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Spherical functions on spherical homogeneous spaces and Rankin-Selberg convolution

In this note, we study spherical functions on certain $p$-adic spherical homogeneous spaces. We show the existence, uniqueness and an explicit formula of the spherical functions, and study its application to Rankin-Selberg convolution. Though we treat only the orthogonal group case in this note, similar results hold for other cases.
§1. Preliminaries

1.1 In this and the next sections, we let $\mathbb{F}$ be a non-archimedean local field of characteristic different from 2, and denote by $\mathcal{O}$ the integer ring of $\mathbb{F}$. Fix a prime element $\pi$ of $\mathbb{F}$ and put $q = \#(\mathcal{O}/\pi\mathcal{O})$. Let $1\cdot1$ be the normalized valuation of $\mathbb{F}$ ($|\pi| = q^{1}$). We denote by $\mathbb{F}_{n}^{m}$ the space of $m \times n$ matrices whose entries are in $\mathbb{F}$. For a symmetric matrix $S$ of degree $m$ and $x \in \mathbb{F}_{n}^{m}$, we put $S[x] = x^t S x$. For a real number $\alpha$, we denote by $[\alpha]$ the integer with $\lfloor \alpha \rfloor \leq \alpha < \lfloor \alpha \rfloor + 1$.

1.2 Let $m$ be a positive integer and put $n = \left\lfloor \frac{m}{2} \right\rfloor$. Let $S_{m}$ be a symmetric matrix of degree $m$ given by

\[
S_{m} = \begin{cases} 
\begin{bmatrix} 0 & J_{n} \\ J_{n} & 0 \end{bmatrix} & \text{if } m \text{ is even} \\
\begin{bmatrix} 0 & 0 & J_{n} \\ 0 & 2 & 0 \\
J_{n} & 0 & 0 \end{bmatrix} & \text{if } m \text{ is odd}
\end{cases}
\]

where $J_{n} = \begin{bmatrix} 0 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \end{bmatrix} \in \mathrm{GL}_{n}(\mathbb{F})$. Denote by $G_{m}$ (or $O(m)$) the orthogonal group of $S_{m}$: $G_{m} = O(m) = \{ g \in \mathrm{GL}_{m}(\mathbb{F}) \mid g^t S_{m} g = S_{m} \}$. Let $K_{m} = G_{m}(\mathcal{O})$ be a maximal open compact subgroup of $G_{m}$. We normalize the Haar measure $dg$ on $G_{m}$ so that $\text{vol}(K_{m}) = 1$.

1.3 We define an embedding $\iota_{m}$ of $G_{m}$ into $G_{m+1}$ as follows:

(a) If $m = 2n$ is even,

\[
\iota_{m}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}
\]

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{m}$ is the block decomposition corresponding to the partition $m = n + n$. 
(b) If $m = 2n + 1$ is odd,

$$t_m\left(\begin{array}{ccc}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{array}\right) = \left[\begin{array}{ccc}
a_1 & a_2 & a_3 \\
\frac{b_2+1}{2} & \frac{b_2-1}{2} & b_3 \\
\frac{b_2-1}{2} & \frac{b_2+1}{2} & b_3 \\
c_1 & \frac{c_2}{2} & \frac{c_2}{2} & c_3
\end{array}\right]$$

where $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in G_m$ is the block decomposition corresponding to the partition $m = n + 1 + n$.

1.4 For an integer $r (1 \leq r \leq n = \lfloor \frac{m}{2} \rfloor)$, let

$$N_{m,r} = \{ v_{m,r}(x, y) := \begin{bmatrix} 1_r - J_r & x S_{m-2r} J_r(y - \frac{1}{2} S_{m-2r}[x]) \\ 0 & 1_{m-2r} \end{bmatrix} | x \in F_n^{m-2r}, y \in Alt_r(F) \}$$

and

$$M_{m,r} = \{ \mu_{m,r}(a, h) := \begin{bmatrix} a & 0 \\ h & 0 \\ 0 & \tilde{a} \end{bmatrix} | a \in GL_r(F), h \in G_{m-2r} \},$$

where $Alt_r = \{ y \in F_r^r | y + y = 0 \}$ and $\tilde{a} = J_r^t a^{-1} J_r$ for $a \in GL_r$. Then $P_{m,r} = N_{m,r} M_{m,r}$ is a maximal parabolic subgroup of $G_m$.

1.5 Let $T_m = \{ d_m(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in F^X \}$ be a maximal $F$-split torus of $G_m$, where $d_m(t_1, \ldots, t_n)$ denotes the matrix $\text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$ if $m$ is even and $\text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$ if $m$ is odd. For $t = d_m(t_1, \ldots, t_n) \in T_m$, put $\delta_m(t) = d(t v t^{-1})/dv = \prod_{i=1}^n |t_i|^{m-2i}$, where $dv$ is a Haar
measure on \( N_{m,n} \) (= a maximal unipotent subgroup of \( G_m \)). Denote by \( X_{unr}(T_m) \) the group of unramified characters of \( T_m \). We let the Weyl group \( W := N_{G_m}(T_m)/T_m \) act on \( X_{unr}(T_m) \) by \((w\chi)(t) = \chi(w^{-1}tw)\).

1.6 Let \( \mathcal{H}_m = \mathcal{H}(G_{m}, K_{m}) \) be the Hecke algebra of \((G_{m}, K_{m})\). For \( \chi \in X_{unr}(T_m) \), let \( \phi_\chi \) be a function on \( G_m \) defined by \( \phi_\chi(vtk) = (\delta_{m}^{1/2}\chi)(t) \) for \( v \in N_{m,n}, t \in T_m, k \in K_m \). Define a \( \mathbb{C} \)-homomorphism \( \chi^\wedge \) of \( \mathcal{H}_m \) to \( \mathbb{C} \) by

\[
\chi^\wedge(\phi) = \int_{G_m} \phi_\chi(g) \phi(g) \, dg \quad (\phi \in \mathcal{H}_m).
\]

Then \( \chi \mapsto \chi^\wedge \) gives rise to a bijection between \( W_m \backslash X_{unr}(T_m) \) and \( \text{Hom}_{\mathbb{C}}(\mathcal{H}_m, \mathbb{C}) \) (cf. [Sa]).

1.7 Let \( T_r^* = \begin{bmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_r \end{bmatrix} \) be a maximal split torus of \( \text{GL}_r(F) \). Let \( \xi \in X_{unr}(T_r^*) \) and \( \chi \in X_{unr}(T_m) \). We often identify \( \xi \) and \( \chi \) with \((\xi_1, \cdots, \xi_r) \in (\mathbb{C}^\times)^r \) and \((\chi_1, \cdots, \chi_n) \in (\mathbb{C}^\times)^n \) determined by \( \xi(\text{diag}(\pi_1^{k_1}, \cdots, \pi_r^{k_r})) = \xi_1^{k_1} \cdots \xi_r^{k_r} \) and \( \chi(\text{diag}(\pi_1^{\ell_1}, \cdots, \pi_n^{\ell_n})) = \chi_1^{\ell_1} \cdots \chi_n^{\ell_n} \) for \((k_1, \cdots, k_r) \in \mathbb{Z}^r \) and \((\ell_1, \cdots, \ell_n) \in \mathbb{Z}^n \), respectively. We define the L-factor \( L(\xi \otimes \chi; s) \) by

\[
L(\xi \otimes \chi; s) = \prod_{1 \leq i < s, 1 \leq j \leq n} (1 - \xi_i \chi_j q^{-s})(1 - \xi_i^{-1} \chi_j^{-1} q^{-s})^{-1}.
\]

We also define the L-factors \( L(\xi, \text{Sym}^2; s) \) and \( L(\xi, \text{Alt}^2; s) \) by

\[
L(\xi, \text{Sym}^2; s) = \prod_{1 \leq i < j \leq r} (1 - \xi_i \xi_j q^{-s})^{-1}, \quad L(\xi, \text{Alt}^2; s) = \prod_{1 \leq i < j \leq r} (1 - \xi_i \xi_j q^{-s})^{-1}.
\]

§2. Local spherical functions

2.1 Let \( m' \) and \( r \) be non-negative integers and put \( m = m' + 2r + 1 \). Let

\[
G = G_m, K = K_m, T = T_m, \mathcal{H} = \mathcal{H}_m, n = \left\lfloor \frac{m}{2} \right\rfloor
\]

\[
G' = G_{m'}, K' = K_{m'}, T' = T_{m'}, \mathcal{H}' = \mathcal{H}_{m'}, n' = \left\lfloor \frac{m'}{2} \right\rfloor
\]
and identify $G'$ with a subgroup of $G$ via $g' \mapsto \mu_{m,r}(1, 1_{m'}(g'))$.

2.2 Let $U = U_{m,r} = N_{m,r} \cdot \{ \mu_{m,r}(z, 1) \mid z \in Z_r \}$ where $Z_r$ is the group of upper unipotent matrices in $GL_r(F)$. Throughout this section, we fix an additive character $\psi$ of $F$ with conductor $\sigma$. We define a character $\psi_U$ of $U$ by

$$\psi_U(v_{m,r}(x, y) \mu_{m,r}(z, 1)) = \psi(x_{n'+1,1} - \epsilon_m x_{n'+2,1} + \sum_{i=1}^{r-1} z_{i,i+1})$$

for $x \in M_{m-2r,r}(F)$, $y \in Alt_r(F)$ and $z \in Z_r$, where we put

$$\epsilon_m = \begin{cases} 1 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

It is easy to see that $G'$ normalizes $U$ and fixes $\psi_U$.

2.3 For $(\chi', \chi) \in X_{unr}(T') \times X_{unr}(T)$, let

$$\Omega(\chi', \chi) = \{ \mathcal{W} : G \mapsto \mathbb{C} \}$$

such that

(i) $\mathcal{W}(uk'gk) = \psi_U(u) \mathcal{W}(g)$ ($u \in U$, $k' \in K'$, $g \in G$, $k \in K$)

(ii) $\varphi'^* \mathcal{W} * \varphi = \chi'^\wedge(\varphi') \chi^\wedge(\varphi) \mathcal{W}$ ($\varphi' \in \mathcal{H}'$, $\varphi \in \mathcal{H}$).

Here

$$(\varphi'^* \mathcal{W} * \varphi)(g) = \int_{G'} dx' \int_G dx \varphi'(x') \mathcal{W}(x'gx) \varphi(x).$$

We call $\Omega(\chi', \chi)$ the space of spherical functions on $G$ attached to $(\chi', \chi)$.

2.4 Remark

(i) Let $G = G' \times G$ and $H = (UG')^{\text{diag}} \subset G$. Then $H$ is a spherical subgroup of $G$ and $\mathcal{W} \in \Omega(\chi', \chi)$ may be regarded as a spherical function of $(G, H)$ (cf. [GP]).

(ii) When $m' = 0$ or $1$, these functions are the usual Whittaker functions. Bump, Friedberg and Furusawa [BFF] have studied the spherical functions in the case $m' = 2$, and Murase and Sugano [MS] considered the case $r = 0$. 

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2.5 Let $L_n = \mathbb{Z}^n$ and $L_n^+ = \{ (\ell_1, \ldots, \ell_n) \in L_n \mid \ell_1 \geq \cdots \geq \ell_n \geq 0 \}$. For $\ell = (\ell_1, \ldots, \ell_n) \in L_n$, put $t_m(\ell) = d_m(\pi^{\ell_1}, \ldots, \pi^{\ell_n}) \in T_m$. We define $t_{m'}(\ell') \in T_{m'}$ for $\ell' \in L_n$ similarly. Let $g_0$ be an element of $G$ given by

$$g_0 = \begin{cases} 
\mu_{m,r}(1, \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}) & \text{if } m \text{ is even} \\
\mu_{m,r}(1, \begin{bmatrix} 1_n' - 2^tX - tXXJ_n' \\
0 & 1 \\
0 & 1_n' \end{bmatrix}) & \text{if } m \text{ is odd}
\end{cases}$$

where $X = (1, \ldots, 1) \in F^n'$ and $A = \begin{bmatrix} 1_n' \end{bmatrix} \in GL_{n'+1}(F)$. For $(\ell', \ell) \in L_n \times L_n$, put $g(\ell', \ell) = t_{m'}(\ell') g_0 t_m(\ell) \in G$.

2.6 Theorem (Cartan decomposition) We have

$$G = \coprod UK' \cdot g(\ell', \ell) \cdot K$$

(disjoint union)

where $\ell'$ runs over $L_n^+$ and $\ell$ over $L_r \times L_{n-r}$.

2.7 Corollary For $\varphi' \in \Omega(\chi', \chi)$, we have

$$\text{Supp } \varphi \subset \coprod UK' \cdot g(\ell', \ell) \cdot K$$

where $\ell'$ runs over $L_n^+$ and $\ell$ over $L_n^+$.

2.8 Using the Cartan decomposition (Corollary 2.7) and a similar method of [Shin] and [Ka], we obtain the following existence and uniqueness of spherical functions:

**Theorem** For $(\chi', \chi) \in X_{\text{unr}}(T') \times X_{\text{unr}}(T)$, there uniquely exists $w_{\chi', \chi} \in \Omega(\chi', \chi)$ with $w_{\chi', \chi}(1) = 1$. In particular, we have $\dim C \Omega(\chi', \chi) = 1$.

2.9 For $\chi \in X_{\text{unr}}(T)$, we put

$$\Delta_m(\chi) = \prod_{1 \leq i < j \leq n} (1 - \chi_i^{-1} \chi_j)(1 - \chi^{-1}_{i} \chi^{-1}_j) \times \begin{cases} 1 & \text{if } m \text{ is even} \\
\prod_{1 \leq i \leq n} (1 - \chi_i^{-2}) & \text{if } m \text{ is odd.} \end{cases}$$
We define \( \Delta_{\mathrm{m}'}(\chi') \) for \( \chi' \in X_{\text{unr}}(T') \) similarly. For \((\chi', \chi) \in X_{\text{unr}}(T) \times X_{\text{unr}}(T')\), we put

\[
D(\chi', \chi) = \Delta_{\mathrm{m}'}(\chi')^{-1} \Delta_{\mathrm{m}}(\chi)^{-1} \prod_{1 \leq i \leq m'} (1 - q^{-1/2}(x_i^\chi)^{-1}) \prod_{1 \leq j \leq n} (1 - q^{-1/2}(x_j^\chi)^{-1})
\]

where \( \eta_{ij} = \begin{cases} 1 & \text{if } j \leq r + i \\ -1 & \text{if } j > r + i \end{cases} \)

Put \( Q_{\mathrm{m}'} = \begin{cases} (1 - q^{n'}) \prod_{1 \leq i \leq n'} (1 - q^{-2i}) & \text{if } m' = 2n' \\ \prod_{1 \leq i \leq n'} (1 - q^{-2i}) & \text{if } m' = 2n' + 1. \end{cases} \)

2. 10 The following explicit formula can be proved by a method similar to that of [CS].

**Theorem** For \((\chi', \chi) \in X_{\text{unr}}(T) \times X_{\text{unr}}(T')\), let \( W_{\chi', \chi} \in \Omega(\chi', \chi) \) be as in Theorem 2.8. Then, for \((\ell', \ell) \in L^+_n \times L^+_n\), we have

\[
W_{\chi', \chi}(g(\ell', \ell)) = \frac{1}{Q_{\mathrm{m}'}} \sum_{w' \in W_{\mathrm{m}'}, w \in W_{\mathrm{m}}} D(w'\chi', w\chi) \\
\times \left( w'\chi' \delta_{\mathrm{m}'}^{1/2} \right) (t_{\mathrm{m}'}(\ell')) \left( w\chi \delta_{\mathrm{m}}^{1/2} \right) (t_{\mathrm{m}}(\ell)).
\]

§3. Application to Rankin-Selberg convolution

3. 1 Let \( G = G_m \) and \( G^* = G_{m-1} \) be the orthogonal group of \( S_m \) and \( S_{m-1} \) defined over \( \mathbb{Q} \). We regard \( G^* \) as a subgroup of \( G \) via \( t_{m-1} \). Let \( r \) be an integer with \( 1 \leq r \leq \left[ \frac{m - 1}{2} \right] \). Let \( P^* = N_{m-1,r} M_{m-1,r} \) be a maximal parabolic subgroup of \( G^* \) and put \( G' = G_{m'} \) with \( m' = m - 2r - 1 \). Then \( \mu^* = \mu_{m-1,r} \) gives an isomorphism of \( \GL_r \times G' \) onto \( M_{m-1,r} \).

3. 2 Let \( \varphi \) be an automorphic form on \( \GL_r(\mathbb{A}) \) with central character \( \omega \). Assume that \( \varphi \) is right-invariant under \( \prod_{p < \infty} \GL_r(\mathbb{Z}_p) \) and square integrable over \( \GL_r(\mathbb{Q}) \backslash \GL_r(\mathbb{A})^1 \), where \( \GL_r(\mathbb{A})^1 = \{ g \in \GL_r(\mathbb{A}) \mid \det(g) \mid_A = 1 \} \). We also
let $f$ be an automorphic form on $G'(A)$ right-invariant under $\prod_{p<\infty} G'(\mathbb{Z}_p)$ and square integrable over $G'(\mathbb{Q}) \backslash G'(A)$. Define a function $\phi(\cdot; \varphi \otimes f)$ on $G^*(A) \times \mathbb{C}$ by

$$\phi(v^* \mu^*(a, g') k^*, s; \varphi \otimes f) = \varphi(a) f(g') \det a^{(m'^* + r - 1)/2},$$

where $v^* \in N_{m-1,r}(A)$, $a \in GL_r(A)$, $g' \in G'(A)$ and $k^* \in K^* \prod_{p<\infty} G^*(\mathbb{Z}_p)$ ($K^*$ is a suitable maximal compact subgroup of $G^*(\mathbb{R})$). The Eisenstein series

$$\sum_{\gamma \in P^*(\mathbb{Q}) \backslash G^*(\mathbb{Q})} \phi(\gamma g^*; s; \varphi \otimes f)$$

is absolutely convergent for $\text{Re}(s) >> 0$ and continued to a meromorphic function of $s$ on the whole $\mathbb{C}$.

3. 3 Let $F$ be a cusp form on $G(A)$ right-invariant under $\prod_{p<\infty} G(\mathbb{Z}_p)$. The object of this section is to study the following Rankin-Selberg convolution

$$Z_{F, \varphi \otimes f}(s) = \int_{G^*(\mathbb{Q}) \backslash G^*(A)} F(g^*) E(g^*, s - \frac{1}{2}; \varphi \otimes f) \, dg^*.$$ 

The function $Z_{F, \varphi \otimes f}(s)$ is continued to a meromorphic function of $s$ on the whole $\mathbb{C}$.

3. 4 Let $U = U_{m,r} \subset G$ and $\psi_U \in \text{Hom}(U(A), \mathbb{C}^\times)$ be as in §2.2 replacing $\psi$ with the additive character $\psi_A$ of $Q \backslash A$ such that $\psi_A(x_\infty) = \exp(2\pi i x_\infty)$ for $x_\infty \in \mathbb{R}$. We set

$$\psi_{f,F}(g) = \int_{U(\mathbb{Q}) \backslash U(A)} \int_{G'(\mathbb{Q}) \backslash G'(A)} dg' f(g') \psi_U(u)^{-1} F(u \mu^*(1, g') g)$$

for $g \in G(A)$ and

$$W_\varphi(x) = \int_{Z_r(\mathbb{Q}) \backslash Z_r(\mathbb{A})} \psi_A(\sum_{i=1}^{r-1} z_{i,i+1}) \varphi(zx) \, dz$$

for $x \in GL_r(A)$. 
3.5 Unfolding the Eisenstein series in the integral of $Z_{F, \varphi \otimes f}(s)$, we get

**Proposition** (The basic identity)

$$Z_{F, \varphi \otimes f}(s) = \int_{(A^\times)^r} W_\varphi(\text{diag}(t_1, \ldots, t_r)) \, \mathcal{W}_{f, F}(\mu^*(\text{diag}(t_1, \ldots, t_r), 1))$$

$$\times \prod_{i=1}^r |t_i|_A^{-(m+r+1)/2+2i} \, dt_1 \cdots dt_r.$$

3.6 We now assume that $\varphi, f$ and $F$ are Hecke eigenform. Let $\xi_p \in X_{\text{unr}}(T_r^*(Q_p))$, $\chi'_p \in X_{\text{unr}}(T_m(Q_p))$, and $\chi_p \in X_{\text{unr}}(T_m(Q_p))$ be the corresponding Satake parameters at $p$. For each $p$, the restriction of $\mathcal{W}_{f, F}$ to $G(Q_p)$ belongs to $\Omega(\chi'_p, \chi_p)$. Then Theorem 2.8 implies that

$$\mathcal{W}_{f, F}(g) = \mathcal{W}^{(\infty)}_{f, F}(g) \prod_{p<\infty} \mathcal{W}_p' \mathcal{W}_p(g_p)$$

for $g = g_{\infty} \prod_{p<\infty} g_p \in G(A)$, where $\mathcal{W}_{f, F}^{(\infty)}$ is the restriction of $\mathcal{W}_{f, F}$ to $G(R)$. It is well-known that a similar fact holds for $W_\varphi$:

$$W_\varphi(x) = W_\varphi^{(\infty)}(x_{\infty}) \prod_{p<\infty} W_{\xi_p}(x_p)$$

for $x = x_{\infty} \prod_{p<\infty} x_p \in G(L_r(A))$, where $W_{\xi_p}$ is the $p$-adic Whittaker function attached to $\xi_p$ on $GL_r(Q_p)$ with $W_{\xi_p}(1) = 1$ (cf. [Shin]) and $W_\varphi^{(\infty)}$ is the restriction of $W_\varphi^{(\infty)}$ to $GL_r(R)$. Therefore we obtain the Euler product decomposition for $Z_{F, \varphi \otimes f}(s)$:

$$Z_{F, \varphi \otimes f}(s) = Z_{F, \varphi \otimes f}^{(\infty)} \prod_{p<\infty} Z_p(s),$$

$$Z_{F, \varphi \otimes f}^{(\infty)}(s) = \int_{(R^\times)^r} W_\varphi^{(\infty)}(\text{diag}(t_1, \ldots, t_r)) \, \mathcal{W}_{f, F}^{(\infty)}(\mu^*(\text{diag}(t_1, \ldots, t_r), 1))$$

$$\times \prod_{i=1}^r |t_i|_\infty^{-(m+r+1)/2+2i} \, dt_1 \cdots dt_r.$$
\[ Z_p(s) = \int_{(\mathbb{R}^*)^r} W_{\xi_p} (\text{diag}(t_1, \ldots, t_r)) \, W_{\chi_{p'}} \, \chi_p \, (\mu^*(\text{diag}(t_1, \ldots, t_r), 1)) \]
\[ \times \prod_{i=1}^{r} |t_i|^\frac{s-(m+r+1)/2+2i}{p} \, d^x t_1 \ldots d^x t_r. \]

3.7 By using Theorem 2.10 and Shintani's explicit formula for \( W_{\xi_p} \) ([Shin]), we obtain the following:

**Theorem**

\[ Z_p(s) = \frac{L(\xi_p \otimes \chi_p, s)}{L(\xi_p \otimes \chi_{p'}, s + 1/2)} \times \begin{cases} L(\xi_{p'}, \text{Sym}^2, 2s)^{-1} & \text{if } m \text{ is even} \\ L(\xi_{p'}, \text{Alt}^2, 2s)^{-1} & \text{if } m \text{ is odd.} \end{cases} \]

3.8 **Remark** Similar results hold for the integral of \( F \) on \( O(m) \) against the restriction to \( O(m) \) of Eisenstein series on \( O(m+1) \).
References


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