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Picard modular surfaces

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1. Introduction

In this paper we give an overview on recent results in [R5] and [R6] where we studied moduli spaces of principally polarized abelian varieties with complex multiplication. In [R5] we studied the special case of complex multiplication by a (polarization preserving) automorphism given by an element M of finite order in the modular group $\Gamma_g = Sp(2g, \mathbb{Z})$.

Then $\mathbb{H}(M) = \{\tau \in \mathbb{H}_g ; M < \tau >= \tau\}$ is a connected complex submanifold of the Siegel upper half space $\mathbb{H}_g$. Let Γ_M be the centralizer of M in the modular group Γ_g. We call M the Picard type, because such varieties occur in the papers [P1],[P2],[P3] by Picard. Then $\Gamma_M \backslash \mathbb{H}(M)$ is the associated Picard moduli variety of Picard type M. Moreover, we study the Satake compactification, i.e. the closure $\mathcal{A}(M)$ of $\Gamma_M \backslash \mathbb{H}(M)$ in the Satake compactification $\mathcal{A}_g = Proj(A(\Gamma_g))$ of $\Gamma_g \backslash \mathbb{H}_g$. Similar to modular forms we define modular forms of Picard type M. The ring $A(M)$ of modular forms of Picard type M defines $\mathcal{A}(M) = Proj(A(M))$ algebraically. For $g \geq 3$ the union of Picard varieties $\Gamma_M \backslash \mathbb{H}(M)$ for the finitely many such M cover the singular locus of $\Gamma_g \backslash \mathbb{H}_g$.

If M satisfies an equation $M^2 + aM + 1 = 0$ $(a = 0, 1)$, the group Γ_M is a unitary group (of some signature (p, q) with $p + q = g$) usually denoted by $U(p, q; R) \subset \Gamma_g$, where $R = \mathbb{Z}[i]$ for $a = 0$ or $R = \mathbb{Z}[\rho]$ (ρ a third root of unity) for $a = 1$. The ring $R = \mathbb{Z}[\rho]$ is called ring of Eisenstein numbers. However, as explained below, the (conjugacy class of the) element M contains more precise information than only the signature (p, q) of the associated hermitian form.

In the elliptic case the modulus $\tau = \exp(2\pi i/6)$ of the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with $j = 0$ is just $\mathcal{A}(M) = \{\tau\}$ where $M = M_+ = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. This M defines a hermitian form of signature $(1,0)$, and $\Gamma_M \cong U(1, 0; \mathbb{Z}[\rho])$. Generalizing this example, any $M \in \Gamma_g$ with $M^2 + M + 1 = 0$ defines an isomorphism $\Gamma_M \cong U(p, q; \mathbb{Z}[\rho])$. The case $(p, q) = (2, 1)$ was studied by Picard. As a complex manifold $\mathbb{H}(M)$ is $SU(p, q; \mathbb{C})/SU(p; \mathbb{C}) \times SU(q; \mathbb{C})$. This is a pq-dimensional manifold. For $(p, q) = (p, 1)$ the Picard variety $\mathcal{A}(M)$ is just a compactification of a p-dimensional ball quotient. We refer to [Sa], [Sh1,2], [Ho1,2] for the general theory.

An important special case with $M^2 + 1 = 0$ is the case of hermitian modular forms. These were studied by Braun [B], Freitag [F1], Matsumoto ([Ma1],[Ma2]), and others ([MSY1],[MSY2]). We give an example of non-conjugate M_i with $M_i^2 + 1 = 0$ and non-isomorphic rings of modular forms $A(M_i)$, to show that the notion "modular form for $U(g, g; \mathbb{Z}[i])$" depends on the embedding of $U(g, g; \mathbb{Z}[i])$ in Γ_{2g}.

A more general approch is given in [R6]. There we constructed an algebraic model for Shimura varieties with a given algebra $L = End(A_\tau) \otimes \mathbb{Q}$ of endomorphisms. It turns out that there is a canonical modular embedding induced by the rational representation for all Shimura varieties of PEL-type. This approach in the case of abelian 3-folds leeds to CM-surfaces usually called Picard modular surfaces. The group of automorphisms preserving the polarization is just $G = L \cap \Gamma_g$. It is usually not true that G generates L as an algebra. However, the case considered by Picard is such a special case.

Finally we compute the ring of Picard modular forms in the case considered by Picard, thus finishing the computation of v. Geemen [G]. This computation implies, that the field of modular function is rational in this case. Rationality was proved by Shimura [Sh2]. Shiga [S] had computed this field by a different method. The ring of modular forms of a certain Nebentypus was studied by Holzapfel [Hol].

2. Notations and first results

Throughout the paper we will use the same notation as in [R1],[R2]. For general facts we refer to [I], [Kr]. So let

$$\begin{aligned}
&\mathbb{H}_g = \{ \tau \in Mat_{g\times g}(\mathbb{C}) |\ \tau \text{ symmetric },\ Im(\tau) > 0\},\\
&\Gamma_g = Sp(2g, \mathbb{Z}),\\
&\Gamma_g(n) = Ker\ (\ \Gamma_g \to Sp\ (\ 2g, \mathbb{Z}/n)).
\end{aligned}$$

Let Γ be a subgroup of finite index of Γ_g. Denote by $A(\Gamma) = \bigoplus_k [\Gamma, k]$ the ring of modular forms for Γ. Let $\mathcal{A}_g(\Gamma) = Proj(A(\Gamma))$ be the corresponding Satake compactification; it contains $\Gamma \backslash \mathbb{H}_g$ as an open dense subset. The open part $\Gamma\backslash\mathbb{H}_g$ is the coarse moduli space for principally polarized abelian varieties with level-Γ structure.

The thetas (of second kind) are given by (we use Mumford's notation f_a)

$$f_a(\tau) = \theta \begin{bmatrix} a \\ 0 \end{bmatrix} (2\tau) = \sum_{x\in\mathbb{Z}^g} \exp 2\pi i \left(\tau \left[x + \frac{1}{2}\ a \right] \right)$$

for $a \in \mathbb{Z}^g$. The functions $f_a(\tau)$ only depend on a mod 2 hence a is regarded as element in $\mathbb{F}_2^g$.

The group $Sp(2g, \mathbb{R})$ acts on $\mathbb{H}_g$ by

$$\sigma < \tau >= (A\tau + B)(C\tau + D)^{-1} \text{ for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}) \text{ and } \tau \in \mathbb{H}_g.$$

This action induces for any $k \in \mathbb{Z}$ a (right) group action on the algebra of holomorphic functions $\{f : \mathbb{H}_g \to \mathbb{C}\}$ by

$$f|_k\sigma(\tau) = \det(C\tau + D)^{-k} f(\sigma < \tau >).$$

A holomorphic function f on $\mathbb{H}_g$ is a modular form of weight k, or in short $f \in [\Gamma_g, k]$, iff $f|_k\sigma = f$ for all $\sigma \in \Gamma_g$. In genus $g = 1$ one has to add a condition for the cusps.

It is well known that the group Γ_g is generated by $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\sigma_S = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ where S runs over all symmetric $g \times g$-matrix and $1 \in Gl(g, \mathbb{Z})$ ([F2]). If we allow for a moment half-integral weight, the modular group Γ_g acts on the thetas by

$$f_a|_{1/2}\sigma_S = i^{S[a]}\ f_a$$
$$f_a|_{1/2}J = \sum_{b\ \in \mathbb{F}_2^g} (T_g)_{a,b}\ f_b$$

where $T_g \in Gl(2^g, \mathbb{C})$ is the matrix

$$T_g = \left(\frac{1+i}{2}\right)^g \left((-1)^{<a,b>}\ \right)_{a,b\in\ \mathbb{F}_2^g}.$$

The equation for J is independent of the choice of the square root $\sqrt{det(-\tau)}$ on the 2-ring, i.e. on the ring $\mathbb{C}[f_a(\tau)]_{(2)} = \{f \in \mathbb{C}[f_a(\tau)] \text{ with } 2|deg(f)\}$. The correct square root is $\sqrt{\det(\tau/i)}$ which is here replaced by $\epsilon^g\sqrt{\det(-\tau)}$ with $\epsilon = \frac{1+i}{\sqrt{2}}$ a primitive 8^{th} root of unity, see [F2]. We use the equation only for mixed products, hence it is well defined.

Take $D_S\ = \operatorname{diag}(\ i^{S[a]}$ for $a\ \in\ \mathbb{F}_2^g)$ and let

$$H_g =<\ T_g,\ D_S \text{ for symmetric } S \in M_g(\mathbb{Z})\ >$$

be the finite subgroup of $Gl\ (2^g, \mathbb{C})$ generated by the elements T_g and all D_S. If we map J to T_g and σ_S to D_S we get the theta representation (of index 1, [R3]).

$$\rho_{theta}\ :\ \Gamma_g \longrightarrow H_g\ /\ (\pm 1).$$

The kernel, denoted by $\Gamma_g^*(2,4)$, is described in [R1].

We recall from [R1] that the ring of modular forms of even weight is given by

$$A(\Gamma_g)_{(2)} = \bigoplus_{2|k}[\Gamma_g, k]\ =\ (\mathbb{C}[f_a(\tau)]^{H_g})^N.$$

Here N denotes the normalization (in its field of fractions). Moreover, $A(\Gamma_1) = \mathbb{C}[f_a(\tau)]^{H_1}$, $A(\Gamma_2)_{(2)} = \mathbb{C}[f_a(\tau)]^{H_2}$ and $A(\Gamma_3) = \mathbb{C}[f_a(\tau)]^{H_3}$. We use binary numbers to index the thetas, i.e. (in genus $g = 2$) $f_0 = f_{\substack{0\\0}}$, $f_1 = f_{\substack{1\\0}}$, $f_2 = f_{\substack{0\\1}}$ and $f_3 = f_{\substack{1\\1}}$.

The Siegel Φ-operator may be defined as follows. Siegel modular forms in even weight are always rational functions in the theta constants of second kind. On them, the Φ-operator is given by

$$\Phi(f_{\substack{a\\0}}) = f_a \text{ and } \Phi(f_{\substack{a\\1}}) = 0.$$

(Here a is considered as element in $\mathbb{F}_2^g$ and $\substack{a\\ *}$ as an element in $\mathbb{F}_2^{g+1}$.)

The important new ingredient for Picard modular forms is to consider a fixed Picard type, i.e. an element $M \in \Gamma_g$ of finite order. The element M is conjugate in $Sp(2g, \mathbb{C})$ to an element $\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}$ with $U = \operatorname{diag}(\zeta_1, \ldots, \zeta_g)$, where ζ_i are roots of unity. Hence $(\zeta_1, \ldots, \zeta_g, \bar{\zeta}_1, \ldots, \bar{\zeta}_g)$ are just the eigenvalues of M. The characteristic polynomial of M is a product of cyclotomic polynomials. The dimension of $\mathbb{H}(M)$ is given by ([F2], p. 197)

$$\dim(\mathbb{H}(M)) = \#\{(i,j); i \le j \text{ and } \zeta_i\zeta_j = 1\}.$$

Definition. A holomorphic function f on $\mathbb{H}_g$ is a Picard modular form of weight k and Picard type M, or in short $f \in [\Gamma_M, k]$, iff $f|_k\sigma = f$ for all $\sigma \in \Gamma_M$.

(If $M \in \Gamma_1$ one has to add a cusp condition, but this case is not interesting for our purpose. The varieties $\mathcal{A}(M)$ are just points for $M \neq 1$.) Denote by $A(M) = \bigoplus_k [\Gamma_M, k]$ the ring of Picard modular forms of Picard type M and by $\mathcal{A}(M) = Proj(A(M))$ the corresponding Satake compactification.

Strictly speaking it is very easy to compute the ring of Picard modular forms. Choose some $\tau \in \mathbb{H}(M)$, put $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and let $\bar{M} = (m_{a,b})_{a,b\in\mathbb{F}_2^g}$ be an image of M in H_g. Then

$$f_a|_{1/2}M(\tau) = \sum_{b\in\mathbb{F}_2^g} m_{a,b} f_b(\tau) = \frac{1}{\sqrt{det(C\tau + D)}} f_a(M < \tau >)$$

produces relations for the restricted functions $\widetilde{f_a(\tau)}$ on $\mathbb{H}(M)$. Because $\mathbb{H}(M)$ is connected, the square root $\sqrt{det(C\tau + D)}$ may be chosen independent of τ. The induced relations on the 2-ring are independent of the choice of the square root. Let $G(M) = \{\sigma \in H_g; \sigma\bar{M} = \bar{M}\sigma\}$ be the centralizer of $\bar{M}$ in H_g. The group $G(M)$ only depends on M. The element i is always contained in $G(M)$, therefore we always get Picard modular forms of even weight. The following theorem is a consequence of the corresponding result for the group $\Gamma_g^*(2,4)$.

Theorem 1. *The ring of Picard modular forms of even weight is given by*

$$A(M)_{(2)} = \bigoplus_{2|k} [\Gamma_M, k] \;=\; (\mathbb{C}[\widetilde{f_a(\tau)}]^{G(M)})^N.$$

The problem is to find all relations and to compute the normalization. However, for small g normalization is often not necessary, hence there remains only the problem of finding all relations.

We consider now the special case, that M satisfies an equation $M^2 + aM + 1 = 0$ with $a = 0$ or $a = 1$. Hence M is an element of order 3 or 4. Let ω be a root of the equation $\omega^2 + a\omega + 1 = 0$. We may choose $\omega = i = \sqrt{-1}$ for $a = 0$ and $\omega = \rho = \exp(2\pi i/3)$ for $a = 1$. In both cases $\mathbb{Z}[\omega]$ is the ring of integers in the quadratic number field $\mathbb{Q}(\omega)$, and it is a principal ideal domain. One defines an $\mathbb{Z}[\omega]$-module structure on $\mathbb{Z}^{2g}$ by $\omega x = Mx$. This induces an isomorphism $\mathbb{Z}^{2g} \cong \mathbb{Z}[\omega]^g$. The associated $\mathbb{Z}$-bilinear form

$$\mathbb{Z}[\omega]^g \times \mathbb{Z}[\omega]^g \to \mathbb{Z}[\omega]$$

which is defined by

$$< x, y >= {}^t xJMy - \omega^t xJy \text{ for } x, y \in \mathbb{Z}^{2g}$$

turns out to be a hermitian form ($< Mx, y >= \omega < x, y >$ and $\overline{< y, x >} =< x, y >$ are easy to check). This leads to the (well known) equality

$$U(M; \mathbb{Z}[\omega]) = \{\sigma \in Gl(g, \mathbb{Z}[\omega]); < \sigma x, \sigma y >=< x, y >\} = \Gamma_M = \{\sigma \in \Gamma_g; \sigma M = M\sigma\}.$$

For any concrete M one easily computes the signature (p, q) with $p + q = g$ of (the $\mathbb{R}$-linear extension of) this hermitian form and denotes $U(M; \mathbb{Z}[\omega]) = U(p, q; \mathbb{Z}[\omega])$. For $(p, q) = (p, 1)$ a hermitian form in a suitable basis is given by $|z_1|^2 + \ldots + |z_p|^2 - |z_{p+1}|^2$. A subspace of $\mathbb{C}^{p+1}$ on which the form is negativ definite, is spanned by a unique $(z_1, \ldots, z_p, 1)$ with $\sum |z_i|^2 < 1$. Hence the variety $\mathbb{H}(M)$ is isomorphic to the p-dimensional ball in $\mathbb{P}^p$ and $U(M; \mathbb{Z}[\omega])$ is an arithmetic subgroup of $U(p, q; \mathbb{C})$.

3. Hermitian modular forms

An important special case (of Picard modular forms) are the hermitian modular forms, which belong to the Picard type

$$M = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \in \Gamma_{2g},$$

where $1 \in GL(g, \mathbb{Z})$. Usually one considers the hermitian upper half space

$$\mathbb{HH}_g = \{X + iY \in Mat_g(\mathbb{C}); Y > 0\},$$

where $X = {}^t\overline{X}$ and $Y = {}^t\overline{Y}$ are the "hermitian real and imaginary part". One gets a sequence of inclusions

$$\mathbb{H}_g \subset \mathbb{HH}_g \subset \mathbb{H}_{2g}.$$

The first inclusion is the natural one. The second inclusion is given by

$$Z \mapsto \widetilde{Z} = \frac{1}{2}\begin{pmatrix} Z + {}^tZ & i(Z - {}^tZ) \\ -i(Z - {}^tZ) & Z + {}^tZ \end{pmatrix}.$$

These inclusions are equivariant under the inclusions of groups

$$\Gamma_g \subset \Gamma_g(\mathbb{Z}[i]) \subset \Gamma_{2g},$$

where $\Gamma_g(\mathbb{Z}[i])$ is the hermitian modular group

$$\Gamma_g(\mathbb{Z}[i]) = \{\sigma \in Mat_{2g}(\mathbb{Z}[i]); {}^t\overline{\sigma}J\sigma = J\}.$$

Again the first inclusion is the natural one. The second inclusion is given by

$$\begin{pmatrix} A + i\alpha & B + i\beta \\ C + i\gamma & D + i\delta \end{pmatrix} \mapsto \begin{pmatrix} A & -\alpha & B & -\beta \\ \alpha & A & \beta & B \\ C & -\gamma & D & -\delta \\ \gamma & C & \delta & D \end{pmatrix}.$$

One checks easily that the image is just $\Gamma_M \subset \Gamma_{2g}$ using the known generators of $\Gamma_g(\mathbb{Z}[i])$ (see [F1]). The automorphism group $Aut(\mathbb{HH}_g)$ is given by

$$Aut(\mathbb{HH}_g) = \Gamma_g(\mathbb{Z}[i])^{sym} = \Gamma_g(\mathbb{Z}[i]) \rtimes < T >,$$

where $T : Z \mapsto {}^tZ$ acts as transpose. This corresponds to the element $T = \mathrm{diag}(1, -1, 1, -1) \in \Gamma_{2g}$ and $\Gamma_g(\mathbb{Z}[i])^{sym} = \langle \Gamma(M), T \rangle = \{g \in \Gamma_{2g}; gM = \pm Mg\}$. As explained above, the hermitian modular group $\Gamma_g(\mathbb{Z}[i])$ is usually denoted by $U(g, g; \mathbb{Z}[i])$, however, the Picard type M is not explicit in this notation, therefore we avoid that.

Because of $\det(\widetilde{Z}) = \det(Z)^2$ is it usual to change the weight convention and call a holomorphic function f on $\mathbb{H}_{2g}$ a hermitian modular form of weight $2k$, or in short $f \in [\Gamma_g(\mathbb{Z}[i]), 2k]$ iff $f|_k\sigma = f$ for all $\sigma \in \Gamma_M$. Hence the restriction of a modular form $f \in [\Gamma_{2g}, k]$ on $\mathbb{HH}_g$ is a hermitian modular form of weight $2k$. Further restriction to $\mathbb{H}_g$ leads to $\Phi(f)^2 \in [\Gamma_g, 2k]$.

As an easy example we compute the ring of hermitian modular forms in genus $g = 1$, i.e. we consider the Picard type

$$M = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \in \Gamma_2 = Sp(4,\mathbb{Z}).$$

For a description of H_2 we refer to [R2]. The element $\bar{M}$ is just the permutation

$$\bar{M} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \in H_2,$$

the centralizer G of $\bar{M}$ in H_2 is a group of order 768 (using the decomposition of Bruhat type in H_2). The relations (given by M) are only

$$f_1 = f_2.$$

On $\mathbb{C}[f_0, f_1, f_3]$ the centralizer G acts as a group $\bar{G}$ generated by (pseudo) reflections of order 384.

$$\bar{G} = \left\langle \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix}, \begin{pmatrix} 1 & & \\ & i & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \left(\frac{1}{2}\right) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \right\rangle$$

The ring of invariants is easily computed as

$$\mathbb{C}[f_0, f_1, f_3]^G = \mathbb{C}[P_4, P_8, P_{12}]$$

with

$$\begin{aligned} P_{12} &= f_0^{12} + f_3^{12} - 64 f_1^{12} \\ &\quad - 66 f_1^4 (f_0^8 + f_3^8) - 33 f_0^4 f_3^4 (f_0^4 + f_3^4) \\ &\quad + 264 f_1^8 (f_0^4 + f_3^4) + 660 f_0^4 f_1^4 f_3^4 \\ &\quad + 792 f_0^2 f_1^4 f_3^2 (f_0^4 + 2 f_1^4 + f_3^4) \\ P_8 &= f_0^8 + 128 f_1^8 + f_3^8 + 28 f_0^6 f_3^2 + 70 f_0^4 f_3^4 + 28 f_0^2 f_3^6 \\ P_4 &= f_0^4 + 8 f_1^4 + f_3^4 + 6 f_0^2 f_3^2 \end{aligned}$$

For dimension reasons we need one more relation. The equation $M < \tau >= \tau$ implies $\tau = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, hence as a point set $\mathbb{H}_1 = \mathbb{HH}_1$, diagonally embedded in $\mathbb{H}_2$. The equation for decomposable points

$$f_1 f_2 = f_0 f_3$$

is the one more relation we need. By ([R1], theorem 2.8.) $\bigoplus[\Gamma_2^*(2,4)\ , k] = \mathbb{C}[f_a(\tau)]_{(2)}[\Theta]$ with $\Theta \ = \prod_{even\ m} \theta_m$. The restriction of Θ on $\mathbb{HH}_1$ vanishes due to

$$\theta^2 \begin{bmatrix} 1,1 \\ 1,1 \end{bmatrix} (\tau) = 2(f_{\substack{0\\0}} f_{\substack{1\\1}} - f_{\substack{1\\0}} f_{\substack{0\\1}}) = 2(f_0 f_3 - f_1 f_2) = 0.$$

Hence the final result is (well known, compare with [B] or [K])

Theorem 2. *The ring of hermitian modular forms in genus* 1 *is*

$$\bigoplus_k [\Gamma_1(\mathbb{Z}[i]), 2k] = \mathbb{C}[\phi_4, \phi_{12}] \cong \bigoplus_k [\Gamma_1, 4k],$$

with

$$\phi_4 = f_0^4 + f_3^4 + 14 f_0^2 f_3^2,$$
$$\phi_{12} = f_0^2 f_3^2 (f_0^2 - f_3^2)^4.$$

Proof. We have $P_8 = P_4^2$, $108\phi_{12} = P_4 P_8 - P_{12}$ on $\mathbb{HH}_1$. □

In genus 2 the ring was computed by Freitag [F1].

Theorem 3. *The ring of hermitian modular forms in genus 2 is*

$$\bigoplus_k [\Gamma_2(\mathbb{Z}[i]), 2k] = \mathbb{C}[\phi_4, \phi_8, \Theta_{10}, \phi_{12}, \phi'_{12}, \phi_{16}],$$

where $\phi_i \in [\Gamma_4^*(2,4), i/2]$ are polynomials in $f_a(\tau)$ of degree i and Θ_{10} is the restriction of the product

$$\Theta_{10} = \prod_{a,b\in\mathbb{F}_2, <a,b>=0} \theta \begin{bmatrix} a,a \\ b,b \end{bmatrix} (\tau) \in [\Gamma_4(4,8), 5].$$

Moreover, there is a relation $\chi = \Theta_{10}^2 P_3$, where P_3 and χ are polynomials in the ϕ_i and $deg(P_3) = 12$. The field of modular functions for $\Gamma_2(\mathbb{Z}[i])$ is rational.

It was observed by van Geemen ([G],10.7.), that the theta map Th defined by the f_a factors as follows

$$\begin{array}{ccc} \mathcal{A}(\Gamma_{2g}^*(2,4)) & \xrightarrow{Th} & \mathbb{P}^{2^{2g}-1} \\ \cup & & \cup \\ \mathcal{A}(\Gamma_g(\mathbb{Z}[i]) \cap \Gamma_{2g}^*(2,4)) & \longrightarrow & \mathbb{P}^{(2^{2g-1}+2^{g-1}-1)} \end{array}$$

where $\mathbb{P}^{(2^{2g-1}+2^{g-1}-1)}$ is defined by the $\binom{2^g}{2} = 2^{g-1}(2^g - 1)$ linear relations produced by M

$$f_{\substack{a\\b}}(\tau) = f_{\substack{b\\a}}(\tau) \text{ for } a, b \in \mathbb{F}_2^g.$$

Moreover, the image of $\mathcal{A}(\Gamma_2(\mathbb{Z}[i]) \cap \Gamma_4^*(2,4))$ in $\mathbb{P}^9$ is a complete intersection of five quadrics ([G],10.11.).

It is often useful to consider the diagram

$$\begin{array}{ccccccccc} 0 \to & \Gamma_{2g}(2) & \longrightarrow & \Gamma_{2g} & \longrightarrow & Sp(4g, \mathbb{F}_2) & \to & 0 \\ & \cup & & \cup & & \cup & & \\ 0 \to & \Gamma_g(\mathbb{Z}[i], 1+i) & \longrightarrow & \Gamma_g(\mathbb{Z}[i]) & \longrightarrow & Sp(2g, \mathbb{F}_2) & \to & 0 \\ & \cup & & \cup & & \| & & \\ 0 \to & \Gamma_g(2) & \longrightarrow & \Gamma_g & \longrightarrow & Sp(2g, \mathbb{F}_2) & \to & 0 \end{array}$$

In the middle exact sequence the group $\Gamma_g(\mathbb{Z}[i], 1+i)$ is defined as the kernel of the natural map $\Gamma_g(\mathbb{Z}[i]) \to Sp(2g, \mathbb{F}_2)$ induced by $\mathbb{Z}[i]/(1+i) \cong \mathbb{F}_2$. Matsumoto computed the ring of hermitian modular forms for $\Gamma_2(\mathbb{Z}[i], 1+i)$ for a certain multiplier system ([Ma1]). Using this diagram, one may compute the ring of hermitian modular forms (for a certain multiplier system) for $\Gamma_2(\mathbb{Z}[i])$ by taking the invariants under $Sp(4, \mathbb{F}_2) \cong S_6$.

4. An example

As a second example we compute the ring of Picard modular forms with Picard type

$$M = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix} \in \Gamma_2 = Sp(4,\mathbb{Z}).$$

Again the hermitian form associated to M has signature $(1,1)$. The element $\bar{M}$ can be shown to be

$$\bar{M} = \left(-\frac{1}{2}\right)\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \in H_2.$$

The centralizer G is a group of order 128 (using the decomposition of Bruhat type in H_2). The relations (given by M) are

$$f_3 = -f_0, f_2 = -2f_0 - f_1.$$

On $\mathbb{C}[f_0, f_1]$ the centralizer G acts as a group $\bar{G}$ generated by (pseudo) reflections of order 32.

$$\bar{G} = \left\langle \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}, \left(\frac{1}{2}\right)\begin{pmatrix} 1-i & 1-i \\ 1+3i & -1+i \end{pmatrix} \right\rangle.$$

The ring of invariants is easily seen to be

$$\mathbb{C}[f_0, f_1]^{\bar{G}} = \mathbb{C}[P_4, P_8]$$

with

$$\begin{aligned} P_8 &= 45f_0^8 + f_1^8 + 98f_0^4 f_1^4 \\ &\quad + 120f_0^7 f_1 + 8f_0 f_1^7 \\ &\quad + 196f_0^6 f_1^2 + 28f_0^2 f_1^6 \\ &\quad + 168f_0^5 f_1^3 + 56f_0^3 f_1^5 \\ P_4 &= 3f_0^4 - f_1^4 - 4f_0^3 f_1 - 4f_0 f_1^3 - 6f_0^2 f_1^2. \end{aligned}$$

Hence the final result is

$$\bigoplus_k [\Gamma_M, k] = \mathbb{C}[P_4, P_8].$$

Therefore we have an example of non-conjugate M_i with $M_i^2 + 1 = 0$ and non-isomorphic rings of modular forms $A(M_i)$ with signature $(1,1)$. Hence the notion "modular form for $U(1,1;\mathbb{Z}[i])$" depends on the embedding of $U(1,1;\mathbb{Z}[i])$ in Γ_2. This phenomenon may be explained as an example of different automorphy factors for the domain $\mathbb{H}_1$. A geometric explanation will be given in the next section.

5. Picard modular forms and moduli spaces of abelian varieties

We recall some standard facts in an explicit form. For proofs we refer to [M]. For any τ in $\mathbb{H}_g$ we have the lattice $\Lambda_\tau = \mathbb{Z}^g + \tau\mathbb{Z}^g$ in $\mathbb{C}^g$ and the abelian variety $A_\tau = \mathbb{C}^g/\Lambda_\tau$. Moreover we have the action of Λ_τ on $\mathbb{C}^g \times \mathbb{C}$ with the automorphy factor

$$e_\tau(z, \lambda) = \exp -2\pi i \left(\frac{1}{2}\tau[\alpha] + < \alpha, z >\right)$$

for $\lambda = \tau\alpha + \beta \in \Lambda_\tau$ via

$$\lambda(z,t) = (z + \tau\alpha + \beta, e_\tau(z,\lambda)t).$$

The cocycle $\{e_\tau(\ ,\ )\} \in H^1(A_\tau, \mathcal{O}_{A_\tau})$ defines an ample symmetric line bundle $L_\tau = \mathbb{C}^g \times \mathbb{C}/\Lambda_\tau$ over A_τ. One has $H^0(A_\tau, L_\tau) = \mathbb{C}\theta$ where $\theta = \theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\tau,z)$ is the Riemann theta function. More generally the addition of a 2-torsion point $t_m : z \mapsto z + \frac{1}{2}(\tau\alpha+\beta) = z + \frac{1}{2}\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ gives an isomorphism of A_τ with $H^0(A_\tau, t_m^* L_\tau) = \mathbb{C}\theta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](\tau,z)$. The space $\mathbb{H}_g/\Gamma_g$ is a coarse moduli space for the pairs (A_τ, L_τ) which are called principally polarized abelian varieties of dimension g (ppav for short).

If we write an arbitrary cocycle as

$$e(z,\lambda) = \exp 2\pi i(f(z,\lambda))$$

and set

$$E(\lambda,\mu) = f(z+\lambda,\mu) + f(z,\lambda) - f(z+\mu,\lambda) - f(z,\mu)$$

we get the (alternating) Riemann form

$$E : \Lambda_\tau \times \Lambda_\tau \to \mathbb{Z},$$

of $L = \{e(z,\lambda)\} \in Pic(A_\tau)$. The map $e(\ ,\ ) \mapsto E$ is just the Chern class map

$$Pic(A_\tau) \twoheadrightarrow NS(A_\tau) \subset H^2(A_\tau, \mathbb{Z}).$$

The principal polarization is mapped to the standard alternating form

$$E_\tau(\lambda,\mu) = <x_1, y_2> - <x_2, y_1> \quad \text{for } \lambda = \tau x_1 + x_2 \text{ and } \mu = \tau y_1 + y_2.$$

One extends the Riemann form to an $\mathbb{R}$-bilinear form on $\mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$. This form satisfies

$$E(i\lambda, i\mu) = E(\lambda,\mu)$$

and defines the (hermitian) Riemann form

$$H(\lambda,\mu) = E(i\lambda,\mu) + iE(\lambda,\mu).$$

One easily computes (for $\lambda = \tau x_1 + x_2$)

$$H_\tau(\lambda,\lambda) = E_\tau(i\lambda,\lambda) = <\Im(\tau)^{-1}(\Re(\tau)x_1 + x_2), (\Re(\tau)x_1 + x_2)> + <\Im(\tau)x_1, x_1>,$$

hence $H_\tau(\ ,\ )$ is positive definite.

An endomorphism $\phi \in End(A_\tau)$ is given by

$$\phi = A + \tau C \text{ with } \phi\tau = B + \tau D$$

where $A, B, C, D \in M_g(\mathbb{Z})$ and one easily checks that

$$End(A_\tau) \ni \phi \mapsto M_\phi = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \in Mat_{2g}(\mathbb{Z})$$

is an $\mathbb{Z}$-algebra embedding. On $Mat_{2g}(\mathbb{Z})$ we have the Rosati anti-involution, defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \hat{M} = J^{-1\,t}MJ = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}.$$

This map satisfies

$$\hat{\hat{M}} = M \text{ and } (\widehat{M_1 M_2}) = \widehat{M_2}\widehat{M_1},$$

hence is an anti-involution. The endomorphism $\widehat{\phi}$ is defined by

$$M_{\widehat{\phi}} = \widehat{M_\phi}.$$

For $g = 1$ the endomorphisms $\widehat{\phi}$ may be defined as the unique isogeny with $\phi\widehat{\phi} = \widehat{\phi}\phi = deg(\phi)$. Moreover, it is easily verified, that

$$E_\tau(\widehat{\phi}\ ,\ ) = E_\tau(\ ,\phi\ ) \text{ and } H_\tau(\widehat{\phi}\ ,\ ) = H_\tau(\ ,\phi\ ),$$

hence $\widehat{\phi}$ is adjoint to ϕ with respect to E_τ and H_τ. We denote by

$$Aut(A_\tau, L_\tau) = \{\sigma \in Sp(2g,\mathbb{Z}); \sigma < \tau >= \tau\}$$

the finite group of isomorphisms preserving the Riemann form(s). Because of

$${}^t(C\tau + D)^{-1} = A - \sigma < \tau > C \text{ for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{R})$$

the action of the endomorphism ring is in accordance with the action of $Sp(2g,\mathbb{R})$ on $\mathbb{H}_g \times \mathbb{C}^g$ by

$$\sigma(\tau, z) = (\sigma < \tau >, {}^t(C\tau + D)^{-1} z).$$

Instead of E_τ we may consider the classes $\mathbb{N}E_\tau$, i.e. we call two Riemann forms equivalent iff $n_1 E_1 = n_2 E_2$. Let $\mathbb{N} = \{n \in \mathbb{Z}; n > 0\}$ and

$$GSp(2g,\mathbb{Z}) = \{\sigma \in Mat_{2g}(\mathbb{Z}); {}^tMJM = n_M J,\ n \in \mathbb{N}\}.$$

The semi-group of isogenies preserving the class $\mathbb{N}E_\tau$ (a homogeneous polarization) is given by

$$End(A_\tau, [L_\tau]) = \{\sigma \in GSp(2g,\mathbb{Z}); \sigma < \tau >= \tau\}.$$

For $g > 1$ an endomorphism need not preserve a homogeneous polarization. We have

$$deg(\phi) = det(M_\phi)$$

for the degree of an endomorphism. Moreover, the positivity of the Riemann form implies, that

$$Tr(\hat{\phi}\phi) = \left(\frac{1}{2}\right) Tr(M_{\hat{\phi}} M_\phi) = Tr(A^t D - C^t B)$$

is positive for $\phi \neq 0$. The positivity of the Rosati involution is essential for the study of the algebra of complex multiplications $End^0(A_\tau) = End(A_\tau) \otimes_{\mathbb{Z}} \mathbb{Q}$ of A_τ. This is a semi-simple algebra and was classified by Albert (see [M],p. 201). The Neron-Severi group $\mathcal{NS}(A_\tau)$ (the image of $Pic(A_\tau)$ in $H^2(A_\tau,\mathbb{Z})$, which can be characterized as the group of alternating Riemann forms E with $E(i\lambda, i\mu) = E(\lambda,\mu)$) is a free abelian group of finite rank. One can embed $\mathcal{NS}(A_\tau) \subset End(A_\tau)$ as follows: Write

$$E(\lambda,\mu) = E_\tau(\lambda, \sigma\mu)$$

for some $\sigma \in M_{2g}(\mathbb{Z})$. Then

$$E_\tau(\lambda, \widehat{\sigma}\mu) = E_\tau(\sigma\lambda,\mu) = -E_\tau(\mu,\sigma\lambda) = -E(\mu,\lambda) = E(\lambda,\mu) = E_\tau(\lambda,\sigma\mu)$$

implies, that $E \mapsto \sigma$ actually lands in $End(A_\tau)^{Rosati}$. This induces the equality

$$NS^0(A_\tau) = NS(A_\tau) \otimes_{\mathbb{Z}} \mathbb{Q} = End^0(A_\tau)^{Rosati},$$

which is a formally real Jordan algebra over $\mathbb{Q}$ of dimension $b(\tau)$ (called the base number). The endomorphism ring $End(A_\tau)$ acts on $NS(A_\tau)$ via

$$\phi^* E(\lambda,\mu) = E(\phi\lambda,\phi\mu) = E_\tau(\phi\lambda,\sigma\phi\mu) = E_\tau(\lambda,\widehat{\phi}\sigma\phi\mu).$$

Hence, considered in $End(A_\tau)^{Rosati}$, the action is given by $\phi^*(\sigma) = \widehat{\phi}\sigma\phi$, in particular $n^*(\sigma) = n^2\sigma$ for $n \in \mathbb{Z}$. We collect the above information by

Proposition 4. *The space $\Gamma_M\backslash\mathbb{H}(M)$ is a coarse moduli space for principally polarized abelian varieties (A_τ, L_τ) with a polarization preserving automorphism $M \in \Gamma_g$. For any $\tau \in \mathbb{H}(M)$ the algebra $\mathbb{Z}[M]$ is contained in $End(A_\tau)$.*

The proposition explains, why the conjugacy class of the element M is intrinsically defined. Hence the embedding $\mathbb{H}(M) \hookrightarrow \mathbb{H}_g$ is not just an arbitrary modular embedding, but is defined by the moduli problem.

If the characteristic polynomial of M is irreducible, $\Gamma_M\backslash\mathbb{H}(M)$ is zero-dimensional and $L = \mathbb{Q}[M]$ is a number field of degree $2g$, which is totally imaginary over the totally real number field L^{Rosati} of degree g.

6. algebraic families of pricipally polarized abelian varieties

For our purpose it turns out to be convenient to consider an algebraic model of the Siegel upper half space. A period matrix τ induces by

$$\phi_\tau \begin{pmatrix} x \\ y \end{pmatrix} = x - \tau y$$

an isomorphism

$$\phi_\tau : \mathbb{R}^{2g} \to \mathbb{C}^g$$

and

$$M_\tau = \phi_\tau^{-1} i \phi_\tau$$

defines the corresponding complex structure on $\mathbb{R}^{2g}$. As a matrix we have

$$\mathbb{H}_g \ni \tau = p + iq \mapsto M_\tau = \begin{pmatrix} -pq^{-1} & q + pq^{-1}p \\ -q^{-1} & q^{-1}p \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Proof. We have $i = \tau q^{-1} - pq^{-1}$, hence

$$i(x - \tau y) = (pq^{-1}py + qy - pq^{-1}x) - \tau(q^{-1}py - q^{-1}x)$$

□

The matrices M_τ are elements in $Sp(2g,\mathbb{R})$ and satisfy $\widehat{M_\tau} = -M_\tau = {M_\tau}^{-1}$. Instead of M_τ we furthermore define $S_\tau = -M_\tau J$ and get an isomorphism of complex manifolds

$$\boxed{\mathbb{H}_g \cong \{S_\tau \in Sp(2g,\mathbb{R}); S_\tau \textbf{ symmetric and positive definite } \}}$$

In terms of matrices the bijection is given by

$$\mathbb{H}_g \ni \tau = p + iq \mapsto S_\tau = \begin{pmatrix} q + pq^{-1}p & pq^{-1} \\ q^{-1}p & q^{-1} \end{pmatrix}$$

The proof is easy and therefore omitted. We call this the real (or algebraic) model of the Siegel upper half space. The Rosati anti-involution restricts to an involution on $\mathbb{H}_g$ (in the standard model $\tau \mapsto -\tau^{-1}$).

The group $Sp(2g,\mathbb{R})$ acts on $\mathbb{H}_g$ by

$$\sigma < \tau >= (A\tau + B)(C\tau + D)^{-1} \text{ for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{R}) \text{ and } \tau \in \mathbb{H}_g.$$

This corresponds to the action

$$\sigma \bullet M_\tau = \sigma M_\tau \sigma^{-1}$$

on matrices of type M_τ and

$$\sigma \circ S_\tau = \sigma S_\tau \sigma^t$$

on the real model. Remark that

$$\sigma \bullet M_\tau = \sigma S_\tau J \sigma^{-1} = \sigma S_\tau \sigma^t J = (\sigma \circ S_\tau)J,$$

hence the actions are equivariant. We will freely use τ, M_τ or S_τ to denote an element of the Siegel upper half space in the standard model or the real model. For our purpose the algebraic model is more appropriate.

One easily checks that for $M_\tau = \begin{pmatrix} -pq^{-1} & q + pq^{-1}p \\ -q^{-1} & q^{-1}p \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and for an element $M = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \in Mat_{2g}(\mathbb{Z})$ we have

$$\begin{aligned} M \in End(A_\tau) &\iff (A + \tau C)\tau = B + \tau D \\ &\iff MM_\tau = M_\tau M \\ &\iff A\alpha - B\gamma = \alpha A - \beta C \\ &\qquad - C\alpha + D\gamma = \gamma A - \delta C \\ &\qquad A\beta - B\delta = -\alpha B + \beta D \\ &\qquad - C\beta + D\delta = -\gamma B + \delta D \end{aligned}$$

This leads to the following definitions for an algebra $L \subset M_{2g}(\mathbb{Q})$

$$\begin{aligned} \mathbb{H}(L) &= \{\tau \in \mathbb{H}_g; lM_\tau = M_\tau l \text{ for all } l \in L\} \\ \Gamma(L) &= \{\sigma \in \Gamma_g; \sigma L = L\sigma\} \end{aligned}$$

Moreover we consider the diagram

$$\begin{array}{ccc} \Gamma(L)\backslash\mathbb{H}(L) & \hookrightarrow & \Gamma_g\backslash\mathbb{H}_g \\ \cap & & \cap \\ \mathcal{A}(L) & \hookrightarrow & \mathcal{A}_g \end{array}$$

where $\mathcal{A}(L)$ denotes the closure of $\Gamma(L)\backslash\mathbb{H}(L)$ in the Satake compactification $\mathcal{A}_g$. We call $\mathcal{A}(L)$ the Shimura variety of type L. We remark that $\mathbb{H}(M)$ is $\mathbb{H}(L)$ for $L = \mathbb{Q}(M)$ in the new notation.

It is proved in [R6] that the map $L \mapsto \Gamma(L)\backslash\mathbb{H}(L)$ is an equivalence of categories of admissible algebras with Rosati-equivariant embeddings in $M_{2g}(\mathbb{Q})$ and irreducible varieties parametrizing principally polarized abelian varieties with $L \subset End^0(A_\tau)$.

The 2-dimensional varieties $\mathcal{A}(K) \subset \mathcal{A}_3$ for an admissible imaginary quadratic number field K are called $CM - surfaces$ or Picard modular surfaces. Using the embedding $\mathbb{Q} \subset K \subset M_6(\mathbb{Q})$ a generator ω of K over $\mathbb{Q}$ may be considered as an element in $M_6(\mathbb{Q})$ or $M_6(\mathbb{R})$. We define $\omega x = Mx$ for $x \in \mathbb{R}^6$, which induces an isomorphism $\mathbb{R}^6 \cong \mathbb{R}[\omega]^3$. Then $< x, y >= {}^t x(JM - \omega J)y$ defines a hermitian form on $\mathbb{R}^6$.

Proof. Let $M^2 + aM + b = 0$ with $a, b \in \mathbb{Q}$ be the minimal polynomial of M over $\mathbb{Q}$, hence $-a = M + \hat{M}$, $b = M\hat{M}$ which implies $JM + {}^tMJ = -aJ$. Hence

$$ {}^tM(JM - \omega J) = (-JM - aJ)M - \omega(-JM - aJ) = \omega(JM - \omega J) $$

and

$$ {}^t(JM - \omega J) = -{}^tMJ + \omega J = JM - \hat{\omega}J $$

□

Then

$$ \mathbb{H}(K) = \mathbb{H}_3 \cap U(3, <,>) \text{ and } \Gamma(K) = \Gamma_3 \cap U(3, <,>) $$

are independent of the choice of ω.

Proof. The element ω is totally complex, hence

$$ {}^t\sigma(JM - \omega J)\sigma = JM - \omega J $$

is equivalent to

$$ {}^t\sigma JM\sigma = JM \text{ and } {}^t\sigma J\sigma = J $$

which is equivalent to

$$ \sigma \text{ symplectic and } M\sigma = \sigma M $$

hence the result. □

The CM-variety $\mathcal{A}(K)$ (the closure of $\Gamma(K)\backslash\mathbb{H}(K)$ in $\mathcal{A}_3$) are algebraic surfaces iff the hermitian form defined by K is indefinite. These surfaces are called Picard modular surfaces. In the next chapter we compute one example.

7. "Picard modular forms"

In this final part we consider the case first studied by Picard in [Pi1,2] and follow [R5]. As a general reference we refer to [Ho1,2]. A Picard curve is defined by the equation

$$ C(x, y) = \{z^3 = t(t-1)(t-x)(t-y)\} $$

or its projective closure in $\mathbb{P}^2$. For $x \neq y, x \neq 0, 1, y \neq 0, 1$ we get a non-hyperelliptic nonsingular curve of genus $g = 3$. We recall the notation $\rho = \exp(2\pi i/3)$ and $\mathbb{Z}[\rho]$ for the ring of Eisenstein numbers with quotient field $K = \mathbb{Q}(\rho)$. The group $< \rho >$ acts on the Picard curve in an obvious manner sending (z, t) to $(\rho z, t)$. This implies that the Jacobian $Jac(C(x, y))$ has complex multiplication by K. It was observed by Picard ([Pi2]), that the period matrix of the Jacobian $Jac(C(x, y)) = A_\tau$ may be written as

$$\tau = \begin{pmatrix} \frac{u^2+2\rho^2 v}{1-\rho} & \rho^2 u & \frac{\rho u^2-\rho^2 v}{1-\rho} \\ \rho^2 u & -\rho^2 & u \\ \frac{\rho u^2-\rho^2 v}{1-\rho} & u & \frac{u^2-2v}{(1-\rho)\rho} \end{pmatrix}.$$

We define the matrix M_{Picard} (with $M_{Picard}^2 + M_{Picard} + 1 = 0$, $M_{Picard} \in \Gamma_3$) by

$$M_{Picard} = \left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right)$$

Then one checks immediately, that $\tau \in \mathbb{H}(M_{Picard})$. The condition for $\tau \in \mathbb{H}_3$ translates into the ball condition in [Ho2]. We recall, that as complex manifold $\mathbb{H}(M_{Picard})$ is isomorphic to a 2-dimensional ball. The above computation solves generically the relative Schottky problem for Picard curves, i.e.

Proposition 5. *The space $\Gamma_M \backslash \mathbb{H}(M)$ for $M = M_{Picard}$ contains the coarse moduli space for Picard curves as a dense subset. For simple $\tau \in \overset{\circ}{\mathbb{H}}(M)$ the algebra of complex multiplication $End^0(A_\tau)$ is a number field of degree 2 or 6, hence isomorphic to $K = \mathbb{Q}(\rho) \cong \mathbb{Q}(M)$ or a cubic extension of K. The subfield $NS^0(A_\tau)$ is isomorphic to $\mathbb{Q}$ or a real cubic extension of $\mathbb{Q}$ and $End^0(A_\tau) = K(NS^0(A_\tau))$.*

Proof. We have $\dim(\Gamma_M \backslash \mathbb{H}(M)) = 2$, hence a generic point $\tau \in \Gamma_M \backslash \mathbb{H}(M)$ comes from an uniquely determined Picard curve with $Jac(C(x,y)) = A_\tau$. The second statement follows easily from the classification of $End^0(A_\tau)$ in [M]. □

Let $M_{Geemen}, \sigma \in \Gamma_3$ be

$$\sigma = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right), M_{Geemen} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right),$$

then

$$\sigma M_{Picard} \sigma^{-1} = M_{Geemen} = M.$$

Picard moduli varieties (and the ring of Picard modular forms) only depend up to isomorphy on the conjugacy class of M. From now on we consider the fixed Picard type M.

We recall that $M_+ = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ defines a hermitian form of signature $(1,0)$. Similarly $M_- = (M_+)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ defines a hermitian form of signature $(0,1)$. The matrix M_{Geemen} is just $M_+ \oplus M_+ \oplus M_-$ in [G]. The group

$$\Gamma_M = U(M;\mathbb{Z}[\rho])$$

is called Picard modular group and denoted by $U(2,1;\mathbb{Z}[\rho])$ in [Ho2]. Let as before $G = G(M) = \{\sigma \in H_3; \sigma\bar{M} = \bar{M}\sigma\}$ be the centralizer of $\bar{M}$ in H_3.

We have

$$M_+ = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = J^3 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

hence

$$\overline{M_+} = \left(\left(\frac{1+i}{2}\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)^3 \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \left(\frac{1+i}{2}\right)\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \in H_1.$$

Therefore

$$\bar{M} = \left(\frac{1+i}{2}\right)\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \otimes \left(\frac{1+i}{2}\right)\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \otimes \left(\frac{1+i}{2}\right)\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}$$

$$= \left(\frac{1+i}{4}\right)\left(\begin{array}{cccc|cccc} i & 1 & 1 & -i & i & 1 & 1 & -i \\ i & -1 & 1 & i & i & -1 & 1 & i \\ i & 1 & -1 & i & i & 1 & -1 & i \\ i & -1 & -1 & -i & i & -1 & -1 & -i \\ \hline -1 & i & i & 1 & 1 & -i & -i & -1 \\ -1 & -i & i & -1 & 1 & i & -i & 1 \\ -1 & i & -i & -1 & 1 & -i & i & 1 \\ -1 & -i & -i & 1 & 1 & i & i & -1 \end{array}\right)$$

The relations produced by M are

$$f_0 = \frac{1}{2}(1+i+\sqrt{3}+i\sqrt{3})f_1 + \frac{1}{2}(1+i+\sqrt{3}+i\sqrt{3})f_2 - (2+\sqrt{3})if_3$$
$$f_4 = \frac{1}{2}(1+i+\sqrt{3}+i\sqrt{3})f_3$$
$$f_5 = \frac{1}{2}(1-i+\sqrt{3}-i\sqrt{3})f_2 - (1+\sqrt{3})f_3$$
$$f_6 = \frac{1}{2}(1-i+\sqrt{3}-i\sqrt{3})f_1 - (1+\sqrt{3})f_3$$
$$f_7 = -if_1 - if_2 + \frac{1}{2}(-3+3i-\sqrt{3}-i\sqrt{3})f_3$$

Proposition 6. *One has an exact diagram*

$$\begin{array}{ccccccccc} 0 \to & N_3 & \longrightarrow & H_3 & \longrightarrow & Sp\,(6,\mathbb{F}_2) & \to 0 \\ & \cup & & \cup & & \cup & \\ 0 \to & <i> & \longrightarrow & G(M) & \longrightarrow & U(3,\mathbb{F}_4) & \to 0 \end{array}$$

Proof. It was observed by van Geemen ([G],6.4.) that (in our notation) $\rho_{theta}(\Gamma_M \cap \Gamma_3(2))$ is central, hence the cyclic group generated by i. The isomorphism $\mathbb{F}_2[\rho] \cong \mathbb{F}_4$ induces an isomorphism $\mathbb{F}_2^6 \cong \mathbb{F}_2[\rho]^3$. The associated $\mathbb{F}_2$-bilinear form

$$\mathbb{F}_2[\rho]^3 \times \mathbb{F}_2[\rho]^3 \to \mathbb{F}_2[\rho]$$

which is defined by

$$< x, y >= {}^t xJMy - \rho^t xJy \text{ for } x, y \in \mathbb{F}_2^6$$

is a hermitian form and induces the map on the right. □

It follows that $G = G(M)$ is a group of order $2592 = 4\#(U(3,\mathbb{F}_4))$.

Corollary 7. *We have*

$$A(\Gamma_M \cap \Gamma_3(2)) = \mathbb{C}[f_1, f_2, f_3]_{(4)} \text{ and } \mathcal{A}(\Gamma_M \cap \Gamma_3(2)) \cong \mathbb{P}^2.$$

Proof. This follows from theorem [R1]2.8. and proposition 6. The Picard type M produces 5 linear relations between the restricted theta constants. There cannot be any further relation for dimension reasons. On the other hand, $\mathbb{C}[f_1, f_2, f_3]$ is already normal, hence the result.

The second statement follows from the first. It was proved by van Geemen ([G],8.5.) using the description of the Satake compactification in [HW]. The difficult point is to show that the theta map Th is an embedding. □

The next steps follow the general procedure as explained in [R1,2]. The group G acts (effectively) on $\mathbb{C}[f_1, f_2, f_3]$ and is generated as a subgroup of $Gl(3,\mathbb{C})$ by

$$G = \left\langle \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2}(3+\sqrt{3})(1+i) & 0 & 1 \end{pmatrix}, \right.$$

$$\frac{1}{2}\begin{pmatrix} -1+i\sqrt{3} & 0 & 0 \\ i(1+\sqrt{3}) & -1+i & 1+i \\ (2+\sqrt{3})(1-i) & -1+i & -1-i \end{pmatrix},$$

$$\left. \frac{1}{4}\begin{pmatrix} 0 & -2+2i\sqrt{3} & 0 \\ -1+\sqrt{3}-i-i\sqrt{3} & 1+\sqrt{3}-i+i\sqrt{3} & -1+\sqrt{3}-i-i\sqrt{3} \\ -1-3\sqrt{3}-i-i\sqrt{3} & -1+\sqrt{3}-i-i\sqrt{3} & -3-\sqrt{3}+i+i\sqrt{3} \end{pmatrix} \right\rangle$$

The Poincaré-series for the action of G on the polynomial ring $\mathbb{C}[f_1, f_2, f_3]$ is given by

$$\begin{aligned} \Phi_G(\lambda) &= \sum_{l \geq 0} \dim_{\mathbb{C}} \mathbb{C}[f_1, f_2, f_3]^G(l) \times \lambda^l \\ &= \frac{1}{\#G} \sum_{\sigma \in G} \frac{1}{\det(1-\lambda\sigma)} \\ &= \frac{1+\lambda^{24}}{(1-\lambda^{12})^2(1-\lambda^{36})} \end{aligned}$$

Now let us consider the subgroup G' of G which is generated by the last four generators of G in the list above. Then it turns out that G' is a group generated by pseudo-reflections and $G = < G', i >$. We recall that an element σ of finite order in $Gl(3, \mathbb{C})$ is a pseudo-reflection iff precisely one eigenvalue of σ is not equal to one. In the list of Shephard and Todd [ST] the group G' is the symmetry group of the regular complex polytope 3(24)3(18)2 and the image in $PGl(3, \mathbb{C})$ is the Hessian group of order 216 which leaves invariant the configuration of inflections of a cubic curve.

The Poincaré-series for the action of G' on the polynomial ring $\mathbb{C}[f_1, f_2, f_3]$ is given by

$$\begin{aligned} \Phi_{G'}(\lambda) &= \sum_{l\geq 0} \dim_{\mathbb{C}} \mathbb{C}[f_1, f_2, f_3]^{G'}(l) \times \lambda^l \\ &= \frac{1}{\#G'} \sum_{\sigma \in G'} \frac{1}{\det(1 - \lambda\sigma)} \\ &= \frac{1}{(1-\lambda^6)(1-\lambda^{12})(1-\lambda^{18})} \end{aligned}$$

In the summation for G' occur 48 characteristic polynomials, in the summation for G occur 96 polynomials.

Hence the ring of invariants for G' is given by ([St], thm. 4.1.)

$$\mathbb{C}[f_1, f_2, f_3]^{G'} = \mathbb{C}[P_6, P_{12}, P_{18}]$$

and is just the ring of Picard modular forms for some Nebentypus $\chi : \Gamma_M / \Gamma_M \cap \Gamma_3^*(2, 4) \to \{\pm 1\}$ with $\chi(-1) = -1$. The final result for G is

Theorem 8. *The ring of Picard modular forms of Picard type M*

$$A(\Gamma_M) = \mathbb{C}[P_6^2, P_{12}, P_6 P_{18}, P_{18}^2]$$

is generated by 4 polynomials. There is a relation of degree 48 *between the generators. The field of modular functions of Picard type M is rational.*

With some linear algebra one may compute the polynomials P_6, P_{12}, P_{18}.

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