}}^{H,st}(Nt; d_H/d'_{T^\theta}).$$

An element t of G is called *θ-semi-simple* if $t\theta$ is semi-simple in the group $G \rtimes \langle\theta\rangle$ (θ is an automorphism of G of order two). Such an element is called *θ-regular* if $Z_{\mathbf{G}}(t\theta)^\circ$, the connected component of the identity in $Z_{\mathbf{G}}(t\theta)$, is a torus. Further it is called *strongly θ-regular* if $Z_{\mathbf{G}}(t\theta)$ is abelian. In this case $Z_{\mathbf{G}}\big(Z_{\mathbf{G}}(t\theta)^\circ\big)$ is a maximal torus $\mathbf{T}$ in $\mathbf{G}$ which is stable under $\mathrm{Int}(t\theta)$, and $Z_{\mathbf{G}}(t\theta) = \mathbf{T}^{\mathrm{Int}(t\theta)}$ (see Kottwitz-Shelstad [KS, 3.3]). According to [KS, Lemma 3.2.A(a)], we may assume that the strongly θ-regular t lies in a θ-stable F-torus $\mathbf{T}$. Thus $t \in T = \theta(T)$.

To define the norm map – which appears in the statement of the Theorem – following [KS] we fix a θ-stable F-pair $(\mathbf{T}^*, \mathbf{B}^*)$ consisting of a minimal θ-stable F-parabolic subgroup $\mathbf{B}^*$ of $\mathbf{G}$, and a maximal θ-stable F-torus $\mathbf{T}^*$ in $\mathbf{B}^*$. Namely we take $\mathbf{B}^*$ to be the upper triangular subgroup of $\mathbf{G}$, and $\mathbf{T}^*$ to be the diagonal subgroup (thus $\mathbf{T}^* = \mathbf{T}_1^* \times \mathbf{G}_m$). Any two θ-stable F-tori $\mathbf{T}^*$ and $\mathbf{T}$ are θ-conjugate in $\mathbf{G}$, thus given $\mathbf{T}$ ($\mathbf{T}^*$ is fixed) there is $h \in \mathbf{G}$ with $\mathbf{T} = h^{-1}\mathbf{T}^*\theta(h)$, and in particular $t^* \in \mathbf{T}^*$ such that $t = h^{-1}t^*\theta(h)$. The norm of t is defined to be the stable conjugacy class in H which is conjugate to Nt^* over $\overline{F}$, where Nt^* is defined as follows.

Put $\mathbf{V} = (1-\theta)\mathbf{T}^*$ and $\mathbf{U} = \mathbf{T}^*_\theta = \mathbf{T}^*/\mathbf{V}$. Here $\mathbf{T}^*$ consists of $(a, b, c, d; e)$ $(= (\mathrm{diag}(a, b, c, d), e))$, and $\theta(a, b, c, d; e) = (d^{-1}, c^{-1}, b^{-1}, a^{-1}; eabcd)$. Then $\mathbf{V}$ consists of $(\alpha, \beta, \beta, \alpha; 1/\alpha\beta)$. Choose the isomorphism $N : \mathbf{U} \tilde{\to} \mathbf{T}^*_H$ given by

$$(x, y, z, t; w) \bmod \{(\alpha, \beta, \beta, \alpha; 1/\alpha\beta)\} \mapsto (xyw, xzw, tyw, tzw; xyztw^2) = (a, b, e/b, e/a; e).$$

It is surjective since $(b, a/b, 1, e/a; 1) \mapsto (a, b, e/b, e/a; e)$. Of course $\mathbf{T}^*_H$ is the diagonal subgroup in $\mathbf{H}$, and any torus $\mathbf{T}_H$ in $\mathbf{H}$ is conjugate to $\mathbf{T}^*_H$ over $\overline{F}$. The stable conjugacy class of a regular element in H is the intersection with H of its conjugacy class over $\overline{F}$. The choice of the isomorphism $\mathbf{U} \tilde{\to} \mathbf{T}^*_H$ is dictated by dual groups considerations, namely that $\mathbf{H}$ is an endoscopic group in $\mathbf{G}$; this we explain in Section F below.

Our explicit computations permit comparing also unstable twisted orbital integrals of 1_K on G with stable orbital integrals on the associated twisted endoscopic groups, as well as reproving Weissauer's transfer of the unstable orbital integrals of 1_K on $GSp(2)$ to its endoscopic group, but this will not be described here.

B. Stable Conjugacy.

Let us recall the structure of the set of (F-rational) conjugacy classes within the stable ($\overline{F}$-) conjugacy class of a regular element t in H. By definition, the centralizer $Z_{\mathbf{H}}(t)$ of t in $\mathbf{H}$ is a maximal F-torus $\mathbf{T}_H$. The elements t, t' of H are *conjugate* if there is g in H with $t' = \mathrm{Int}(g^{-1})t (= g^{-1}tg)$. They are *stably conjugate* if there is such g in $\mathbf{H}(= \mathbf{H}(\overline{F}))$. Then $g_\sigma = g\sigma(g^{-1})$ lies in $\mathbf{T}_H$ for every σ in the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$, and $g \mapsto \{\sigma \mapsto g_\sigma\}$ defines an isomorphism from the set of conjugacy classes within the stable conjugacy class of t to the pointed set $D(T_H/F) = \ker[H^1(F, \mathbf{T}_H) \to H^1(F, \mathbf{H})]$. In our case $H^1(F, \mathbf{H})$ is trivial, hence $D(T_H/F)$ is a group.

1. Lemma. *The set of stable conjugacy classes of F-tori in $\mathbf{H}$ injects naturally in the image in $H^1(F, W)$ of* $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{H})]$, *where* $\mathbf{N} = \mathrm{Norm}(\mathbf{T}^*_H, \mathbf{H})$, *and W is the Weyl group of $\mathbf{T}^*_H$ in $\mathbf{H}$. This map is an isomorphism when $\mathbf{H}$ is quasi-split. Note that the image is $H^1(F, W)$ when $H^1(F, \mathbf{H})$ is trivial, and $H^1(F, W)$ is the group of continuous homomorphisms $\rho : \Gamma \to W$, when Γ acts trivially on W.*

In our case of $\mathbf{H} = GSp(2)$, the Weyl group W is the dihedral group D_4, generated by the reflections $s_1 = (12)(34)$ and $s_2 = (23)$. Its other elements are $1, (12)(34)(23) = (3421)$ (which takes 1 to 2, 2 to 4, 4 to 3, 3 to 1), $(23)(12)(34) = (2431)$, $(23)(3421) = (42)(31)$,

$(3421)^2 = (23)(41)$, $(23)(23)(41) = (41)$. We list the F-tori $\mathbf{T}$ according to the subgroups of W, the split torus corresponding to $\{1\}$, and conclude the following.

2. Lemma. *We have that $H^1(F,\mathbf{T})$ is trivial except when $\rho(\Gamma)$ is the subgroup of W of the form $\langle(14)(23)\rangle$ or $\langle(14)(23),(12)(34),(13)(24)\rangle$, where $H^1(F,\mathbf{T}) = \mathbb{Z}/2$.*

In the proof we note that if $\mathbf{T}_H$ splits over the Galois extension E of F then $H^1(F,\mathbf{T}_H) = H^1\big(\mathrm{Gal}(E/F),\mathbf{T}_H^*(E)\big)$, where $\mathbf{T}_H^*(E) = \{\mathrm{diag}(a,b,\lambda/b,\lambda/a); a,b,\lambda \in E^\times\}$, and $\mathrm{Gal}(E/F)$ acts via ρ. Thus H^1 is the quotient of the group C^1 of cocycles: $a_\tau \in \mathbf{T}_H^*(E)$ with $a_1 = 1$ and $a_{\sigma\tau} = a_\sigma\sigma^*(a_\tau)$ for all $\sigma,\tau \in \mathrm{Gal}(E/F)$, by the group of coboundaries: $c\sigma^*(c^{-1}), c \in \mathbf{T}_H^*(E)$. Here $\sigma^* = \rho(\sigma)\circ\sigma$, thus $\sigma^*(a) = g_\sigma \cdot \sigma a \cdot g_\sigma^{-1}$ if $\rho(\sigma) = \mathrm{Int}(g_\sigma)$. When $\rho(\Gamma) = \{1\}$, the group H^1 is trivial since $E = F$. The other cases are: (1) $\rho(\Gamma) = \langle(23)\rangle, [E:F] = 2$; (2) $\rho(\Gamma) = \langle(12)(34)\rangle, [E:F] = 2$; (3) $\rho(\Gamma) = \langle(13)(24)\rangle, [E:F] = 2$. These tori are not elliptic – their quotient by the center of H is not compact. The elliptic tori are:
(I) $\rho(\Gamma) = \langle(14)(23)\rangle, [E:F] = 2$;
(II) $\rho(\Gamma) = \langle(14)(23),(12)(34),(13)(24)\rangle$, E is the composition of the different quadratic extensions E_1, E_2, E_3 of F, and so $\mathrm{Gal}(E/F) = \mathbb{Z}/2\times\mathbb{Z}/2$ is generated by σ and τ whose fixed fields are $E_3 = E^{\langle\sigma\rangle}$, $E_2 = E^{\langle\sigma\tau\rangle}$, $E_1 = E^{\langle\tau\rangle}$.
(III) $\rho(\Gamma) = \langle(14),(23)\rangle$, again $E = E_1E_2$ and $\mathrm{Gal}(E/F) = \mathbb{Z}/2\times\mathbb{Z}/2$ is generated by σ and τ, with fixed fields $E_3 = E^{\langle\sigma\rangle}, E_2 = E^{\langle\sigma\tau\rangle}$ and $E_1 = E^{\langle\tau\rangle}$, and $\rho(\tau) = (23), \rho(\tau\sigma) = (14)$.
(IV) $\rho(\Gamma)$ contains an element of order 4. There are two cases here. If $\rho(\Gamma) = W$, then the splitting field E is a Galois extension of F with Galois group $W = D_4$. The other case is when $\rho(\Gamma)$ is $\mathbb{Z}/4$, say $\rho(\sigma) = (3421)$. The splitting field E is a cyclic extension of F of degree 4. □

A standard integration formula from the group to a Levi subgroup containing the torus, reduces the study of orbital integrals of regular elements to that of the study in the case of elliptic elements, and their centralizers, the elliptic tori. These are the cases (I – IV).

C. Explicit representatives.

It is important for us to describe a set of representatives for $t \in T_H$ and for their stably conjugate but not conjugate elements.

Example. Case of $SL(2)$. As a preliminary example, let us consider the case of an elliptic torus $\mathbf{T}$ in $\mathbf{G} = SL(2)/F$ which splits over the quadratic extension $E = F(\sqrt{D})$ of F. If $\mathbf{T}^*$ is the diagonal torus, then a representative of such $\mathbf{T}$ is $\mathbf{T} = h_D^{-1}\mathbf{T}^*h_D, h_D = \left(\begin{smallmatrix}1 & \sqrt{D}\\ 1 & -\sqrt{D}\end{smallmatrix}\right)$. Note that $h_D' = \mathrm{diag}(\|h_D\|^{-1},1)h_D$, where $\|h_D\| = \det h_D$, lies in $SL(2,E)$. If σ is the generator of $\mathrm{Gal}(E/F)$, then $\sigma(h_D) = h_D\varepsilon = wh_D, \varepsilon = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), w = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. The elements of $\mathbf{T}$ are $t = h_D^{-1}ah_D (a \in \mathbf{T}^*)$, and we have $\sigma t = h_D^{-1}w\sigma(a)wh_D$, hence the action of σ on $\mathbf{T}$ induces the action $\sigma^*(a) = \mathrm{Int}(w)\big(\sigma(a)\big)$ on $\mathbf{T}^*$.

If $t, t_1 \in G$ are stably conjugate then $t_1 = g^{-1}tg = \sigma g^{-1}\cdot t \cdot \sigma g$, hence $g_\sigma = g\sigma(g)^{-1} = h_D^{-1}a_\sigma h_D$ lies in $\mathbf{T}$ $(= Z_{\mathbf{G}}(t); \sigma t = t$ and $\sigma t_1 = t_1$ since $t, t_1 \in G)$. Now $1 = g_\sigma\sigma(g_\sigma) = \mathrm{Int}(h_D^{-1})\big(a_\sigma w\sigma(a_\sigma)w\big) = a_\sigma\sigma(a_\sigma)^{-1}$, thus $a_\sigma = \mathrm{diag}(R,R^{-1})$ with $R = \sigma R \in F^\times$. Of course the cocycle g_σ or $a_\sigma \in \mathbf{T}^*$, can be modified by $c\sigma^*(c)^{-1} = (\gamma,\gamma^{-1})(\sigma\gamma,\sigma\gamma^{-1})$, hence R ranges over $F^\times/N_{E/F}E^\times$. The relation $g\sigma(g)^{-1} = h_D^{-1}a_\sigma h_D = h_D^{-1}a_\sigma w\sigma(h_D)$ implies

$$h_Dg = a_\sigma w\sigma(h_Dg) = \left(\begin{smallmatrix}x & y\\ z & t\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & R\\ R^{-1} & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}\bar{x} & \bar{y}\\ \bar{z} & \bar{t}\end{smallmatrix}\right) = \left(\begin{smallmatrix}R\bar{z} & R\bar{t}\\ \bar{x}R^{-1} & \bar{y}R^{-1}\end{smallmatrix}\right) = \left(\begin{smallmatrix}R\bar{z} & R\bar{t}\\ z & t\end{smallmatrix}\right)$$

where we wrote $\overline{x}$ for σx. To have g of determinant 1 we note that $1 = ||g|| = -R(\overline{z}t - z\overline{t})/2\sqrt{D}$ has the solution $z = 1$ and $t = -\sqrt{D}/R$. Then

$$g = g_R = \frac{1}{2\sqrt{D}}\begin{pmatrix} \sqrt{D} & \sqrt{D} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} R & \sqrt{D} \\ 1 & -\sqrt{D}/R \end{pmatrix} = \frac{1}{2}\begin{pmatrix} R+1 & (R-1)\sqrt{D} \\ \frac{R-1}{\sqrt{D}} & R+1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix} \in SL(2,E).$$

Moreover,

$$t = \begin{pmatrix} a & bD \\ b & a \end{pmatrix}, \quad t_1 = g^{-1}tg = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} a & bD \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix} = \begin{pmatrix} a & bD/R \\ Rb & a \end{pmatrix}$$

make a complete set of representatives for the conjugacy classes within the stable conjugacy class of $t \in T \subset G$.

We next similarly describe representatives for the elliptic elements in $H = GSp(2,F)$, and for elements stably conjugate but not conjugate to these representatives.

Notation. Write $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right]$ for $\begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & \delta \end{pmatrix}$.

The tori $\mathbf{T}_H$ of $\mathbf{H} = GSp(2)$ of type (I) split over a quadratic extension $E = F(\sqrt{D})$ of F, whose Galois group is generated by σ.

1. Lemma. *A torus $\mathbf{T}_H$ of type* (I) *is given by*

$$\mathbf{T}_H = \widetilde{h}'^{-1}_D \mathbf{T}^*_H \widetilde{h}'_D = \{t = [\mathbf{a}, \mathbf{b}] = \widetilde{h}'^{-1}_D (a, b, \sigma b, \sigma a)\widetilde{h}'_D;$$
$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 D \\ a_2 & a_1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 & b_2 D \\ b_2 & b_1 \end{pmatrix}, ||\mathbf{a}|| = ||\mathbf{b}||\},$$

where $a = a_1 + a_2\sqrt{D}, b = b_1 + b_2\sqrt{D}$, and $\widetilde{h}'_D = [h'_D, h'_D]$. Moreover $t_1 = \mathrm{Int}(\widetilde{g}^{-1})t = \mathrm{Int}\left(\left[I, \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}\right]\right)t, R \in F - N_{E/F}E$, is stably conjugate but not conjugate to t in H, where $\widetilde{g} = [I, g]$, and $g = g_R$ is as described in the example of $SL(2)$ above.

Analogous descriptions apply to tori of the other types.

D. Stable θ-conjugacy.

Similarly, we describe the (F-rational) θ-conjugacy classes within the stable ($\overline{F}$-) θ-conjugacy class of a strongly θ-regular element t in G. Fix a θ-invariant F-torus $\mathbf{T}^*$; in fact we take $\mathbf{T}^*$ to be the diagonal subgroup. The stable θ-conjugacy class of t in G intersects $\mathbf{T}^*$ ([KS, Lemma 3.2.A]). Hence there is $h \in \mathbf{G}$ and $t^* \in \mathbf{T}^*$, such that $t = h^{-1}t^*\theta(h)$. The centralizers are related by $Z_{\mathbf{G}}(t\theta) = h^{-1}Z_{\mathbf{G}}(t^*\theta)h$. Further $Z_{\mathbf{G}}(t^*\theta) = \mathbf{T}^{*\theta}$, the centralizer of $Z_{\mathbf{G}}(t\theta)$ in $\mathbf{G}$ is an F-torus $\mathbf{T}$ which is $\theta_t = \mathrm{Int}(t) \circ \theta$ invariant, and $Z_{\mathbf{G}}(t\theta) = \mathbf{T}^{\theta_t}$. The θ-conjugacy classes within the stable θ-conjugacy class of t can be classified as follows.

(1) Suppose that $t_1 = g^{-1}t\theta(g)$ and t are stably θ-conjugate in G. Then $g_\sigma = g\sigma(g)^{-1} \in Z_G(t\theta) = T^{\theta_t}$. The set $D(F,\theta,t) = \ker[H^1(F, \mathbf{T}^{\theta_t}) \to H^1(F, \mathbf{G})]$ parametrizes, via $(t_1, t) \mapsto \{\sigma \mapsto g_\sigma\}$, the θ-conjugacy classes within the stable θ-conjugacy class of t. The Galois action on $\mathbf{T}, \sigma(t) = \sigma\big(h^{-1}t^*\theta(h)\big) = h^{-1} \cdot h\sigma(h)^{-1} \cdot \sigma(t^*) \cdot \theta\big(\sigma(h)h^{-1}\big)\theta(h)$ induces a Galois action σ^* on $\mathbf{T}^*$, given by $\sigma^*(t^*) = h\sigma(h)^{-1}\sigma(t^*)\theta\big(\sigma(h)h^{-1}\big)$, and $H^1(F, \mathbf{T}^{\theta_t}) = H^1(F, \mathbf{T}^{*\theta})$.

(2) The norm map $N : \mathbf{T}^* \to \mathbf{T}^*_H$ factorizes via the projection $\mathbf{T}^* \to \mathbf{T}^*/\mathbf{V}, \mathbf{V} = (1-\theta)\mathbf{T}^*$, and the isomorphism $\mathbf{U} = \mathbf{T}^*_\theta = \mathbf{T}^*/\mathbf{V} \tilde{\to} \mathbf{T}^*_H$. Suppose that the norm Nt^* of $t^* \in \mathbf{T}^*$ is defined over F. Then for each $\sigma \in \Gamma$ there is $\ell \in \mathbf{T}^*$ such that $\sigma^*(t^*) = \ell t^*\theta(\ell)^{-1}$. Then

$$h^{-1}t^*\theta(h) = t = \sigma(t) = \sigma h^{-1} \cdot \sigma t^* \cdot \theta(\sigma h) = \sigma(h)^{-1}\ell t^*\theta\big(\ell^{-1}\sigma(h)\big),$$

hence
$$t^* = h_\sigma \ell \cdot t^* \cdot \theta(h_\sigma \ell)^{-1}, \;\; h_\sigma = h\sigma(h)^{-1},$$
and $h_\sigma \ell \in Z_{\mathbf{G}}(t^*\theta) = \mathbf{T}^{*\theta}$, so that $h_\sigma \in \mathbf{T}^*$. Moreover, $(1-\theta)(h_\sigma) = t^*\sigma(t^*)^{-1}$. Hence (h_σ, t^*) lies in $H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{T}^*)$, in a subset isomorphic to $H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V})$; this invariant parametrizes the (strongly θ-regular) θ-conjugacy classes which have the same norm (see [KS, Appendix A] (or Section G below) for a definition and properties of these hypercohomology groups; the lines preceding Lemma 6.3.A, for the definition of obs(δ); (6.2), for the definition of inv$'(\delta, \delta')$; and the page prior to Theorem 5.1D, for the definition of inv(δ, δ'): if $t_1 = g^{-1}t\theta(g)$ as in (1) above, then $\mathbf{T}_t = Z_{\mathbf{G}}(Z_{\mathbf{G}}(t\theta)^\circ)$ is a maximal torus in $\mathbf{G}$. Denote its inverse image under the natural homomorphism $\pi : \mathbf{G}_{sc} \to \mathbf{G}$ by $\mathbf{T}_t^{sc}$ ($\mathbf{G}_{sc}$ is the simply connected covering F-group of the derived group of $\mathbf{G}$), and write $g = \pi(g_1)z$, g_1 in $\mathbf{G}_{sc}$, z in $Z(\mathbf{G})$. Then $\sigma(g_1)g_1^{-1}$ lies in $\mathbf{T}_t^{sc}$, $(1-\theta_t)\pi(\sigma(g_1)g_1^{-1}) = \sigma(b)b^{-1}$, where $b = \theta(z)z^{-1} = (1-\theta_t)(z^{-1}) \in \mathbf{V}_t = (1-\theta_t)(\mathbf{T}_t)$. Hence $(\sigma \mapsto \sigma(g_1)g_1^{-1}, b)$ defines the element inv(t, t_1) of $H^1(F, \mathbf{T}_t^{sc} \stackrel{(1-\theta_t)\circ\pi}{\longrightarrow} \mathbf{V}_t)$. It parametrizes the θ-conjugacy classes under G_{sc} within the stable θ-conjugacy class of t. The image in $H^1(F, \mathbf{T}_t \stackrel{1-\theta_t}{\to} \mathbf{V}_t)$, under the map $[\mathbf{T}_t^{sc} \to \mathbf{V}_t] \to [\mathbf{T}_t \to \mathbf{V}_t]$ (induced by $\pi : \mathbf{T}_t^{sc} \to \mathbf{T}_t$), is denoted inv$'(t, t_1)$. It parametrizes the θ-conjugacy classes within the stable θ-conjugacy class of t, as noted in (1) above).

Note that there is an exact sequence
$$H^0(F, \mathbf{T}^*) = \mathbf{T}^{*\Gamma} = T^* \stackrel{1-\theta}{\to} H^0(F, \mathbf{V}) = V \to H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V}) \to H^1(F, \mathbf{T}^*) \stackrel{1-\theta}{\to} H^1(F, \mathbf{V}).$$
Moreover, the exact sequence $1 \to \mathbf{T}^{*\theta} \to \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V} \to 1$ induces the exact sequence
$$H^0(F, \mathbf{T}^*) \stackrel{1-\theta}{\to} H^0(F, \mathbf{V}) \to H^1(F, \mathbf{T}^{*\theta}) \to H^1(F, \mathbf{T}^*) \stackrel{1-\theta}{\to} H^1(F, \mathbf{V}).$$
Hence, $H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V})$ and $D(F, \theta, t)$ is $\ker[H^1(F, \mathbf{T}^{*\theta}) \to H^1(F, \mathbf{G})] \simeq \ker[H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V}) \to H^1(F, \mathbf{G})]$.

In our case the group $H^1(F, \mathbf{G})$ is trivial $(\mathbf{G} = GL(4) \times GL(1))$, and so is $H^1(F, \mathbf{T}^*)$. Hence $D(F, \theta, t) = H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V}) = V/(1-\theta)T^*$. The θ-invariant F-tori $\mathbf{T}$ determine homomorphisms $\rho : \Gamma \to W(\mathbf{T}^{*\theta}, \mathbf{G}^\theta) = W(\mathbf{T}^*, \mathbf{G})^\theta$. We can describe a set of representatives for the F-tori $\mathbf{T}$ in $\mathbf{G}$, and the groups $H^1(F, \mathbf{T}^* \to \mathbf{V}) = H^1(F, \mathbf{T}^{*\theta})$ which parametrize the θ-conjugacy classes within the stable θ-conjugacy classes of strongly θ-regular elements in G, which are represented by elements of T. Since $W(\mathbf{T}^*, \mathbf{G})^\theta = W(\mathbf{T}_H^*, \mathbf{H})$, our list of θ-invariant tori $\mathbf{T}$ is obtained from the list of tori $\mathbf{T}_H$, where $\mathbf{T}$ is the centralizer of $\mathbf{T}_H$.

A useful fact would be that we can choose $h \in \mathbf{G}$ such that $\theta(h) = h$. Then the stable θ-conjugacy classes of strongly θ-regular elements are represented by $t = h^{-1}t^*\theta(h) = h^{-1}t^*h, t^* \in \mathbf{T}^*$, and we also exhibit a complete list of representatives for the θ-conjugacy classes within the stable θ-conjugacy class of such a strongly θ-regular element t.

Then we list the θ-invariant F-tori in $\mathbf{G}$ up to F-isomorphism; they are parametrized by the homomorphisms $\rho : \Gamma \to W = W(\mathbf{T}^{*\theta}, \mathbf{G}^\theta) = W(\mathbf{T}^*, \mathbf{G})^\theta$. Note that $\mathbf{G}^\theta = Sp(2)$. Further we compute $H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \stackrel{1-\theta}{\to} \mathbf{V})$, we give an explicit realization of

$\mathbf{T} = h^{-1}\mathbf{T}^*h$ (and $h = \theta(h)$), and for $t \in T$, strongly θ-regular, a set of representatives in G for the θ-conjugacy classes in the stable θ-conjugacy class of t. Note that the only significant difference from the non twisted case is that we work with $\mathbf{G}^\theta = Sp(2)$ instead of with $\mathbf{H} = GSp(2)$.

Let us clarify that $t \in G$ is strongly θ-regular means that $t = h^{-1}t^*\theta(h), h \in \mathbf{G}$, where t^* is such that $Z_\mathbf{G}(t^*\theta)$ is $\mathbf{T}^{*\theta}$. Then $Z_\mathbf{G}(t\theta) = h^{-1}Z_\mathbf{G}(t^*\theta)h$ is the torus $\mathbf{T}^{\mathrm{Int}(t)\circ\theta}$, where $\mathbf{T}$ is $Z_\mathbf{G}(Z_\mathbf{G}(t\theta))$, an $\mathrm{Int}(t)\circ\theta$-invariant maximal torus in $\mathbf{G}$. If $u = h^{-1}u^*h \in T$, where $u^* \in T^{*\theta}_{reg}$, then $h_\sigma\sigma(u^*)h_\sigma^{-1} = u^* = \theta(u^*) = \theta(h_\sigma)\sigma(u^*)\theta(h_\sigma)^{-1}$ implies that $h_\sigma = h\sigma(h^{-1})$ is a θ-invariant element in the Weyl group $W(\mathbf{T}^*, \mathbf{G})$ of $\mathbf{T}^*$, hence it can be represented by an element of $W = W(\mathbf{T}^{*\theta}, \mathbf{G}^{*\theta})$, and the tori $\mathbf{T}$ in $\mathbf{G}$ so obtained define $\rho : \Gamma \to W$. Hence we consider the centralizers of the tori in $\mathbf{G}^{*\theta}$.

F. Endoscopic groups.

Our Theorem is the "fundamental lemma" for the lifting of representations from $GSp(2)$ to $GL(4)$. It is compatible with a dual group situation, which we proceed to describe.

Let $\mathbf{G}$ be the F-group $\mathbf{G}_1 \times \mathbf{G}_m$, where $\mathbf{G}_1 = GL(4)$ and $\mathbf{G}_m = GL(1)$. Let $\hat{G} = \hat{G}_1 \times \hat{G}_m = GL(4,\mathbb{C}) \times GL(1,\mathbb{C})$ be its connected dual group. Put $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$, and $\hat{\theta}(g_1) = \theta(g_1) = J\,{}^tg_1^{-1}J^{-1}$ for $g_1 \in \mathbf{G}_1$, where ${}^tg_1$ is the transpose of g_1. For $g = (g_1, t)$ in $\hat{G}$, write $\hat{\theta}(g) = \hat{\theta}(g_1, t) = \big(t\theta(g_1), t\big)$. This is an automorphism of $\hat{G}$ of order 2. We often attach a subscript 1 to indicate the $GL(4)$-component of an object in $\mathbf{G} = GL(4) \times GL(1)$, and sometimes abuse notations and ignore the $GL(1)$-component.

Denote by $\hat{T}$ the diagonal subgroup in $\hat{G}$ (thus $\hat{T} = \hat{T}_1 \times \mathbb{C}^\times$), and by $\mathbf{T}^*$ the diagonal subgroup of $\mathbf{G}$. Let $\hat{B}$ and $\mathbf{B}$ be the upper triangular subgroups in $\hat{G}$ and $\mathbf{G}$. Then the group $X_*(\hat{T}) = \mathrm{Hom}(\mathbf{G}_m, \hat{T}) = \{(a,b,c,d;e)\}$ is isomorphic to $X^*(\mathbf{T}^*) = \mathrm{Hom}(\mathbf{T}^*, \mathbf{G}_m)$, and $X^*(\hat{T}) = \{(x,y,z,t;u)\} = X_*(\mathbf{T}^*)$. The automorphism $\hat{\theta}$ induces an automorphism θ on $\mathbf{G}$ (fixing $\mathbf{B}$), given on $\mathbf{T}^*$ as follows.

$$\begin{aligned}\big(\theta(x,y,z,t;u)\big)(a,b,c,d;e) &= (x,y,z,t;u)\big(\hat{\theta}(a,b,c,d;e)\big)\\ &= (x,y,z,t;u)(e/d, e/c, e/b, e/a; e) = a^{-t}b^{-z}c^{-y}d^{-x}e^{x+y+z+t+u}\\ &= (-t,-z,-y,-x,x+y+z+t+u)(a,b,c,d;e).\end{aligned}$$

Then for $(g,t) \in \mathbf{G}, \theta(g,t) = \big(\theta(g), t\|g\|\big)$, where $\|g\|$ denotes the determinant of g.

We are concerned with lifting of representations and transfer of orbital integrals between $\mathbf{G}$ and its endoscopic groups, in fact its twisted (by θ) such groups. The twisted endoscopic groups of $(\hat{G}, \hat{\theta})$ are determined by $\hat{H} = Z_{\hat{G}}(\hat{s}\hat{\theta})^0$ (superscript zero for "connected component of the identity"), where this centralizer is

$$Z_{\hat{G}}(\hat{s}\hat{\theta}) = \{(x,t) \in \hat{G}; x\hat{s}\theta(x)^{-1} = t\hat{s}\} \subset Z_{GL(4,\mathbb{C})}\big(\hat{s}\hat{\theta}(\hat{s})\big) \times GL(1,\mathbb{C}),$$

and by a Galois action $\rho : \Gamma = \mathrm{Gal}(\overline{F}/F) \to Z_{\hat{G}}(\hat{s}\hat{\theta})$. Here $\hat{s}$ is a semi-simple element in $\hat{G}$ (which can and will be taken to be $\hat{s} = (\hat{s}_1, 1)$), which can and will be taken to be diagonal, chosen up to $\hat{\theta}$-conjugacy, namely $\hat{T} \ni \hat{s} \equiv g\hat{s}\hat{\theta}(g^{-1})$. Using a diagonal g we conclude that $\hat{s} = \mathrm{diag}(1,1,c,d)$. Taking g to be a representative in $\hat{G}$ of the reflections $(23), (14), (12)(34)$

in the Weyl group of $\hat{G}$ (these elements are fixed by $\hat{\theta}$), we conclude that the $\hat{\theta}$-conjugacy class of $\hat{s}$ does not change if c is replaced by c^{-1}, d by d^{-1}, and (c,d) by (d,c). Let us list the possibilities. Recall ([KS, 2.1]) that an endoscopic group $\mathbf{H}$ is called *elliptic* if $(Z(\hat{H})^{\Gamma})^0$ is contained in the center $Z(\hat{G})$ of $\hat{G}$.

A list of the twisted endoscopic groups of $(\hat{G},\hat{\theta})$ is as follows.

1. $\hat{s} = I$, $Z_{\hat{G}}(\hat{\theta}) = GSp(2,\mathbb{C})$ is connected, hence equal to $\hat{H}$, the Galois action is trivial, and the endoscopic group is $\mathbf{H} = GSp(2)$ over F. Since $Z(\hat{H}) = \mathbb{C}^\times = Z(\hat{G})$, $\mathbf{H}$ is elliptic.

An endoscopic group $\mathbf{C}$ of $\mathbf{H}$ is determined by a semi-simple (diagonal, up to conjugacy) element s in $\hat{H}$. The only proper elliptic endoscopic group of $\mathbf{H}$ is determined by $s = \mathrm{diag}(1,-1,-1,1)$, whose centralizer in $\hat{H}$ is $\hat{C}_0 = \begin{pmatrix} \bullet & 0 & 0 & \bullet \\ 0 & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0 \\ \bullet & 0 & 0 & \bullet \end{pmatrix} = \{(a,b) \in GL(2,\mathbb{C})^2;\ \det a = \det b\}$. Note that the connected component of $Z(\hat{C}_0) = \langle Z(\hat{H}), s\rangle$ is $Z(\hat{H})$, so that $\mathbf{C}_0$ is elliptic. Also, $X_*(\hat{T}_0) = \{(a,b,c,d); a+d = b+c\} = X^*(T_0^*)$ has dual $X_*(T_0^*) = X^*(\hat{T}_0) = \{(x,y,z,t)/(u,-u,-u,u)\}$, hence $\mathbf{C}_0 = GL(2) \times GL(2)/GL(1)$, where $GL(1)$ embeds via $u \mapsto (u,u^{-1})$.

The dual group of $\mathbf{H}_0 = Sp(2)$ is $\hat{H}_0 = PGSp(2,\mathbb{C})$. Its proper elliptic endoscopic groups are obtained as follows. (i) The centralizer of $s = \mathrm{diag}(1,-1,-1,1)$ in $\hat{H}_0$ is generated by the reflection $\mathrm{diag}(w,w)$ and its connected component $\hat{C}_0/\hat{Z} = (GL(2,\mathbb{C}) \times GL(2,\mathbb{C}))'/\mathbb{C}^\times$, the prime indicates equal determinants. The corresponding endoscopic group is $(GL(2) \times GL(2))'/GL(1)$, unless there is a quadratic extension E/F whose galois group permutes the two factors, in which case $\mathrm{Res}_{E/F}\, GL(2)'/GL(1)$ is obtained (its group of F-points is $GL(2,E)'/F^\times$, where the prime indicates here determinant in $F^\times$. (ii) The centralizer of $s_1 = \mathrm{diag}(1,1,-1,-1)$ in $\hat{H}_0$ is generated by $\left(\begin{smallmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{smallmatrix}\right)$ (where $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) and its connected component $\hat{C}_1^0 = \{\mathrm{diag}(x, \lambda\varepsilon x \varepsilon);\ x \in PGL(2,\mathbb{C}),\ \lambda \in \mathbb{C}^\times\}$. The endoscopic group is elliptic only when there is a quadratic extension E/F such that $\mathrm{Gal}(E/F)$ acts via $\mathrm{Int}\left(\begin{smallmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{smallmatrix}\right)$ on this connected component, thus by $\sigma(x,\lambda) = (x,\lambda^{-1})$ on $(x,\lambda) \in PGL(2,\mathbb{C}) \times \mathbb{C}^\times$, and then the endoscopic group is $SL(2) \times U(1,E/F)$, where $U(1,E/F)$ is the unitary group with F-points $E^1 = \{x \in E^\times; x\overline{x} = 1\}$.

2. $\hat{s} = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, $Z_{\hat{G}}(\hat{s}\hat{\theta}) = GO(\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right);\mathbb{C})$ is $\{(x,t) \in \hat{G};\ x\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right){}^t x \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right) = t\}$, or

$$\langle (A, B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = ((\begin{smallmatrix} aA & bA\varepsilon \\ c\varepsilon A & d\varepsilon A\varepsilon \end{smallmatrix}), \|AB\|), (\mathrm{diag}(1,w,1),1)\rangle = \langle ((\begin{smallmatrix} A & 0 \\ 0 & t\varepsilon A\varepsilon \end{smallmatrix}), t\|A\|), (\mathrm{diag}(1,w,1),1)\rangle$$

has connected component $\hat{C} = GL(2,\mathbb{C})^2/\mathbb{C}^\times$, with $\mathbb{C}^\times$ embedding via $z \mapsto (z,z^{-1})$. Note that $Z(\hat{C}) = \mathbb{C}^\times$ is $Z(\hat{G})$, hence $\mathbf{C}$ is elliptic. Now

$$X^*(T_C^*) = X_*(\hat{T}_C) = \{(a,b;c,d)/(u,u;u^{-1},u^{-1})\}$$

has dual $X_*(T_C^*) = X^*(\hat{T}_C) = \{(x,y;z,t); x+y = z+t\}$, thus $\mathbf{C} = \big(GL(2) \times GL(2)\big)'$, where the prime means the subgroup of (A,B) with $\|A\| = \|B\|$, when Γ acts trivially. If there is a quadratic field extension E/F and $\rho(\sigma) \in \mathrm{diag}(1,w,1)\hat{C}$ for σ in $\mathrm{Gal}(E/F)$, then σ acts on $\mathbf{C} = \mathbf{C}_E = \mathrm{Res}_{E/F}\, GL(2)'$ by permuting the two factors. In particular, $C_E = \mathbf{C}_E(F) = GL(2,E)'$, the prime indicating determinant in $F^\times$. Note that the centralizer of

$(\varepsilon,\varepsilon)=\mathrm{diag}(1,-1,-1,1)$ in $\hat{C}$ is generated by the scalars and $(\varepsilon,\varepsilon)$, hence C has no elliptic endoscopic groups.

3. $\hat{s}=\mathrm{diag}(1,1,1,-1)$, $Z_{\hat{G}}(\hat{s}\hat{\theta})=\langle(\mathrm{diag}(a,B,b),\|B\|),(\iota,1);\ \iota=\begin{pmatrix}0&0&1\\0&I&0\\1&0&0\end{pmatrix}$, $B\in GL(2,\mathbb{C})$, $a,b\in\mathbb{C}^\times, ab=\|B\|\rangle$ has connected component $\hat{C}_+=(GL(2,\mathbb{C})\times GL(1,\mathbb{C})^2)'$ (the prime indicates (a,B,b) with $ab=\|B\|$), with center $Z(\hat{C}_+)=\mathbb{C}^{\times 2}$, which will not be elliptic unless the Galois action is non trivial, namely there is a quadratic extension E/F with $\rho(\sigma)=\iota$, $\langle\sigma\rangle=\mathrm{Gal}(E/F)$. In this case $\left(Z(\hat{C}_+)^\Gamma\right)^0=\mathbb{C}^\times$ is $Z(\hat{G})$. We have $X_*(\hat{T}_+)=\{(a,b,c,b+c-a;b+c)\}=X^*(T_+^*)$, with dual $X^*(\hat{T}_+)=\{(x,y,z,t;w)\}/\{(u,v,v,u;-u-v)\}=\{(x,y,z,t)\}/\{(u,-u,-u,u)\}=X_*(T_+^*)$. We conclude that $\mathbf{C}_+=\mathbf{C}_+^E=(GL(2)\times \mathrm{Res}_{E/F}GL(1))/GL(1)$, $GL(1)$ embeds as (z,z^{-1}), and $C_+=\mathbf{C}_+(F)=GL(2,F)\times E^\times/F^\times\simeq GL(2,F)\times E^1$.

4. $\hat{s}=\begin{pmatrix}I&0\\0&cI\end{pmatrix}$, $c\neq\pm1$, $Z_{\hat{G}}(\hat{s}\hat{\theta})=\langle(\begin{pmatrix}A&0\\0&t\varepsilon A\varepsilon\end{pmatrix},t\|A\|)\rangle$ is connected but not elliptic.

5. $\hat{s}=\mathrm{diag}(1,1,1,d), d\neq\pm1, Z_{\hat{G}}(\hat{s}\hat{\theta})=\langle(\mathrm{diag}(a,A,\|A\|/a;\|A\|)\rangle$ is connected but not elliptic.

6. $\hat{s}=\mathrm{diag}(1,1,-1,d), d\neq\pm1, Z_{\hat{G}}(\hat{s}\hat{\theta})=\langle(\mathrm{diag}(a,b,t/b,t/a),t),(\mathrm{diag}(1,w,1),1)\rangle$ is not elliptic.

7. $\hat{s}=\mathrm{diag}(1,1,c,d), c^2\neq1\neq d^2, c\neq d,d^{-1}, Z_{\hat{G}}(\hat{s}\hat{\theta})=\langle(\mathrm{diag}(a,b,t/b,t/a),t)\rangle$ is connected but not elliptic.

The **norm map** is defined as follows. Put $\mathbf{V}=(1-\theta)\mathbf{T}^*$ and $\mathbf{U}=\mathbf{T}^*_\theta=\mathbf{T}^*/\mathbf{V}$. Since $\mathbf{T}^*$ consists of $(a,b,c,d;e)$ and $\theta(a,b,c,d;e)=(d^{-1},c^{-1},b^{-1},a^{-1};eabcd)$, we have that $\mathbf{V}$ consists of $(\alpha,\beta,\beta,\alpha;1/\alpha\beta)$. The isomorphism $\hat{U}=\hat{T}^{\hat{\theta}}\simeq\hat{T}_H$, where $\mathbf{T}^*_H$ is the diagonal torus in $\mathbf{H}=GSp(2)$, defines a morphism

$$X_*(\mathbf{T}^*)\to X_*(\mathbf{T}^*)/X_*(V)=X^*(\hat{T})/X^*(\hat{V})=X^*(\hat{U}=\hat{T}^{\hat{\theta}})=X^*(\hat{\mathbf{T}}_H)\tilde{\to}X_*(\mathbf{T}^*_H),$$

the last arrow being defined by

$$(x,y,z,t;w)\mapsto(x+y+w,x+z+w,t+y+w,t+z+w;x+y+z+t+2w),$$

and a norm map $N:\mathbf{T}^*\to\mathbf{T}^*_H$, given by

$$(x,y,z,t;w)\,\mathrm{mod}(\alpha,\beta,\beta,\alpha;1/\alpha\beta)\mapsto(xyw,xzw,tyw,tzw;xyztw^2)=(a,b,e/b,e/a;e),$$

which is surjective since $(b,a/b,1,e/a;1)\mapsto(a,b,e/b,e/a;e)$.

To describe the norm for the twisted endoscopic group $\mathbf{C}$ (of (2) above), note that $\hat{T}_C\tilde{\to}\hat{T}_H$ by $((a,d),(b,c))\mapsto(ab,ac,bd,cd)$. Hence $X^*(\hat{T}_H)\tilde{\to}X^*(\hat{T}_C)$ via $(x,y,z,t)\,\mathrm{mod}\{(\alpha,\beta,\beta,\alpha)\}\mapsto\left((x+y,z+t),(x+z,y+t)\right)$, and the composition $X_*(\mathbf{T}^*)\to X^*(\hat{T}_H)\simeq X^*(\hat{T}_C)$ defines the norm map

$$N_C:\mathbf{T}^*\to\mathbf{T}^*_C,\ (x,y,z,t;w)\mapsto\left((xyw,ztw);(xzw,ytw)\right)\left(=\left(\begin{pmatrix}xyw&0\\0&ztw\end{pmatrix},\begin{pmatrix}xzw&0\\0&ytw\end{pmatrix}\right)\right).$$

Let us also describe the norm map for the twisted endoscopic group $\mathbf{C}_+$ of (3) above. Since the map $X^*(\hat{T}^{\hat{\theta}})\tilde{\to}X_*(T_+^*)$ is the identity, the norm is defined by

$$N:X_*(\mathbf{T}^*)\to X_*(\mathbf{T}^*)/X_*(V)=X^*(\hat{T})/X^*(\hat{V})=X^*(\hat{U}=\hat{T}^{\theta}=\hat{T}_+)=X_*(\mathbf{T}^*_+),$$

$N(x,y,z,t)=(x,y,z,t)\,\mathrm{mod}(u,u^{-1},u^{-1},u)$.

G. Instability.

Recall that the set of θ-conjugacy classes within the stable θ-conjugacy class of a strongly θ-regular element t in G is parametrized by the set $D(F,\theta,t) = \ker[H^1(F,\mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) \to H^1(F,\mathbf{G})] = \ker[H^1(F,\mathbf{T}^{*\theta}) \to H^1(F,\mathbf{G})]$, which is a group in our case, as $H^1(F,\mathbf{G})$ is trivial. There is an exact sequence

$$H^0(F,\mathbf{T}^*) = T^* \xrightarrow{1-\theta} H^0(F,\mathbf{V}) = V \to D(F,\theta,t) \to H^1(F,\mathbf{T}^*) \xrightarrow{1-\theta} H^1(F,\mathbf{V}).$$

In our case of $\mathbf{G} = GL(4) \times GL(1)$, we have $H^1(F,\mathbf{T}^*) = \{1\}$ for all tori (or Galois actions, namely subgroups of the symmetric group S_4 on four letters), hence $D(F,\theta,t) = V/(1-\theta)T^*$.

There is a dual five term exact sequence, useful when stabilizing the twisted trace formula. Let $\phi : \hat{V} \to \hat{T}$ be the homomorphism dual to $\mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}$. Thus $\phi : X_*(\hat{V}) = X^*(\mathbf{V}) \to X^*(\mathbf{T}^*) = X_*(\hat{T})$ takes $\chi = (x,y,z,t;w)$ to $(\phi(\chi))(a,b,c,d;e) = \chi(ad,bc,bc,ad;1/abcd) = (ad)^{x+t-w}(bc)^{y+z-w}$. Namely, ϕ takes $(x,y,z,t;w)$ in $\hat{V} = \hat{T}/\hat{U} = \hat{T}/\hat{T}^{\hat{\theta}}$ to $(xt/w, yz/w, yz/w, xt/w; 1)$ in $\hat{T}$. Recall that $\hat{T}^{\hat{\theta}} = \{(a,b,e/b,e/a;e)\}$.

To obtain the dual sequence recall the Langlands isomorphism $H^1(W_F,\hat{T}) = \mathrm{Hom}_{cts}(T,\mathbb{C}^\times)$ ($T = \mathbf{T}(F)$; [KS, about a page after Lemma A.3.A]), and its hypercohomology analogue ([KS, Lemma A.3.B]): $H^1(W_F,\hat{V} \xrightarrow{\phi} \hat{T})$ is isomorphic to the group $\mathcal{K}(F,\theta,T^*)$ of characters $\mathrm{Hom}_{cts}\big(H^1(F,\mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}),\mathbb{C}^\times\big)$. Since the Weil group W_F of F acts on $\hat{T}$ and $\hat{V}$ via the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$, one has $H^0(W_F,\hat{V}) = \hat{V}^\Gamma \xrightarrow{\phi} H^0(W_F,\hat{T}) = \hat{T}^\Gamma \to \mathcal{K}(F,\theta,T^*) = H^1(W_F,\hat{V} \xrightarrow{\phi} \hat{T}) \to H^1(W_F,\hat{V}) \xrightarrow{\phi} H^1(W_F,\hat{T})$. This is the exact sequence [KS, A.1.1], for $\phi : \hat{V} \to \hat{T}$, which is dual to the previous five terms exact sequence for $1-\theta : \mathbf{T}^* \to \mathbf{V}$.

Definition. The *stable θ-orbital integral* Φ^{st} at a strongly θ-regular element t in T, where $\mathbf{T}$ is any F-torus in $\mathbf{G}$, is the sum of the θ-orbital integrals on the θ-conjugacy classes within the stable θ-conjugacy class of t.

The set of such θ-conjugacy classes (for some t) is parametrized by the group $H^1(F,\mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) = H^1(F,\mathbf{T}^{*\theta})$ computed in Section D (of the text). For each character κ of this group (into the group of roots of unity in $\mathbb{C}^\times$), we can also make the:

Definition. The *κ-orbital integral* is the linear combination of the θ-orbital integrals weighted by the values of κ at the element of $H^1(F,\mathbf{T}^* \to \mathbf{V})$ parametrizing the θ-conjugacy class.

These weighted (by κ) combinations of the θ-orbital integrals are to be compared with stable orbital integrals on the θ-endoscopic groups $\mathbf{H}$ of $(\mathbf{G},\theta)$. The θ-endoscopic group $\mathbf{H}$ is determined from κ, by [KS, Lemma 7.2.A], via the surjection $H^1(W_F,\hat{V} \xrightarrow{\phi} \hat{T}) \to \mathrm{Hom}_{cts}\big(H^1(F,\mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}),\mathbb{C}^\times\big)$ (see [KS, Lemma A.3.B]). Recall ([KS, A.1]) that:

Definition. The *first (abelian) hypercohomology group* $H^1(G, A \xrightarrow{f} B)$ is the quotient of the group of 1-hypercocycles, by the subgroup of 1-hypercoboundaries. A 1-*hypercocycle* is a pair (a,b) with a being a 1-cocycle of G in A, and $b \in B$ such that $f(a) = \partial b$ (∂b is the 1-cocycle $\sigma \mapsto b^{-1}\sigma(b)$ of G in B). A 1-hypercoboundary is a pair $\big(\partial a, f(a)\big), a \in A$.

Thus $H^1(W_F, \hat{V} \xrightarrow{\phi} \hat{T})$ consists of elements represented by pairs $(a,b), a \in H^1(W_{K/F}, \hat{V})$, where K/F is a galois extension over which T splits and $\hat{V} = \hat{T}/\hat{U}, \hat{U} = (\hat{T}^{\hat{\theta}})^0$. Here $\phi : \hat{V} \to \hat{T}$ is the map dual to $1 - \theta : \mathbf{T}^* \to \mathbf{V}$, thus $\phi(x,y,z,t;w) = (xt/w, yz/w, yz/w, xt/w; 1)$, and $b \in \hat{T}$ satisfies $\phi(a) = \partial b$. The θ-endoscopic group $\mathbf{H}$ has a dual group whose connected component $\hat{H}$ is $Z_{\hat{G}}(b\hat{\theta})^0$, the connected centralizer of $b\hat{\theta}$ in $\hat{G}$ ([KS, Lemma 7.2.A]).

We then compute the 1-hypercocycles representing the non trivial characters κ on $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$, according to our listing of tori.

In the comparison of the unstable (κ-)θ-orbital integral at a strongly θ-regular element t, and the stable orbital integral on the endoscopic group H_κ determined by κ, a transfer factor appears. It is a product of a sign and of a Jacobian factor $\Delta_{G,H_\kappa} = \Delta_G/\Delta_{H_\kappa}$, denoted Δ_{IV} in [KS, 4.5], which we also describe in the main cases.

H. Kazhdan's decomposition.

A main ingredient in our proof of the matching is the (twisted analogue [F7] of) Kazhdan's decomposition [K, p. 226], which we now recall. Let $\mathbf{H}$ be a connected reductive R-group, where R is the ring of integers of F, and put $H = \mathbf{H}(F), K_H = \mathbf{H}(R)$.

Definition ([K]). An element $k \in H$ is called *absolutely semi simple* if $k^a = 1$ for some positive integer a which is prime to the residual characteristic p of R. A $k \in H$ is called *topologically unipotent* if $k^{q^N} \to 1$ as $N \to \infty$, $q = \#(R/\boldsymbol{\pi} R)$, $\boldsymbol{\pi}$ generates the maximal ideal in R.

1. Proposition ([K]). *Any element $k \in K_H$ has a unique decomposition $k = su = us$, where s is absolutely semi simple, u is topologically unipotent, and s, u lie in K_H. For any $k \in K_H$ and $x \in H$, if $\mathrm{Int}(x)k(= xkx^{-1})$ lies in K_H, then x is in $K_H Z_H(s)$, where $Z_H(s)$ denotes the centralizer of s in H.* □

In fact [K] proves this only for $\mathbf{H} = GL(n)$, but since s is defined as a limit of a sequence of the form k^{q^m}, both s and u lie in K_H.

The twisted analogue which we need is reproduced next (from [F7]). Let $\mathbf{G}$ be a reductive connected R-group and θ an automorphism of $G = \mathbf{G}(F)$ of order ℓ $\big((\ell, p) = 1\big)$, whose restriction to $K = \mathbf{G}(R)$ is an automorphism of K of order ℓ. Denote by $\langle K, \theta \rangle$ the group generated by K and θ.

Definition. The element $k\theta$ of $G\theta \subset \langle G, \theta \rangle$ is called *absolutely semi-simple* if $(k\theta)^a = 1$ for some positive integer a indivisible by p.

2. Proposition ([F7]). *Any $k\theta \in K\theta$ has a unique decomposition $k\theta = s\theta \cdot u = u \cdot s\theta$ with absolutely semi simple $s\theta$ (called the absolutely semi simple part of $k\theta$) and topologically unipotent u (named the topologically unipotent part of $k\theta$). Both s and u lie in K. In particular, $Z_G(s\theta \cdot u)$ lies in $Z_G(s\theta)$.* □

3. Proposition ([F7]). *Given $k \in K$, put $\tilde{\theta}(h) = s\theta(h)s^{-1}$, where $k\theta = s\theta \cdot u$. This $\tilde{\theta}$ is an automorphism of order ℓ on $Z_K\big((s\theta)^\ell\big)$. Suppose that the first cohomology set $H^1\Big(\langle\tilde{\theta}\rangle, Z_K\big((s\theta)^\ell\big)\Big)$, of the group $\langle\tilde{\theta}\rangle$ generated by $\tilde{\theta}$, with coefficients in the centralizer*

$Z_K\big((s\theta)^\ell\big)$ *in* K, *injects in* $H^1\Big(\langle\tilde\sigma\rangle, Z_G\big((s\theta)^\ell\big)\Big)$. *Then any* $x \in G$ *such that* $\mathrm{Int}(x)(k\theta)$ *lies in* $K\theta$, *must lie in* $KZ_G(s\theta)$. □

The supposition of this proposition can be verified for our group $G = GL(4,F)\times GL(1,F)$ and our automorphism θ in the same way it is verified in [F7] for $GL(3,F)$. Note also (see [F7]) that if the elements $k\theta = s\theta\cdot u$ and $k'\theta = s'\theta\cdot u'$ of $K\theta$ are conjugate by $\mathbf{G}(\overline{F})$ ($\overline{F}$ is a separable closure of F) then so are $s\theta$ and $s'\theta$, and if $s = s'$ then u, u' are conjugate in $Z_{\mathbf{G}(\overline{F})}(s\theta)$.

Our argument uses the function

$$1_{s\theta}(u) = |K/K\cap Z_G(s\theta)|1_K(s\theta\cdot u) = \int_{K/K\cap Z_G(s\theta)} 1_K\big(\mathrm{Int}(x)(s\theta\cdot u)\big)dx.$$

Then the orbital integral $\Phi_{1_K}(k\theta) = \int_{G/Z_G(k\theta)} 1_K\big(\mathrm{Int}(x)(k\theta)\big)dx$ is equal – by Proposition 3 – to $= \int_{Z_G(s\theta)/Z_G(s\theta\cdot u)} 1_{s\theta}\big(\mathrm{Int}(x)u\big)dx = \Phi_{1_{s\theta}}(u)$, the orbital integral of the characteristic function $1_{s\theta}$ of the compact subgroup $Z_K(s\theta) = K\cap Z_G(s\theta)$ of $Z_G(s\theta)$ (multiplied by $|K/Z_K(s\theta)|$) at the topologically unipotent element u in $Z_K(s\theta)$.

Since $(k\theta)^2 = s\theta(s)\cdot u^2$, where in our case $\theta(g,t) = \big(\theta(g), t\det g\big), g\in GL(4,F), t\in F^\times$, $\theta(g) = J^tg^{-1}J^{-1}$, we shall deal with various cases according to the values of $s\theta(s)$ (s denotes also the $GL(4,F)$-component of s).

4. Lemma. *If* $x = s\theta(s)$ *has the eigenvalue* λ, *then it has the eigenvalue* λ^{-1} *too.*

J. Decomposition for $Sp(2)$**.**

In computing the orbital integrals of 1_K on $H = GSp(2,F)$, we use the following decomposition.

1. Lemma. *We have a disjoint decomposition* $H = GSp(2,F) = \bigcup_{n\geq 0} Ku_nC_A = \bigcup_{n\geq 0} C_Au_nK$, *where* $A\in F - F^2, u_n = \begin{pmatrix} 1&0&0&\pi^{-n}/A\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}$, $C_A = \left\{\left(\begin{smallmatrix}\mathbf{a}&\mathbf{b}\\ \mathbf{c}&\mathbf{d}\end{smallmatrix}\right)\in H; \mathbf{a} = \left(\begin{smallmatrix}a_1&a_2\\ Aa_2&a_1\end{smallmatrix}\right), \mathbf{b} = \dots\right\}$, $K = GSp(2,R)$, *and* $|A| = 1$ *or* $= |\pi|$.

There is an analogous decomposition for $Sp(2,F)$.

2. Lemma. *We have a disjoint decomposition* $Sp(2,F) = \bigcup_{m\geq 0} C_A^1u_mK^1$, *where the superscript* 1 *stands for the subgroup of elements with determinant one.*

We need an analogous result for $A\in F^{\times 2}$. Note that for $A\in F - F^2$, the subgroup C_A of H is isomorphic to $GL(2,E)'$, $E = F(\sqrt{A})$, where the prime indicates elements with determinant in $F^\times$. The isomorphism is given by $\mathbf{a}\mapsto\tilde a = a_1 + a_2\sqrt{A}$. Put $C_0 = \left\{\left[\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right), \left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)\right] = \begin{pmatrix} a&0&0&b\\ 0&\alpha&\beta&0\\ 0&\gamma&\delta&0\\ c&0&0&d\end{pmatrix}\in H\right\}$; it is isomorphic to the group $GL(2, F\oplus F)' = \{(g,g') = (\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right), \left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\varepsilon\end{smallmatrix}\right)); \det g = \det g'\}$.

5. Lemma. *We have a disjoint decomposition*

$$H = GSp(2,F) = \bigcup_{m\geq 0} Kz(m)C_0, \qquad z(m) = \begin{pmatrix} 1 & 0 & \pi^{-m} & 0 \\ 0 & 1 & 0 & \pi^{-m} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Put $H^1 = Sp(2,F)$, $C_0^1 = C_0 \cap H^1 \simeq SL(2,F) \times SL(2,F)$, $K^1 = K \cap H^1$.

6. Lemma. $H^1 = \bigcup_{m\geq 0} C_0^1 z(m) K^1$, *where the union is disjoint.*

Part II. Main comparison. A. Strategy.

Let us review our strategy in computing the θ-orbital integrals of 1_K. It is based on the twisted Kazhdan decomposition. Given a semi-simple $t\theta \in K \rtimes \langle\theta\rangle$, $G = GL(4,F) \times GL(1,F)$, $K = GL(4,R) \times GL(1,R)$, it has the decomposition $t\theta = u \cdot s\theta = s\theta \cdot u$, where $s\theta$ is absolutely semi simple, and u is topologically unipotent. Then $\Phi_{1_K}^G(t\theta) = \Phi_{1_{Z_K(s\theta)}}^{Z_G(s\theta)}(u)$. The associated stable θ-orbital integral we wish to relate to the stable orbital integral $\Phi_{1_{K_H}}^{H,st}(Nt)$, where H is the endoscopic group $GSp(2,F)$, and Nt is the stable orbit of the norm of t. To compute the norm we write $t = h^{-1}t^*\theta(h)$, where $h \in \mathbf{G}$ $(= \mathbf{G}(\overline{F}))$, and $t^* \in T^*$, where $\mathbf{T}^*$ is the diagonal subgroup and $T^* = \mathbf{T}^{*\Gamma}$. On T^* the norm is defined by $T^* \to T^*/(1-\theta)T^* \simeq T_H^*$, thus $N(a,b,c,d;e) = (abe, ace, bde, cde; e^2abcd)$. A θ-semi-simple t ($t\theta$ is semi simple in $G \rtimes \langle\theta\rangle$) is called *strongly θ-regular* if $Z_G(t\theta)$ is abelian, in which case the centralizer $Z_G(Z_G(t\theta)^0)$ of $Z_G(t\theta)^0$ in G is an F-torus T in G which is invariant under $\mathrm{Int}(t) \circ \theta$, and $Z_G(t\theta) = T^{\mathrm{Int}(t)\circ\theta}$. The θ-orbit of t intersects T^*, thus there is $h \in \mathbf{G}$ and $t^* \in T^*$ with $t = h^{-1}t^*\theta(h)$, and $Z_G(t\theta) = h^{-1}Z_G(t^*\theta)h = h^{-1}T^{*\theta}h$. Then $T = Z_G(h^{-1}T^{*\theta}h) = h^{-1}T^*h$, and $Z_G(t\theta) = T^{\mathrm{Int}(t)\circ\theta}$ consists of the $x \in T$ with $t\theta(x)t^{-1} = x$, thus $x^{-1}t\theta(x) = t$.

An F-torus T in G is determined by $h \in \mathbf{G}$ and the Galois action on $\mathbf{T}^*$. Namely, for $t = h^{-1}t^*h \in T = h^{-1}T^*h$ we have $h^{-1}t^*h = t = \sigma t = \sigma h^{-1}\sigma t^*\sigma h$, and so $\sigma t^* = h_\sigma^{-1}t^*h_\sigma$, where $\mathrm{Int}(h_\sigma^{-1}) \in \mathrm{Norm}(T^*,G)$ has the image w_σ in $W = W(T^*,G) = \mathrm{Norm}(T^*,G)\,/\,\mathrm{Cent}(T^*,G)$. If T^* is a θ-invariant F-torus, taking $t^* \in T^{*\theta}$ we conclude that $\mathrm{Int}(h_\sigma^{-1}) = \mathrm{Int}\,(\theta(h_\sigma)^{-1})$, thus $w_\sigma \in W^\theta$, and the torus determines a cocycle $\langle w_\sigma\rangle$ in $H^1(F, W^\theta)$. We denoted the homomorphism $\Gamma \to W^\theta$, $\sigma \mapsto w_\sigma$, by ρ, and classified the tori according to the image of $\rho : \mathrm{Gal}(\overline{F}/F) \to W^\theta$, as types (1) – (3) and (I) – (IV). We explicitly realized T in the form $T = h^{-1}T^*h$, with $h = \theta(h)$. Thus in each stable θ-conjugacy class of strongly θ-regular elements we have a representative $t = h^{-1}t^*h$, and further we found representatives for the θ-conjugacy classes within its stable θ-conjugacy class, of the form $g^{-1}tg, g = g_R$ with $g = \theta(g)$.

A θ-semi-simple $t \in G$ is called *θ-elliptic* if $Z_G(t\theta)^0/Z(G)^0$ is anisotropic. The associated tori $T = Z_G(Z_G(t\theta)^0)$ are called θ-elliptic. A complete set of representatives for the θ-elliptic tori is given by the tori of type (I)-(IV). The computations of θ-orbital integrals of non θ-elliptic strongly θ-regular elements can be reduced – using a standard integration formula – to the case of the θ-elliptic elements, so we deal only with t in tori T of types (I) - (IV).

B. Twisted orbital integrals of type (I).

Let $u = \theta(u)$ be a topologically unipotent element in $GL(4,R) \times GL(1,R)$. Then $\Phi^G_{1_K}(u\theta) = \Phi^{Z_G(\theta)}_{1_{Z_K(\theta)}}(u)$, where $Z_G(\theta) = H^1 = Sp(2,F)$ and $Z_K(\theta) = K^1 = K \cap H^1$. We compute the value of this integral at u in a torus of type (I). To consider also the integrals at stably θ-conjugate but not θ-conjugate elements, we look at a complete set of representatives, parametrized by ρ_1, ρ_2. Here $\rho_i \in \{1, \boldsymbol{\pi}\}$ if E/F is unramified, and $\rho_i \in \{1, \varepsilon\} = R^\times/R^{\times 2}$ if E/F is ramified. Thus take s_ρ in the torus $T_\rho = \{s_\rho = [\phi^D_{\rho_1}(a_1 + b_1\sqrt{D}), \phi^D_{\rho_2}(a_2 + b_2\sqrt{D})] \in C^1_0\}$, where $\phi^D_\rho(a + b\sqrt{D}) = \left(\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix}\right)$. If $E^1 = \{x \in E^\times; N_{E/F}x = 1\}$, then T_ρ is isomorphic to $E^1 \times E^1$. By Lemma I.J.6 we have

$$\begin{aligned}\Phi^{H^1}_{1_{K^1}}(s_\rho) &= \int_{T_\rho\backslash H^1} 1_{K^1}(g^{-1}s_\rho g)dg \\ &= \sum_{m\geq 0} |K^1|_{H^1} \int_{T_\rho\backslash C^1_0/C^1_0\cap z(m)K^1z(m)^{-1}} 1_{K^1}\big(z(m)^{-1}h^{-1}s_\rho h z(m)\big)dh.\end{aligned}$$

The integrand in the last integral is non zero precisely when $h^{-1}s_\rho h$ lies in $z(m)K^1z(m)^{-1} \cap C^1_0 = K^{C^1_0}_m$. Using Lemma I.J.7 we obtain

$$= \sum_{m\geq 0} \frac{|K^1|_{G^1}}{|K^{C^1_0}_m|_{C^1_0}} \int_{T_\rho\backslash C^1_0} 1_{K^{C^1_0}_m}(h^{-1}s_\rho h)dh = \sum_{m\geq 0}[K^1_0 : K^1_m] \int_{T_{\rho m}\backslash C^1_0} 1_{K^1_m}(h^{-1}s_{\rho m}h)dh,$$

where now $C^1_0 = SL(2,F) \times SL(2,F)$, and $\rho \mapsto \rho_m$ is a permutation of the set of ρ (trivial e.g. when m is even). Using the double coset decomposition for $SL(2,F)$ of Lemma I.I.3:

$$= \sum_{m\geq 0} \sum_{r\in R_{\rho m}} [R^1_{T_{\rho m}} : T_{\rho m} \cap rK^1_0 r^{-1}][K^1_0 : K^1_m] \int_{K^1_0} 1_{K^1_m}(k^{-1}r^{-1}s_{\rho m} rk)dk.$$

Here $R^1_{T_{\rho m}} = T_{\rho m} \cap K^1_0 = T^1_{\rho m}(R)$. Let $\mathbf{j}$ signify (j_1, j_2).

The proof consists of computing the various terms which appear in the last sum, then computing the analogous sum which is obtained in the non twisted case, then taking the suitable linear combinations weighted by κ, multiplying by the transfer factor, and comparing the resulting sums. This is done in the context of each of the elliptic tori, of types (I)-(IV). In particular we obtain explicit formulas for all of the orbital integrals of the unit element 1_K. All this is done in Part II of the text for a topologically unipotent element. Part III deals with elements whose absolutely semi simple part is non trivial, where the integral is reduced to one in a smaller group.

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