Title: A generalization of Kohnen's estimates for Fourier coefficients of Siegel cusp forms

Author(s): Horie, Taro

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A generalization of Kohnen's estimates for Fourier coefficients of Siegel cusp forms

Taro Horie (堀江 太郎)

Graduate school of Polymathematics, Nagoya University
Chikusa-ku, Nagoya 464-01, Japan
E-mail:t-horie@math.nagoya-u.ac.jp

The purpose of this article is to show that the main result of [K] is valid for any level.

Theorem. Let $F$ be a cusp form of integral or half integral weight $k(>2)$ with respect to the subgroup $\Gamma_2(N)$ of $\text{Sp}_2(\mathbb{Z})$, where

$$\Gamma_2(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \mid C \equiv 0 \pmod{N} \right\}.$$ 

And let its Fourier expansion be given by

$$F(Z) = \sum_T a(T) \exp(2\pi i \text{tr} T(Z)),$$

where $T$ runs over positive definite symmetric half-integral $2 \times 2$-matrices. Then we have

$$a(T) \ll \epsilon, F(\min T)^{5/18+\epsilon} (\det T)^{(k-1)/2+\epsilon} \quad (\forall \epsilon > 0), \quad (1)$$

where $\min T$ is the smallest positive integer represented by $T$.

The idea to prove Theorem is the same as in [K], that is a combination of appropriate estimates for both Fourier coefficients of Jacobi Poincaré series and Petterson norms of Fourier-Jacobi coefficients of Siegel modular forms.

$\mathcal{H}_i$ denotes the Siegel upper half space of degree $i$ consisting of complex $i \times i$-matrices with positive definite imaginary part. We often write

$$Z = X + iY = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \begin{pmatrix} u + iv & x + iy \\ x + iy & u' + iv' \end{pmatrix} \in \mathcal{H}_2.$$

For simplicity, we consider only the integral weight case.

Proposition 1. We let $\Gamma_1'(N)$ be the Jacobi group which is the semi direct product of $\Gamma_1(N)$ and $\mathbb{Z}^2$, and let $J_{k,m}^{\text{cusp}}(N)$ be the space of holomorphic Jacobi cusp forms on $\mathcal{H}_1 \times \mathbb{C}$ of weight $k$ and index $m$ with respect to $\Gamma_1'(N)$ (cf. e.g. [E-Z]).
For $\phi$ in $J_{k,m}^{\text{cusp}}(N)$, let $c(n, r)$ be the $(n, r)$-th Fourier coefficient of $\phi$ ($n, r \in \mathbb{Z}, r^2 < 4mn$). Put $D = r^2 - 4mn$. Then we have

$$c(n, r) \ll_{\epsilon, k} (m + |D|^{1/2+\epsilon})^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} ||\phi|| \quad (\forall \epsilon > 0)$$

where the constant implied in $\ll$ depends only on $\epsilon$ and $k$ (not on $m$).

Proof. Let $P_{n,r} = P_{k,m,n,r}$ be the $(n, r)$-th Jacobi Poincaré series in $J_{k,m}^{\text{cusp}}(N)$ characterized by

$$\langle \psi, P_{n,r} \rangle = \lambda_{k,m,D} b_{n,r}(\psi) \quad (\forall \psi \in J_{k,m}^{\text{cusp}}(N))$$

where $b_{n,r}(\psi)$ denotes the $(n, r)$-th Fourier coefficients of $\psi$ and

$$\lambda_{k,m,D} := \frac{1}{2} \Gamma(k - \frac{3}{2}) \pi - k + \frac{3}{2} k - m \frac{2|D|^{-k/2} + \frac{3}{2}}{m^{(k-1)/2}}$$

Then the Cauchy-Schwarz inequality gives

$$|c(n, r)|^2 \leq \lambda_{k,m,D}^{-2} ||\phi||^2 \langle P_{n,r}, P_{n,r} \rangle v = \lambda_{k,m,D}^{-1} b_{n,r}(P_{n,r}) ||\phi||^2$$

We can show that the Fourier coefficient of $P_{n,r}$ as follows (cf. [G-K-Z], p.519);

$$b_{n,r}(P_{n,r}) = 1 + (-1)^k \delta_m(r) + \frac{i^k \pi \sqrt{2}}{\sqrt{m}} \sum_{N|c \geq 1} c^{-3/2} (\exp(r^2/2mc) H_{m,c}^+(n, r) + (-1)^k \exp(-r^2/2mc) H_{m,c}^-(n, r)) J_{k-3/2} \left( \frac{\pi |D|}{mc} \right),$$

where

$$\delta_m(r) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod{m} \\ 0 & \text{otherwise} \end{cases}$$

$J_{k-3/2}$ is the modified Bessel function of order $k - 3/2$, and

$$H_{m,c}^{\pm}(n, r) := \sum_{x(c), y(c)} e_c((mx^2 + rx + n)y + ny \pm rx),$$

where $x$ resp. $y$ run through $\mathbb{Z}/c\mathbb{Z}$ resp. $(\mathbb{Z}/c\mathbb{Z})^*$, $\bar{y}$ denotes an inverse of $y \pmod{c}$, $e_c(b) := \exp(2\pi ib/c)$ for $c \in \mathbb{N}$, $b \in \mathbb{Z}/c\mathbb{Z}$, $\varepsilon(y) = 1$ or $i$ according as $y \equiv 1 \pmod{4}$ or $\equiv 3 \pmod{4}$, and $(\cdot \mid \cdot)$ means the Kronecker symbol. $H_{m,c}^{\pm}(n, r)$ is a certain character sum, which is Gauss sum for $x$ and Kloosterman sum for $y$, and by factorizing $c$ to prime powers, for $D := r^2 - 4mn$ we can prove an estimate

$$H_{m,c}^{\pm}(n, r) \ll \epsilon c^{-1-\epsilon} (D, c) \quad (\forall \epsilon > 0).$$

From this and the estimate

$$J_{k-3/2}(x) \ll_k \min \{x^{-1/2}, x^{k-3/2} \} \quad (x > 0)$$

(cf. e.g. [B], p.4 and p.74), we easily find

$$b_{n,r}(P_{n,r}) \ll_{\epsilon,k} 1 + \frac{|D|^{1/2+2\epsilon}}{m}$$

for any $\epsilon > 0$ and complete the proof.
To estimate Petterson norm $||\phi||$, for an analogue of the Rankin convolution series

$$D_{F,F}(s) := \zeta(2s - 2k + 4) \sum_{n \geq 1} \langle \phi_n, \phi_n \rangle n^{-s}$$

where

$$F(z) = \sum_{n \geq 1} \phi_n(z) \exp(2\pi i n \tau'),$$

we want to use the following Landau's Theorem;

**Theorem** (Landau-Shintani). Suppose that

$$\xi(s) = \sum_{n \geq 1} c(n) n^{-s}, \quad \xi_i(s) = \sum_{n \geq 1} c_i(n) n^{-s} \quad (1 \leq i \leq I)$$

are Dirichlet serieses with non-negative coefficients which converge for $\Re(s) > \sigma_0$, have meromorphic continuation to $\mathbb{C}$ with finitely many poles and satisfy a functional equation

$$\xi^*(\delta - s) = \sum_{i=1}^{I} \xi_i^*(s)$$

where

$$\xi_i^*(s) = B A^i \prod_{j=1}^{J} \Gamma(a_j s + b_j) \xi(s) \quad (A \in \mathbb{C}, B \in \mathbb{C}, a_j > 0, b_j \in \mathbb{R}),$$

$$\xi_i^*(s) = B_i A_i^s \prod_{j=1}^{J} \Gamma(a_j s + b_j) \xi(s) \quad (A_i \in \mathbb{C}, B_i \in \mathbb{C}, a_j and b_j are same as above).$$

Suppose

$$\kappa := (2\sigma_0 - \delta) \sum_{j=1}^{J} a_j - \frac{1}{2} > 0.$$ 

Then we have

$$\sum_{n \leq x} c(n) = \sum_{s \text{all poles}} \text{Res} \left( \frac{\xi(s)}{s} x^s \right) + O_\eta(x^\eta)$$

for any $\eta > \eta_0 := (\delta + \sigma_0(k - 1))/(k + 1)$.

For the proof, see Theorem 3 and its proof in [S-S].

The central extension of $\Gamma_1^J(N)$ by $\mathbb{Z}$ is embedded into $\Gamma_2(N)$ via

$$\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu, \kappa \right) \mapsto \begin{pmatrix} a & b & \mu' \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -\lambda' \end{pmatrix}, \quad (\lambda, \mu) = (\lambda', \mu') \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and we denote by $C_N$ the image in $\Gamma_2(N)$. Denote the left upper entry of $Z \in \mathcal{H}_2$ by $Z_1$. For a natural number $N$, $Z \in \mathcal{H}_2$ and $s \in \mathbb{C}$ with $\text{Re}(s) \gg 2$ we define a Klingen-Siegel type Eisenstein series

$$E_{s,N}(Z) := \sum_{M \in C_N \Gamma_2(N)} \left( \frac{\det \text{Im} M \langle Z \rangle}{\text{Im} M \langle Z \rangle_1} \right)^s.$$  

It is easily seen that this series is well defined, absolutely convergent, and invariant under the action of $\Gamma_2(N)$. We put

$$E_{s,N}^*(Z) := \pi^{-s} \Gamma(s) \zeta(2s) E_{s,N}(Z).$$

By Main Lemma on p.545 in [K-S], we know $E_{s,1}(Z)$ has a meromorphic continuation to $\mathbb{C}$, has only two poles at $s = 0, 2$ which are simple, and satisfies a functional equation

$$E_{2-s,1}^*(Z) = E_{s,1}^*(Z).$$

By the method of Rankin-Selberg convolution

$$\pi^{-k+2} \langle F E_{s-k+2,N}, F \rangle = D_{F,F}^*(s)$$

(2)
can be proved, and analytic properties of $D_{F,F}^*(s)$ follow from those of $E_{s,N}^*(s)$. But the functional equations are complicated.

The idea to prove Theorem for any level $N$ is to write the functional equations satisfied by Eisenstein series as a form

$$E_{2-s,N}^*(Z) = \text{a linear combination of } E_{s,m}^*(Z)$$

where $m$ is a natural number with $m|N$. This is necessary to apply Rankin’s method.

**Lemma 1.** $E_{s,N}(Z)$ has a meromorphic continuation to $\mathbb{C}$. Its poles are $s = 0$ and $2$, which are simple. And it satisfies a functional equation

$$E_{2-s,N}^*(Z) = \text{a finite sum of } \frac{\pm n^s}{P(s)} E_{s,m}^*(Z),$$

where $m, n$ are natural numbers with $m|N$ and $P(s)$ is a finite product of $1 - \tilde{m}^{2(2-s)}$ with $\tilde{m}|m$.

**Proof.** For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2(N)$, we notice that

$$\frac{\det \text{Im} M \langle Z \rangle}{\text{Im} M \langle Z \rangle_1} = \frac{|Y|}{Y \begin{pmatrix} c_4 & -c_3 \\ -d_3 & -d_4 \end{pmatrix}}$$

$$(Y \begin{pmatrix} a \\ b \end{pmatrix}) := (\bar{a}, \bar{b}) Y \begin{pmatrix} a \\ b \end{pmatrix}, Z^*$ means the adjoint matrix of $Z) and the mapping

$$\begin{pmatrix} * & * & * & * \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \mapsto (c_3, c_4, d_3, d_4)$$
induces a bijection between

\[ C_N \backslash \Gamma_2(N) \text{ and } \{(c_3, c_4, d_3, d_4) \in \mathbb{Z}^4 \mid \text{primitive and } c_3 \equiv c_4 \equiv 0 \pmod{N}\}. \]

In the following sums, \( c = \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \), \( d = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} \) run over \( \mathbb{Z}^2 \) under the condition that 
\( c_3, c_4, d_3, d_4 \) are relatively prime. In general, for a square free integer \( m \) and a natural number \( l = p_1^{e_1}p_2^{e_2} \ldots p_r^{e_r} \in \mathbb{N} \) (where \( p_1, p_2, \ldots, p_r \) are different prime numbers and \( e_i > 0 \)) it holds

\[
\frac{1}{l^s} E_{s,m}(lZ) = \sum_{\substack{(t_c, t_d) = 1 \\ c \equiv 0 \pmod{m}}} \frac{|Y|^s}{(Y[Z^s t_c + d]^s)}
\]

\[
= \left( \sum_{(t_c, t_d) = 1 \atop (t_f, t_d) \neq 1} \frac{|Y|^s}{(Y[Z^s t_c + d]^s)} + \sum_{(t_c, t_d) = 1 \atop c \equiv 0 \pmod{m}} \frac{|Y|^s}{(Y[Z^s t_c + d]^s)} \right)
\]

\[
\sum_{\substack{(t_c, t_d) = 1 \atop c \equiv 0 \pmod{m}}} \frac{1}{(l/p_i)^{2s}} \sum_{t_c \equiv 0 \pmod{m}} \frac{|Y|^s}{(Y[Z^s ((l/p_i)c + d)]^s)}
\]

\[
- \sum_{i \neq j} \frac{1}{(l/p_i p_j)^{2s}} \sum_{t_c \equiv 0 \pmod{m}} \frac{|Y|^s}{(Y[Z^s ((l/p_i p_j)c + d)]^s)}
\]

\[
+ \ldots
\]

\[
= \sum_{i} \frac{1}{p_i^{2s}} \left( \sum_{t_c \equiv 0 \pmod{m}} - \sum_{t_c \equiv 0 \pmod{m}} \right) \frac{|Y|^s}{(Y[Z^s ((l/p_i)c + d)]^s)}
\]

\[
- \ldots
\]

\[
= \sum_{i} \frac{1}{(p_i)^{2s}} \{E_{s,m}((l/p_i)Z) - E_{s,1,c.m.(m,p_i)}((l/p_i)Z)\}
\]

\[
- \sum_{i \neq j} \frac{1}{(l/p_i p_j)^{s}} \{E_{s,m}((l/p_i p_j)Z) - E_{s,1,c.m.(m,p_i)}((l/p_i p_j)Z) - E_{s,1,c.m.(m,p_j)}((l/p_i p_j)Z)
\]

\[
+ E_{s,1,c.m.(m,p_i p_j)}((l/p_i p_j)Z)\}
\]

+ \ldots
We apply (3) for \( m = 1 \) and \( l = N \); if \( N \) is not square-free the last term is \( E_{s,N}(Z) \), otherwise the last two terms are \((-N^{-2s} - 1)E_{s,N}(Z)\), and in the both cases the rests are \( \pm \tilde{n}^{-s}E_{s,\overline{m}}(lZ) \) where \( \overline{l}, \tilde{m}, \overline{n} \) are natural numbers with \( \overline{l}\tilde{m}|N, \tilde{m} < N \). Hence for a non-square-free number \( N \) we have

\[
E_{s,N}(Z) = \text{a finite sum of } \pm n^s E_{s,m}^*(IZ)
\]

where \( l, m, n \) are natural numbers with \( lm|N, m < N \), and for a square-free number \( N \) we have

\[
(1 - N^{2s})E_{s,N}(Z) = \text{a finite sum of the same type as above.}
\]

So, by induction on \( N \) we deduce that \( E_{s,N}(Z) \) has a meromorphic continuation to \( \mathbb{C} \), has poles only at \( s = 0, 2 \) and satisfies a functional equation

\[
E_{2-s,N}^*(Z) = \text{a finite sum of } \frac{\pm n^s}{P_1(s)} E_{s,m}^*(IZ)
\]

where \( l, m, n \) are natural numbers with \( lm|N \) and \( P_1(s) \) is a finite product of \( 1 - \tilde{m}^{2(2-s)} \) with \( \tilde{m}|m \). Now we notice that (3) makes \( l \) smaller, and apply (3) repeatedly in all terms in this right-hand side until \( l \) becomes 1, then finally we get the functional equation in Lemma 1.

Then we can use Rankin's method and deduce

**Lemma 2.** Let the notations be as above, and take a natural number \( m \) with \( m|N \). For \( L \in \Gamma_2 \), we write the Fourier expansions of \( F(L^{-1}(Z)) \) as

\[
F(L^{-1}(Z)) = \sum_{n \geq 1} \phi_{n,L}(\tau, z) \exp \left( \frac{2\pi in\tau'}{N} \right).
\]

We define a Dirichlet series \( D_{F,F,m}(s) \) as \( \zeta(2s - 2k + 4) \) times

\[
\sum_{n \geq 1} \left\{ \sum_{L \in \Gamma_2(N) \setminus \Gamma_2(m)} \int_{\mathcal{F}} |\phi_{n,L}(\tau, z)|^2 \exp \left( -\frac{4\pi ny^2}{vN} \right) v^{k-3} du dv dx dy \right\} n^{-s}
\]

where \( \mathcal{F} \) is a fundamental domain \( \Gamma_1^1(m) \backslash \mathcal{H}_1 \times \mathbb{C} \) (so \( D_{F,F,N}(s) = D_{F,F}(s) \)), and put

\[
D_{F,F,m}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F,F,m}(s).
\]

Then we have

\[
\pi^{-k+2} \langle FE^* - k + 2, m' FS \rangle = N^s D_{F,F,m}^*(s).
\]
From (2), (4) and Lemma 1 we have proved

**Proposition 2.** $D_{F,F,m}(s)$ is a Dirichlet series which has a meromorphic continuation to $\mathbb{C}$, possibly has a unique pole at $s = k$, and satisfies a functional equation

$$D^{*}_{F,F}(2k - 2 - s) = D^{*}_{F,F,N}(2k - 2 - s)$$

a finite sum of $\frac{\pm n^{s}}{P(s)}D^{*}_{F,F,m}(s)$

where $m, n$ are natural numbers with $m|N$ and $P(s)$ is a finite product of $1 - \tilde{m}^{2(k-s)}$ with $\tilde{m}|m$.

Now we can use Landau's Theorem for $D_{F,F,m}(s)'s$, because $D_{F,F,m}(s)/(1 - p^{2(k-s)})$ has non-negative coefficients and has a unique pole at $s = k$, hence it converge for $s > k$. Therefore we have

$$\sum_{n \leq x} ||\phi_{n}||^{2} = \left( \text{Res}_{s=k} \frac{D_{F,F}(s)}{s} \right) x^{k} + O_{\epsilon}(x^{k-4/9 + \epsilon}) \quad (\forall \epsilon > 0)$$

where $\phi_{n}$ is the n-th Fourier-Jacobi coefficient of $F(Z)$. Taking $x = m$ and $x = m - 1$ and subtracting, we find

$$||\phi_{m}||^{2} \ll_{\epsilon,F} m^{k-4/9 + \epsilon},$$

hence

$$||\phi_{m}|| \ll_{\epsilon,F} m^{k/2 - 2/9 + \epsilon} \quad (\forall \epsilon > 0). \quad (5)$$

By Proposition 2 and (5), we obtain

$$c(n, r) \ll_{\epsilon,k} (m + |D|^{1/2 + \epsilon})^{1/2} |D|^{5/18 + \epsilon}.$$ 

Both sides of (1) are invariant if $T$ is replaced by $^tUTU \quad (U \in GL_2(\mathbb{Z}))$. Hence we may assume that

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \quad m = \min T,$$

so that $a(T) = c(n, r)$. By reduction theory we have $m = \min T \leq \frac{2}{\sqrt{3}}|D|^{1/2}$ and complete the proof of Theorem.

**Remark.**

1. When $N = 1$, the Rankin convolution series $D_{F,F}(s)$ is a linear combination of spinor zeta functions of Hecke eigen forms, as shown in [K-S]. In order to deduce estimates for eigenvalues of Hecke operators, we need find a relation between $D_{F,F,m}(s)'s$ and spinor zeta functions.
2. When we generalize Kohnen's method to higher genus, we should cut $Z$ as follows;

$$Z = \begin{pmatrix}
* & \ldots & * & * \\
\vdots & \ddots & \vdots & \vdots \\
* & \ldots & * & *
\end{pmatrix}_{\tau'}$$

References


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