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On the exterior problem of compressible Navier-Stokes equation

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§0. Introduction.

Let Ω be an exterior domain in R^3 with compact smooth boundary $\partial\Omega$. The motion of a compressible viscous and heat-conductive fluid is described by the following system

$$
(0.1)\qquad
\begin{cases}
\rho_t + (v\cdot\nabla)\rho + \rho\cdot\mathrm{div}v = 0 & \text{in } [0,\infty)\times\Omega,\\
v_t + (v\cdot\nabla)v = \frac{\mu}{\rho}\cdot\Delta v + \frac{\mu+\mu'}{\rho}\cdot\nabla(\mathrm{div}v) - \frac{\nabla P(\rho,\theta)}{\rho} & \text{in } [0,\infty)\times\Omega,\\
\theta_t + (v\cdot\nabla)\theta + \frac{\theta\cdot\partial_\theta P}{\rho\cdot c}\cdot\mathrm{div}v = \frac{k}{\rho\cdot c}\Delta\theta + \frac{\Psi}{\rho\cdot c} & \text{in } [0,\infty)\times\Omega,\\
v|_{\partial\Omega} = v|_\infty = 0,\ \theta|_{\partial\Omega} = \theta|_\infty = \overline{\theta} & \text{on } [0,\infty)\times\partial\Omega,\\
(\rho,v,\theta)(0,x) = (\rho_0,v_0,\theta_0)(x) & \text{in } \Omega,
\end{cases}
$$

where ρ is the density, $v = (v_1,v_2,v_3)$ the velocity, θ the absolute temperature, $P = P(\rho,\theta)$ the pressure, μ and μ' the viscosity coefficients, k the coefficient of the heat conduction, c the heat capacity at constant volume and Ψ is the dissipation function:

$$\Psi = \frac{\mu}{2}(\partial_k v_j+\partial_j v_k)^2 + \mu'(\partial_k v_j)^2.$$

In this lecture we consider the following linearized equations for the system (0.1), i.e.,

$$
(0.2)\qquad
\begin{cases}
\rho_t + \gamma\mathrm{div}v = f_1 & \text{in } [0,\infty)\times\Omega,\\
v_t - \alpha\Delta v - \beta\nabla(\mathrm{div}v) + \gamma\nabla\rho + \omega\nabla\theta = f_2 & \text{in } [0,\infty)\times\Omega,\\
\theta_t - \kappa\Delta\theta + \omega\mathrm{div}v = f_3 & \text{in } [0,\infty)\times\Omega,\\
v|_{\partial\Omega} = v|_\infty = 0,\ \theta|_{\partial\Omega} = \theta|_\infty = 0 & \text{on } [0,\infty)\times\partial\Omega,
\end{cases}
$$

$$(\rho, v, \theta)(0,x) = (\rho_0, v_0, \theta_0)(x) \qquad \text{in } \Omega,$$

where α, γ, κ, ω are positive numbers and β is a nonnegative number.

Our main results are the following. Let $1 < q < \infty$, m be an integer and $W_q^m(\Omega) = \{W_q^m(\Omega)\}^3$ be the usal Sobolev space. Set

$$X_q^m(\Omega) = \{{}^T\mathbb{u};\ \mathbb{u} \in W_q^{m+1}(\Omega)\times W_q^m(\Omega)\times W_q^m(\Omega)\},\ X_q(\Omega) = X_q^0(\Omega),$$

where ${}^T\mathbb{u}$ means the transposed $\mathbb{u}$. Define the 5×5 matrix operator $\mathbb{A}$ by the relation:

$$\mathbb{A} = \begin{pmatrix} 0 & \gamma\mathrm{div} & 0 \\ \gamma\nabla & -\alpha\Delta-\beta\nabla\mathrm{div} & \omega\nabla \\ 0 & \omega\mathrm{div} & -\kappa\Delta \end{pmatrix}, \tag{0.3}$$

with the domain:

$$\mathcal{D}(\mathbb{A}) = \{{}^T\mathbb{u};\ \mathbb{u} = \{\rho, v, \theta\} \in W_q^1(\Omega)\times W_q^2(\Omega)\times W_q^2(\Omega),$$
$$v|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}.$$

Let $\mathbb{P}$ be the projection from $\mathcal{D}(\mathbb{A})$ into $\{{}^T\{v,\theta\};\ \{v,\theta\} \in W_q^2(\Omega)\times W_q^2(\Omega),$ $v|_{\partial\Omega} = 0,\ \theta|_{\partial\Omega} = 0$ on $\partial\Omega\}$ and $\rho(-\mathbb{A})$ be the resolvent set of the operator $-\mathbb{A}$. Then

Theorem A. Let $1 < q < \infty$ and m be an integer ≥ 0. Then $-\mathbb{A}$ is a closed linear operator in $X_q(\Omega)$ and

$$\rho(-\mathbb{A}) \supset \Sigma = \{\lambda \in \mathbb{C};\ C\mathrm{Re}\lambda + (\mathrm{Im}\lambda)^2 > 0\}, \tag{0.4}$$

where C is a constant depending only on α, β, γ, κ and ω. Moreover, the following properties are valid: There exist positive constants λ_0 and $\delta < \frac{\pi}{2}$ such that

$$|\lambda|\,\|(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{X_q^m(\Omega)} + \|\mathbb{P}(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{2+m,q,\Omega} \leq C(\lambda_0,\delta,m)\|\mathbb{f}\|_{X_q^m(\Omega)}$$

for any $\lambda - \lambda_0 \in \Sigma_\delta = \{\lambda \in \mathbb{C};\ |\arg\lambda| \leq \pi - \delta\}$ and any $\mathbb{f} \in X_q^m(\Omega)$.

Theorem A means that $-\mathbb{A}$ generates an analytic semigroup $e^{-t\mathbb{A}}$ on $X_q(\Omega)$. Then setting

$$X_{q,b}(\Omega) = \{\mathbb{u} \in X_q(\Omega);\ \mathbb{u}(x) = 0 \text{ for } x \in R^3 \setminus B_b\}, \tag{0.5}$$

we have

Theorem B. (local energy decay) Let $1 < q < \infty$ and let b_0 be a fixed number such that $B_{b_0} \supset R^3 \setminus \Omega$. Suppose that $b > b_0$ and $\Omega_b = \Omega \cap B_b$. Then for $\varphi \in C_0^\infty(\Omega_b)$ such that $\int_{\Omega_b} \varphi(x)dx = 1$ and $\mathbb{u} = {}^T\{\rho, v, \theta\} \in X_{q,b}(\Omega)$, we have the following representation

$$e^{-t\mathbb{A}}\mathbb{u} = \mathbb{T}_1(b,\varphi,t)\mathbb{u} + \mathbb{T}_2(b,\varphi,t)\mathbb{u} \tag{0.6}$$

where $\mathbb{e}_j$ $(j = 1,2, \cdots,5)$ are unit row vectors, $N_D\mathbb{u} = \int_D \rho(x)dx$ and

$$\mathbb{T}_1(b,\varphi,t)\mathbb{u} = e^{-t\mathbb{A}}\{\mathbb{u} - (N_{\Omega_b}\mathbb{u})\cdot\varphi\mathbb{e}_1\},$$

$$\mathbb{T}_2(b,\varphi,t)\mathbb{u} = (N_{\Omega_b}\mathbb{u})\{\varphi\cdot\mathbb{e}_1 - \int_0^t e^{-s\mathbb{A}}\begin{pmatrix}0\\ \nabla\varphi\\ 0\end{pmatrix}ds\}.$$

Moreover, the following estimates are valid: for $M \geq 0$ integer, $\mathbb{u} \in X_{q,b}(\Omega)$ and $t > 0$

$$\|\partial_t^M \mathbb{T}_1(b,\varphi,t)\mathbb{u}\|_{X_q(\Omega_b)} + \|\partial_t^M \mathbb{T}_1(b,\varphi,t)\mathbb{u}\|_{2,q,\Omega_b} \leq C(q,b,\varphi,M)t^{-3/2-M}\|\mathbb{u}\|_{X_q(\Omega)}, \tag{0.7}$$

$$\|\partial_t^{M+1} \mathbb{T}_2(b,\varphi,t)\|_{X_q(\Omega_b)} + \|\partial_t^{M+1} \mathbb{T}_2(b,\varphi,t)\|_{2,q,\Omega_b} \leq C(q,b,\varphi,M)t^{-3/2-M}\|\mathbb{u}\|_{X_q(\Omega)}. \tag{0.8}$$

System (0.2) was given by Matsumura and Nisida [12] and Ponce [17]. They seek solutions for the system (0.1) in a neighborhood of a constant state $(\rho, v, \theta) = (\bar{\rho}_0, 0, \bar{\theta}_0)$ where $\bar{\rho}_0$, $\bar{\theta}_0$ are positive constants under the following assumptions:

(1) μ, μ' are constants $\mu > 0$ and $\frac{2}{3}\mu + \mu' \geq 0$.

(2) c, k are positive constants.

(3) P is a known function of ρ, θ, smooth in a neighborhood of $(\bar{\rho}_0, \bar{\theta}_0)$ where $\frac{\partial P}{\partial \rho}$, $\frac{\partial P}{\partial \theta} > 0$.

Note that the assumption (1) is stronger than ours because they also study the Neumann boundary condition.

In equations (0.1), put $\alpha = \dfrac{\mu}{\bar{\rho}_0}$, $\beta = \dfrac{\mu+\mu'}{\bar{\rho}_0}$, $\gamma = \{\dfrac{\partial P}{\partial \rho}(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}$, $\kappa = \dfrac{k}{c\bar{\rho}_0}$ and put $\omega = \dfrac{1}{\bar{\rho}_0}\cdot\dfrac{\partial P}{\partial \theta}(\bar{\rho}_0, \bar{\theta}_0)\{\bar{\theta}_0/C\}^{1/2}$. Then using the notation (ρ, v, θ) for the vector $(\dfrac{1}{\bar{\rho}_0}\{\dfrac{\partial P}{\partial \rho}(\bar{\rho}_0, \bar{\theta}_0)\}^{1/2}\rho, v, \{\dfrac{C}{\bar{\theta}_0}\}^{1/2}\theta)$ and separating the linear part of the equations (0.1) becomes the equations (0.2).

The existence theorems of unique solution local in time for the system (0.1) are obtained by Nash [15], Itaya [7,8] for the initial value problem, and by Tani [20] for the initial boundary value problem. On the other hand the existence theorem of global solution in time for the system (0.1) are obtained by Matsumura and Nishida [12,13], Ponce [17] for the initial problem, and by Matsumura and Nishida [14] for the initial boundary value problem in L_2-framework for sufficiently small initial data. In addition Matumura and Nishida [14] show that this solution approaches the stationary state as $t \to \infty$, and also Deckelnick [3,4] gives explicit estimates for the decay rate. Concerning the linearized equations (0.2), Matsumura and Nishida [12] give the spectral analysis and energy estimates of solutions in L_2-sense and Ponce [17] the L_p-L_q estimates for solutions in R^3, respectively.

In this note, we shall continue to study linearlized equations (0.2) in order to obtain the L_q-theory for the system (0.1). We shall show that the operator defined on this system generates analytic semigroup although the system (0.2) is hyperbolic-parabolic type. In particular, the resolvent set of this operator contains a parabolic region. We shall also show the local energy decay of solutions for the system (0.2). The local energy decay plays an important role in obtaining the solutions of nonlinear problems, in particular, L_p-L_q estimates of solutions. For instance, Iwashita [9] for Navier-Stokes equations, Kobayashi and Shibata [11] for Oseen equations, Iwashita and Shibata [10] and Shibata [19] for elastic and wave equations, they prove the existence of solutions for nonlinear problems by using this methods.

§1. Stationary problem in a bounded domain

In this section we consider the stationary problem in a bounded domain D in R^3 with smooth boundary ∂D;

$$\text{(1.1a)} \qquad \lambda\rho + \gamma\cdot\mathrm{div}v = f_1 \qquad \text{in } D,$$

$$\text{(1.1b)} \qquad \lambda v - \alpha\Delta v - \beta\nabla(\mathrm{div}v) + \gamma\cdot\nabla\rho + \omega\cdot\nabla\theta = f_2 \qquad \text{in } D,$$

$$\text{(1.1c)} \qquad \lambda\theta - \kappa\Delta\theta + \omega\cdot\mathrm{div}v = f_3 \qquad \text{in } D,$$

$$\text{(1.1d)} \qquad v|_{\partial D} = 0 \qquad \text{on } \partial D,$$

$$\text{(1.1e)} \qquad \theta|_{\partial D} = 0 \qquad \text{on } \partial D.$$

here λ is a complex parameter. Our goal of this section is to show the following theorem concerning a unique existence of solutions to (1.1). Let $1 < q < \infty$, m be an integer and let

$$Y^m_q(D) = \{{}^T\{f_1, f_2, f_3\} \in X^m_q(D);\ \int_D f_1(x)dx = 0\},\ Y_q(D) = X^0_q(D),$$

where $X^m_q(D)$ is the same symbol as in (0.7). Set $\mathbb{A}_D$ be the maximal restriction to closed subspace $Y_q(D)$. Applying this notation to (1.1), we have $(\lambda+\mathbb{A}_D)\mathbb{u} = \mathbb{f}$ in $Y_q(D)$ where $\mathbb{u} = {}^T\{\rho, v, \theta\}$ and $\mathbb{f} = {}^T\{f_1, f_2, f_3\}$. Then

Theorem 1.1. Let $1 < q < \infty$, m be a nonnegative integer. Then $\mathbb{A}_D$ is a closed linear operator in $Y_q(D)$ and

$$\rho(-\mathbb{A}_D) \supset \{0\} \cup \Sigma'$$

where $\Sigma' = \{\lambda \in \mathbb{C};\ 6(\gamma^2+\omega^2)\mathrm{Re}\lambda + \alpha(\mathrm{Im}\lambda)^2 > 0\}$. Moreover, the following properties are valid: There exists a number $0 < \delta < \frac{\pi}{2}$ such that

$$|\lambda|\,\|(\lambda+\mathbb{A}_D)^{-1}\mathbb{f}\|_{Y_q^m(D)} + \|\mathbb{P}(\lambda+\mathbb{A}_D)^{-1}\mathbb{f}\|_{m+2,q,D} \le C(q,m,\delta,D)\|\mathbb{f}\|_{Y_q^m(D)}$$

for any $\lambda \in \Sigma_\delta \cup \{0\}$ and any $\mathbb{f} \in Y_q^m(\Omega)$.

To prove this theorem we shall use the following properties. First proposition is concerned the existence theorem of solutions to the Stokes equations.

Proposition 1.2. ([2]) Let $1 < q < \infty$, $0 < \delta < \frac{\pi}{2}$, m be an integer ≥ 0. Then for every $f \in W_q^m(D)$ and every $g \in W_q^{m+1}(D)$ with $\int_D g(x)dx = 0$ there exists a unique $u \in W_q^{m+2}(D)$ which together with some $p \in W_q^{m+1}(D)$ satisfying

$$-\Delta u + \nabla p = f,\ \mathrm{div} u = g \text{ in } D,$$

$$u = 0 \text{ on } \partial D.$$

Here p is unique up to an additive constant. Furthermore, the following estimate is valid:

$$\|u\|_{m+2,q,D} + \|\nabla p\|_{m,q,D} \le C\{\|f\|_{m,q,D}+\|g\|_{m+1,q,D}\},$$

where $C = C(D,q,\varepsilon)$ is a constant.

The next proposition is well-known as a general Poincaré's inequality.

Proposition 1.3. (cf., eg. [3]) Let $1 \le q < \infty$. There exists a constant $C > 0$ such that the inequality

$$\|u\|_{q,D} \le C\{\|\nabla u\|_{q,D}+|\int_D u(x)dx|\},$$

holds for any $u \in W_q^1(D)$. Furthermore, if $q \ne 1$, D is bounded and if $u \in W_q^1(D)$ with $u = 0$ on ∂D, then we have

$$\|u\|_{q,D} \le C\|\nabla u\|_{q,D}.$$

The following proposition is concerned the existence theorem of solutions to the elastic equations.

Proposition 1.3. Let $1 < q < \infty$, m be an integer ≥ 0. Let α be a positive number, η be a complex number such that $\alpha + \eta \neq 0$. Then there exist positive numbers λ_0 and $0 < \delta < \frac{\pi}{2}$ satisfying the following conditions: For every $\lambda - \lambda_0 \in \Sigma_\delta$, every $f \in W_q^m(D)$ there exists a unique $u \in W_q^{m+2}(D)$ such that

$$\lambda u - \alpha\Delta u - \eta\nabla \mathrm{div} u = f \text{ in } D,\ u|_{\partial D} = 0 \text{ on } \partial D.$$

Furthermore the following estimates is valid:

$$|\lambda| \|u\|_{m,q,D} + \|u\|_{m+2,q,D} \leq C\|f\|_{m,q,D},$$

where $C = C(D,q,m,\delta,\lambda_0,\alpha,\eta)$ is a constant.

Remark 1.4. In Theorem 1.1 when $\int_D f_1 dx \neq 0$, taking $\varphi \in C_0^\infty(D)$ such that $\int_D \varphi(x)dx = 1$ and define the operators $\mathbb{N}_j = \mathbb{N}_j(\varphi,D)$ $(j = 1, 2,3)$ from $X_q(D)$ into itself by the notations:

$$\begin{aligned} &\mathbb{N}_1 \mathbb{f} = \mathbb{f} - (N_D \mathbb{f})\cdot \mathbb{e}_1 \\ (1.2) \qquad &\mathbb{N}_2 \mathbb{f} = -(N_D \mathbb{f})\begin{pmatrix} 0 \\ \nabla\varphi \\ 0 \end{pmatrix} \text{ for } \mathbb{f} = {}^T\{f_1, f_2, f_3\} \in X_q(D), \\ &\mathbb{N}_3 \mathbb{f} = (N_D \mathbb{f})\varphi\cdot \mathbb{e}_1 \end{aligned}$$

where $N_D\mathbb{f}$ is the same symbol as in Theorem B. Then we can write $(\lambda+\mathbb{A})^{-1}$ as follows:

$$(1.3) \qquad (\lambda+\mathbb{A})^{-1} = (\lambda+\mathbb{A}_D)^{-1}\mathbb{N}_1 + \frac{1}{\lambda}(\lambda+\mathbb{A}_D)^{-1}\mathbb{N}_2 + \frac{1}{\lambda}\mathbb{N}_3,$$

which implies that $-\mathbb{A}$ is a closed linear operator in $X_q(D)$, $\rho(-\mathbb{A}) \supset \Sigma'$ and the following properties are valid:

$$|\lambda| \|(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{X_q^m(D)} + \|\mathbb{P}(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{m+2,q,D}$$
$$\leq C(\delta,m,D)\{\|\mathbb{f}\|_{X_q(D)} + \frac{1}{|\lambda|}\|f_1\|_{q,D}\}$$

for any $\lambda \in \Sigma_\delta$ and any $\mathbb{f} \in X_q^m(D)$. Also note that this estimate holds even if D is an exterior domain because usual ellptic estimates hold.

§2. On the stationary problem in R^3

In this section, we shall show the basic estimations of solutions to the following stationary linearized equations in R^3 with a complex parameter λ:

$$\begin{aligned} &\lambda\rho + \gamma\cdot \mathrm{div}v = f_1, \\ (2.1)\qquad &\lambda v - \alpha\Delta v - \beta\nabla(\mathrm{div}v) + \gamma\cdot\nabla\rho + \omega\cdot\nabla\theta = f_2 \text{ in } R^3, \\ &\lambda\theta - \kappa\Delta\theta + \omega\cdot \mathrm{div}v = f_3. \end{aligned}$$

By taking Fourier transform on (2.1) we obtain

$$[\lambda\cdot I + \hat{\mathbb{A}}(\xi)]\hat{\mathbb{u}} = \hat{\mathbb{f}},$$

where I is the identity, $\mathcal{F}(f) = \hat{f}$ stand for the Fourier transforms of f, $\mathbb{u} = {}^T(\rho, v, \theta)$, $\mathbb{f} = {}^T(f_1, f_2, f_3)$. Here $\hat{\mathbb{A}}(\xi)$ is 5×5 symmetric matrix as follows:

$$\hat{\mathbb{A}}(\xi) = \begin{pmatrix} 0 & i\gamma\xi_k & 0 \\ i\gamma\xi_j & \delta_{jk}\alpha|\xi|^2 + \beta\xi_j\xi_k & i\omega\xi_j \\ 0 & i\omega\xi_k & \kappa|\xi|^2 \end{pmatrix}$$

where $i = \sqrt{-1}$ and $\delta_{jk} = 0$ when $k \neq j$ and $= 1$ when $k = j$. Then we have

$$(2.2a)\qquad [\lambda\cdot I + \hat{\mathbb{A}}(\xi)]^{-1} = \{\det[\lambda\cdot I + \hat{\mathbb{A}}(\xi)]\}^{-1}\cdot\tilde{\mathbb{A}}(\lambda;\xi),$$

$$(2.2b)\qquad \det[\lambda\cdot I + \hat{\mathbb{A}}(\xi)] = (\lambda + \alpha|\xi|^2)^2 F(\lambda;|\xi|),$$

where

$$(2.2c)\quad F(\lambda;|\xi|) = \lambda^3 + (\alpha+\beta+\kappa)|\xi|^2\lambda^2 + [(\alpha+\beta)\kappa|\xi|^2 + \gamma^2 + \omega^2]|\xi|^2\lambda + \gamma^2\kappa|\xi|^4,$$

and $\tilde{\mathbb{A}}(\lambda;\xi) = (\tilde{a}_{ij}(\lambda;\xi))$ is the 5×5 matrix and the components are

$$\begin{aligned} &\tilde{a}_{11} = (\lambda+\alpha|\xi|^2)^2\{\lambda^2 + (\alpha+\beta+\kappa)|\xi|^2\lambda + [\omega^2 + (\alpha+\beta)\kappa|\xi|^2]\cdot|\xi|^2\}, \\ &\tilde{a}_{15} = \tilde{a}_{51} = -\gamma\omega(\lambda+\alpha|\xi|^2)^2|\xi|^2, \\ (2.2d)\qquad &\tilde{a}_{1,j} = \tilde{a}_{j,1} = -i\gamma(\lambda+\alpha|\xi|^2)^2(\lambda+\kappa|\xi|^2)\xi_{j-1} \quad (j = 2,3,4), \\ &\tilde{a}_{5,j} = \tilde{a}_{j,5} = -i\omega\lambda(\lambda+\alpha|\xi|^2)^2\xi_{j-1} \quad (j = 2,3,4), \\ &\tilde{a}_{55} = (\lambda+\alpha|\xi|^2)^2\{\lambda^2 + (\alpha+\beta)|\xi|^2\lambda + \gamma^2|\xi|^2\}, \end{aligned}$$

$$\tilde{a}_{ij} = (\lambda+\alpha|\xi|^2)\{\lambda(\lambda+\alpha|\xi|^2)(\lambda+\kappa|\xi|^2)\delta_{ij} + (\delta_{ij}|\xi|^2-\xi_{i-1}\xi_{j-1})(\beta\lambda^2+[\beta\kappa|\xi|^2+\omega^2+\gamma^2]\lambda+\gamma^2\kappa|\xi|^2)\},$$
$$(i,j = 2,3,4).$$

From the spectral analysis of $\hat{\mathbb{A}}(\xi)$ given by Matsumura and Nishida [12] (cf. Ponce [17]) we have

Lemma 2.1. Let $\{\lambda_j(\xi)\}_{j=1}^5$ be the roots of $\det[\lambda\cdot I+\hat{\mathbb{A}}(\xi)] = 0$, where $\lambda_4(\xi) = \lambda_5(\xi) = -\alpha|\xi|^2$. Then it follows that:

(i) $\lambda_j(\xi)$ depends on $|\xi|$ only, $\lambda_j(0) = 0$ and $\mathrm{Re}\lambda_j(\xi) < 0$ for any $|\xi| > 0$, $j = 1,\cdots,5$.

(ii) $\lambda_j(\xi) \neq \lambda_k(\xi)$, $j \neq k$ and j, $k = 1, 2, 3, 4$ for all $|\xi|$ except at most four points of $|\xi| > 0$.

(iii) There exist positive constants r_1 such that $\lambda_j(\xi)$ has a Taylor series expansion for $|\xi| < r_1$ as follows: $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$ is a complex number, $\lambda_3(\xi)$ is a real number and

$$\lambda_1(\xi) = (\gamma^2+\omega^2)^{1/2}(i|\xi|) + \frac{(\gamma^2+\omega^2)(\alpha+\beta)+\omega^2\kappa}{2(\gamma^2+\omega^2)}(i|\xi|)^2 + \cdots\cdot,$$

$$\lambda_3(\xi) = \frac{\gamma^2\kappa}{\gamma^2+\omega^2}(i|\xi|)^2 + \frac{\gamma^2\omega^2\kappa^2\{(\gamma^2+\omega^2)(\alpha+\beta)-\gamma^2\kappa\}}{(\gamma^2+\omega^2)^4}(i|\xi|)^4 + \cdots\cdot.$$

Similarly, there exist positive constants $r_2 > r_1$ such that $\lambda_j(\xi)$ has a Laurent series expansion for $|\xi| > r_2$ as follows: If $\alpha + \beta \neq \kappa$, then $\lambda_j(\xi)$ are real numbers and

$$\lambda_1(\xi) = (\alpha+\beta)(i|\xi|)^2 + \frac{\gamma^2\kappa - (\gamma^2+\omega^2)(\alpha+\beta)}{(\alpha+\beta)(\alpha+\beta-\kappa)} + \cdots\cdots\cdot,$$

$$\lambda_2(\xi) = \kappa(i|\xi|)^2 - \frac{\omega^2}{\kappa-\alpha-\beta} + \cdots\cdots\cdots\cdot,$$

$$\lambda_3(\xi) = -\frac{\gamma^2}{\alpha+\beta} + \cdots\cdots\cdots.$$

If $\alpha + \beta = \kappa$, then $\lambda_1(\xi) = \overline{\lambda_2(\xi)}$ is a complex number, $\lambda_3(\xi)$ is a real number and

$$\lambda_1(\xi) = \kappa(i|\xi|)^2 + \sqrt{\omega}(i|\xi|) + \cdots\cdots\cdot,$$

$$\lambda_3(\xi) = -\frac{\gamma^2}{\kappa} + \cdots\cdots\cdots.$$

(iv) There exists a positive constants β_0, β_1, β_2 and r_1 such that $-\beta_0|\xi|^2 \le \mathrm{Re}\lambda_j(\xi) \le -\beta_1|\xi|^2$ for $|\xi| < r_1$ and $\mathrm{Re}\lambda_j(\xi) < -\beta_2$ for $|\xi| > r_2$, $j = 1, 2, \cdots, 5$.

Now we set for $\mathbb{f} \in X_q(R^3)$, $\mathbb{f} = {}^T\{f_j\}_{j=1}^5$

$$\mathbb{R}_0(\lambda)\mathbb{f}(x) = \mathcal{F}^{-1}\{[\lambda\cdot I+\hat{\mathbb{A}}(\xi)]^{-1}\hat{\mathbb{f}}(\xi)\}(x) = {}^T\{\sum_{i=1}^5 R_{ij}(\lambda)f_i(x)\}_{j=1}^5, \tag{2.3}$$

where $R_{ij}(\lambda) = \mathcal{F}^{-1}\{[\lambda\cdot I+\hat{\mathbb{A}}(\xi)]^{-1}\tilde{a}_{ij}(\lambda;\xi)\mathcal{F}\}$. When $\mathbb{f} = {}^T\{f_1, \mathrm{f}_2, f_5\}$ where $\mathrm{f}_2 = (f_2, f_3, f_4)$ we shall use the representation as follows:

$$\mathbb{R}_0(\lambda)\mathbb{f}(x) = {}^T\{R_{0,\rho}(\lambda)\mathbb{f}(x), R_{0,v}(\lambda)\mathbb{f}(x), R_{0,\theta}(\lambda)\mathbb{f}(x)\}. \tag{2.4}$$

Then we shall have the following estimates of $\mathbb{R}_0(\lambda)\mathbb{f}$ which is the core of our argument.

Theorem 2.2. Let $1 < q < \infty$, b be a positive number and $X_{q,b}(R^3)$ be the same symbol as in (0.5). Then for any $\mathbb{f} \in X_{q,b}(R^3)$ any $\lambda \in \{\lambda \in \mathbf{C};\ \mathrm{Re}\lambda \ge 0,\ 0 < |\lambda| \le 1\}$

$$\|(\frac{d}{d\lambda})^k\mathbb{R}_0(\lambda)\mathbb{f}\|_{X_q(B_b)} + \|(\frac{d}{d\lambda})^k\mathbb{P}\mathbb{R}_0(\lambda)\mathbb{f}\|_{2,q,B_b} \le C\max\{1, |\lambda|^{1/2-k}\}\|\mathbb{f}\|_{X_q(R^3)},$$

where k are integers ≥ 0 and $C = C(q,b,k)$ is a constant.

Finally in this section, we shall investigate the continuity as $\lambda \to 0$ for the operator $\mathbb{R}_0(\lambda)$ and the properties for $\mathbb{R}_0(0)$.

Lemma 2.4. Let $1 < q < \infty$ and b be a positive number and let $\mathbb{f} \in X_{q,b}(R^3)$. Then ${}^T\mathbb{R}_0(0)\mathbb{f} \in W^1_{q,loc}(R^3)\times W^2_{q,loc}(R^3)\times W^2_{q,loc}(R^3)$ and

$$\lim_{R\to\infty} R^{-3}\int_{R<|x|<2R} |\mathbb{R}_0(0)\mathbb{f}|^q dx = 0. \tag{2.5}$$

Moreover,

$$\|{}^T\mathbb{R}_0(\lambda)\mathbb{f} - {}^T\mathbb{R}_0(0)\mathbb{f}\|_{W^1_{q,loc}(R^3)\times W^2_{q,loc}(R^3)\times W^2_{q,loc}(R^3)} \to 0 \tag{2.6}$$

as $\lambda \to 0$ and $\mathrm{Re}\lambda \geq 0$.

§3. Proof of Theorem A.

First note that by Lemma 2.1 (iii)

$\det[\lambda+\hat{\mathbb{A}}(\xi)] \neq 0$ for $\lambda \in \Sigma'' = \{\lambda \in \mathbf{C};\ C_1\mathrm{Re}\lambda + (\mathrm{Im}\lambda)^2 > 0\}$

where C_1 is a constant depending only on α, β, γ, κ, and ω. Combining this with Theorem 1.1 we take the constant C in the parabolic region $\Sigma = \{\lambda \in \mathbf{C};\ C_1\mathrm{Re}\lambda + (\mathrm{Im}\lambda)^2 > 0\}$ so that $\Sigma \subset \Sigma' \cap \Sigma''$. In view of Remark 1.4, we only show (0.5). Now we shall construct parametrix to (1.1) in Ω. Let $\partial\Omega \subset B_{R_0}$, b be a fixed constant $b > R_0 + 3$ and let $\Omega_b = \Omega \cap B_b$. Given $\lambda \in \Sigma$ and $g \in X_q(\Omega_b)$, let ${}^T w \in W_q^1(\Omega_b)\times W_q^2(\Omega_b)\times W_q^2(\Omega_b)$ be solutions to the problem:

$$(\lambda+\mathbb{A})w = g \text{ in } \Omega_b,$$
$$\mathbb{P}w = 0 \text{ on } \partial\Omega_b.$$

The existence of such w is guaranteed by Remark 1.4. In terms of w, let us define the operator $\mathbb{L}(\lambda)$ by relations:

$$(3.1)\qquad w = \mathbb{L}(\lambda)g = \{L_\rho(\lambda)g, L_v(\lambda)g, L_\theta(\lambda)g\}.$$

Here and hereafter, for $\mathbb{f} \in X_q(\Omega)$, we put $\mathbb{f}_0(x) = \mathbb{f}(x)$ for $x \in \Omega$ and $= 0$ for $x \in R^3 \setminus \Omega$, $\pi_b\mathbb{f}$ stands for the restriction of $\mathbb{f}$ to Ω_b. By Remark 1.4 and (3.1) we have

$$(3.2)\qquad \|\mathbb{L}(\lambda)\pi_b\mathbb{f}\|_{X_q(\Omega_b)} + \|\mathbb{P}\mathbb{L}(\lambda)\pi_b\mathbb{f}\|_{2,q,\Omega_b} \leq C(q,b,\lambda)\|\mathbb{f}\|_{X_q(\Omega)} \text{ for any } \mathbb{f} \in X_q(\Omega).$$

Let $\mathbb{R}_0(\lambda)$, $R_{0,\rho}(\lambda)$, $R_{0,v}(\lambda)$ and $R_{0,\theta}(\lambda)$ be the same symbol as in (2.3) and (2.4). Since $\det[\lambda+\hat{\mathbb{A}}(\xi)] \neq 0$ whenever $\xi \in R^3$ and $\lambda \in \Sigma$, by Theorem 7.9.5 of [6], we see that

$$(3.3)\qquad \|\mathbb{R}_0(\lambda)\mathbb{f}_0\|_{X_q(R^3)} + \|\mathbb{P}\mathbb{R}_0(\lambda)\mathbb{f}_0\|_{2,q,R^3}$$

$$\leq C(q,\lambda)\|\mathbb{f}\|_{X_q(\Omega)} \text{ for any } \mathbb{f} \in X_q(\Omega).$$

Let $\varphi \in C^\infty(R^3)$ such that $\varphi(x) = 0$ for $|x| \leq b - 2$ and $= 1$ for $|x| \geq b - 1$. We introduce the operator $\mathbb{Q}_1(\lambda)$ by the relations:

$$(3.4) \qquad \mathbb{Q}_1(\lambda)\mathbb{f} = {}^T\{\mathbb{Q}_{1,\rho}(\lambda)\mathbb{f}, Q_{1,v}(\lambda)\mathbb{f}, \mathbb{Q}_{1,\theta}(\lambda)\mathbb{f}\}$$
$$:= \varphi\mathbb{R}_0(\lambda)(\mathbb{f}_0) + (1-\varphi)\mathbb{L}(\lambda)\pi_b\mathbb{f} \text{ for any } \mathbb{f} \in X_q(\Omega),$$

Then by (3.2) and (3.3) we have

$$(3.5) \qquad {}^T\mathbb{Q}_1(\lambda)\mathbb{f} \in W^1_q(\Omega)\times W^2_q(\Omega)\times W^2_q(\Omega) \text{ for any } \mathbb{f} \in X_q(\Omega),$$

$$(3.6) \qquad \|\mathbb{Q}_1(\lambda)\mathbb{f}\|_{X_q(\Omega)} + \|\mathbb{P}\mathbb{Q}_1(\lambda)\mathbb{f}\|_{2,q,\Omega}$$
$$\leq C(q,\lambda,b)\|\mathbb{f}\|_{X_q(\Omega)} \text{ for any } \mathbb{f} \in X_q(\Omega),$$

and

$$(3.7a) \qquad (\lambda+\mathbb{A})\mathbb{Q}_1(\lambda)\mathbb{f} = \mathbb{f} + \mathbb{V}(\lambda)\mathbb{f} \text{ in } \Omega,$$

$$(3.7b) \qquad \mathbb{P}\mathbb{Q}_1(\lambda)\mathbb{f} = 0 \text{ on } \partial\Omega,$$

where $\mathbb{V}(\lambda)\mathbb{f} = {}^T\{V_\rho(\lambda)\mathbb{f}, V(\lambda)\mathbb{f}, V(\lambda)\mathbb{f}\}$ and

$$(3.8a) \qquad V_\rho(\lambda)\mathbb{f} = \gamma\nabla\varphi[R_{0,v}(\lambda)(\mathbb{f}_0) - L_v(\lambda)\pi_b\mathbb{f}],$$

$$(3.8b) \qquad V_v(\lambda)\mathbb{f} = -\alpha[\Delta\varphi + 2(\partial_j\varphi)\partial_j][R_{0,v}(\lambda)(\mathbb{f}_0) - L_v(\lambda)\pi_b\mathbb{f}]$$
$$- \beta\nabla\{\partial_j\varphi[R_{0,v}(\lambda)(\mathbb{f}_0) - L_v(\lambda)\pi_b\mathbb{f}]_j\}$$
$$- \beta\nabla\varphi\{\mathrm{div}[R_{0,v}(\lambda)(\mathbb{f}_0) - L_v(\lambda)\pi_b\mathbb{f}]\}$$
$$+ \gamma\nabla\varphi[R_{0,\rho}(\lambda)(\mathbb{f}_0) - L_\rho(\lambda)\pi_b\mathbb{f}]$$
$$+ \omega\partial_j\varphi[R_{0,\theta}(\lambda)(\mathbb{f}_0) - L_\theta(\lambda)\pi_b\mathbb{f}]_j,$$

$$(3.8c) \qquad V_\theta(\lambda)\mathbb{f} = -\kappa[\Delta\varphi + 2\partial_j\varphi\partial_j][R_{0,\theta}(\lambda)(\mathbb{f}_0) - L_\theta(\lambda)\pi_b\mathbb{f}]$$
$$+ \omega\partial_j\varphi[R_{0,v}(\lambda)(\mathbb{f}_0) - L_v(\lambda)\pi_b\mathbb{f}]_j.$$

Our task is to prove that $I + \mathbb{V}(\lambda)$ has the bounded inverse from $X_q(\Omega)$ onto itself. It follows from (3.2), (3.3) and (3.8) that ${}^T\mathbb{V}(\lambda) \in \mathcal{B}(X_q(\Omega), W^2_q(\Omega)\times W^1_q(\Omega)\times W^1_q(\Omega))$ for each $\lambda \in \Sigma$. Since $\mathrm{supp}\mathbb{V}(\lambda)\mathbb{f} \subset D_{b-1} = \{x \in R^3;\ b-2 < |x| < b-1\}$, by Rellich's compactness theorem $\mathbb{V}(\lambda)$ is a compact operator from $X_q(\Omega)$ onto itself. Thus by Fredholm's alternative theorem, it suffices to show that $I + \mathbb{V}(\lambda)$ is injective in $X_q(\Omega)$ in order to prove that $I + \mathbb{V}(\lambda)$ has the bounded inverse. Let $(I+\mathbb{V}(\lambda))\mathbb{f} = 0$ in Ω, $\mathbb{f} \in X_q(\Omega)$. Then it follows from (3.5), (3.7) and $\mathrm{Ker}\{\lambda+\mathbb{A}\} = 0$ that

$$\mathbb{Q}_1(\lambda)\mathbb{f} = 0 \text{ in } \Omega,$$

$$\mathbb{P}\mathbb{Q}_1(\lambda)\mathbb{f} = 0 \text{ on } \partial\Omega,$$

which together with (3.4) and implies that

(3.9a) $$\mathbb{R}_0(\lambda)(\mathbb{f}_0) = 0 \text{ for } |x| \geq b - 1,$$

(3.9b) $$\mathbb{L}(\lambda)\pi_b\mathbb{f} = 0 \text{ for } |x| \leq b - 2.$$

Put $\mathbb{z} = \pi_b\mathbb{R}_0(\lambda)(\mathbb{f}_0) - \mathbb{w}$ where $\mathbb{w} = \mathbb{L}(\lambda)\pi_b\mathbb{f}$ in Ω_b and $= 0$ in $R^3 \setminus \Omega$. By (3.9b) we know that ${}^T\mathbb{w} \in W^1_q(B_b)\times W^2_q(B_b)\times W^2_q(B_b)$ and

$$(\lambda+\mathbb{A})\mathbb{w} = \pi^0_b\mathbb{f}_0 \text{ in } B_b,\ \mathbb{P}\mathbb{w} = 0 \text{ on } |x| = b,$$

where $\pi^0_b\mathbb{f}_0$ stands for the restriction of $\mathbb{f}_0$ to B_b, and hence we see that

$$(\lambda+\mathbb{A})\mathbb{z} = 0 \text{ in } B_b,\ \mathbb{P}\mathbb{u} = 0 \text{ on } |x| = b,$$

which with the help of Theorem 1.1 means that $\mathbb{z} = 0$ in B_b. As a results, we have

(3.10) $$\mathbb{R}_0(\lambda)(\mathbb{f}_0) = \mathbb{L}(\lambda)\pi_b\mathbb{f} \quad \text{in } \Omega_b.$$

Combining (3.4) and (3.10), we see that

(3.11) $$\begin{aligned}\mathbb{R}_0(\lambda)(\mathbb{f}_0) &= \varphi\{\mathbb{R}_0(\lambda)(\mathbb{f}_0)-\mathbb{L}(\lambda)\pi_b\mathbb{f}\} + \mathbb{R}_0(\lambda)(\mathbb{f}_0)\\ &= \mathbb{Q}_1(\lambda)\mathbb{f} = 0 \text{ in } \Omega_b.\end{aligned}$$

It follows from (3.9) and (3.11) that $\mathbb{R}_0(\lambda)(\mathbb{f}_0) = 0$ in Ω, which together with (2.1) implies that $\mathbb{f}_0 = \mathbb{f} = 0$ in Ω. Therefore, we have prove that $(I+\mathbb{V}(\lambda))$ has the bounded inverse $(I+\mathbb{V}(\lambda))^{-1}$ from $X_q(\Omega)$ onto itself. Given $\mathbb{f} \in X_q(\Omega)$, if we put $\mathbb{u} = \mathbb{Q}_1(\lambda)(I+\mathbb{V}(\lambda))^{-1}$, by (3.7) and (3.6) we see that $(\lambda+\mathbb{A})\mathbb{u} = \mathbb{f}$ in $X_q(\Omega)$ and $\mathbb{u} \in \mathcal{D}(\mathbb{A})$, which means that the inverse $(\lambda+\mathbb{A})^{-1}$ of $(\lambda+\mathbb{A})$ exists, and it is bounded, that is by (3.6)

$$\|(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{X_q(\Omega)} + \|\mathbb{P}(\lambda+\mathbb{A})^{-1}\mathbb{f}\|_{2,q,\Omega}$$
$$\le C(q,b,\lambda)\|(I+\mathbb{V}(\lambda))^{-1}\|_{\mathscr{B}(X_q(\Omega))}\|\mathbb{f}\|_{X_q(\Omega)}$$

for any $\mathbb{f} \in X_q(\Omega)$, which completes the proof.

§4. Behaviour of $(\lambda+\mathbb{A})^{-1}$ near $\lambda = 0$

In this section we shall discuss behaviour of $(\lambda+\mathbb{A})^{-1}$ near $\lambda = 0$. Our goal of this section is to prove the following theorem. Set

$$Y_{q,b}(\Omega) = \{\mathbb{f} \in Y_q(\Omega_b);\ \mathbb{f}(x) = 0 \text{ for } x \in R^3 \setminus B_b\}.$$

Theorem 4.1. Let $1 < q < \infty$, b_0 be a number such that $B_{b_0} \supset R^3 \setminus \Omega$ and let $b > b_0$. Put $D_\varepsilon = \{\lambda \in \mathbf{C};\ \mathrm{Re}\lambda \ge 0,\ 0 < |\lambda| \le \varepsilon\}$, $\mathscr{Y} = \mathscr{B}(Y_{q,b}(\Omega);\mathscr{D}(\mathbb{A}))$ and $\mathscr{A}(D_\varepsilon;\mathscr{Y})$ is the set of all $\mathscr{Y}$-valued holomorphic functions in D_ε. Then, there exists a positive number ε and $\mathbb{R}(\lambda) \in \mathscr{A}(D_\varepsilon;\mathscr{Y})$ such that $\mathbb{R}(\lambda)\mathbb{f} = (\lambda+\mathbb{A})^{-1}\mathbb{f}$ and

$$\|(\tfrac{d}{d\lambda})^k\mathbb{R}(\lambda)\mathbb{f}\|_{X_q(\Omega_b)} + \|(\tfrac{d}{d\lambda})^k\mathbb{P}\mathbb{R}(\lambda)\mathbb{f}\|_{2,q,\Omega_b}$$
$$\le C(q,b,k,\varepsilon)\max\{1,|\lambda|^{1/2-k}\}\|\mathbb{f}\|_{X_q(\Omega)},\quad k = 0,1,2.$$

for any $\lambda \in D_\varepsilon$ and $\mathbb{f} \in Y_{q,b}(\Omega)$.

In Theorem 4.1, in view of proof of Remark 1.4, taking $\psi \in C_0^\infty(\Omega_b)$ such that $\int_{\Omega_b}\psi(x)dx = 1$, we have the following corollary:

Corollary 4.2. Let $1 < q < \infty$, b_0 be a number such that $B_{b_0} \supset R^3 \setminus \Omega$ and let $b > b_0$. Put $\mathscr{X} = \mathscr{B}(X_{q,b}(\Omega);\mathscr{D}(\mathbb{A}))$. Then, there exists a positive number ε and $\mathbb{R}(\lambda) \in \mathscr{A}(D_\varepsilon;\mathscr{X})$ such that $\mathbb{R}(\lambda)\mathbb{f} = (\lambda+\mathbb{A})^{-1}\mathbb{f}$ and

$$\|(\tfrac{d}{d\lambda})^k\mathbb{R}(\lambda)\mathbb{f}\|_{X_q(\Omega_b)} + \|(\tfrac{d}{d\lambda})^k\mathbb{P}\mathbb{R}(\lambda)\mathbb{f}\|_{2,q,\Omega_b}$$
$$\le C(q,b,k,\varepsilon)\max\{1,|\lambda|^{1/2-k}\}\{\|\mathbb{f}\|_{X_q(\Omega)}+|\lambda|^{-1}\|f_1\|_{q,\Omega}\}$$

for $k = 0, 1, 2$, any $\lambda \in D_\varepsilon$ and $\mathbb{f} = \{f_1,f_2,f_3\} \in X_{q,b}(\Omega)$. Moreover,

$$\mathbb{R}(\lambda) = \mathbb{R}(\lambda)\mathbb{N}_1 + \frac{1}{\lambda}\mathbb{R}(\lambda)\mathbb{N}_2 + \frac{1}{\lambda}\mathbb{N}_3$$

where $\mathbb{N}_j = \mathbb{N}_j(\psi, \Omega_b)$ $j = 1, 2, 3$, are the same symbols as in (1.2).

To prove Theorem 4.1, in the same way to the proof of Theorem A we shall construct a parametrix near $\lambda = 0$. The key in our argument is the following proposition concerning the uniqueness, which was proved by Iwashita [9].

Proposition 4.3. Let $1 < q < \infty$ and let $W^m_{q,E}(\Omega) = \{u;$ there exists a $U \in W^m_{q,loc}(R^3)$ such that $u = U$ in $\Omega\}$. Suppose that $\mathbb{u} \in W^1_{q,E}(\Omega)\times W^2_{q,E}(\Omega)\times W^2_{q,E}(\Omega)$ satisfies the homogeneous equation:

$$\mathbb{A}\mathbb{u} = 0 \text{ in } \Omega, \ \mathbb{P}\mathbb{u} = 0 \text{ in } \partial\Omega$$

and satisfies

$$\lim_{R\to\infty} \frac{1}{R^3}\int_{R<|x|<2R} |\mathbb{u}(x)|^q dx = 0.$$

Then $\rho = 0$, $v = 0$ and $\theta = 0$ in Ω.

§5. Proof of Theorem B

In this section, we shall prove Theorem B. To do this we prepare the following lemma, which was proved by Shibata. (see Theorem 3.2 and 3.7 of [18])

Lemma 5.1. Let X be a Banach space with norm $|\cdot|_X$. Let $f(\tau)$ be a function of $C^\infty(R-\{0\};X)$ such that $f(\tau) = 0$, $|\tau| \geq a$ with some $a > 0$. Assume that there exists a constant $C(f)$ depending on f such that for any $0 < |\tau| \leq a$,

$$|(\frac{d}{d\tau})^k f(\tau)|_X \leq C(f)|\tau|^{-1/2-k}, \ k = 0, 1.$$

Put $g(t) = \int_{-\infty}^{\infty} f(\tau)e^{-it\tau}d\tau$. Then

$$|g(t)|_X \leq C(1+t)^{-1/2}C(f).$$

Now we shall prove Theorem B. In view of the facts that when $0 <$ ≤ 1 by Theorem A we have

$$\|\partial_t^M e^{-t\mathbb{A}}\mathbb{u}\|_{X_q(\Omega)} + \|\mathbb{P}\partial_t^M e^{-t\mathbb{A}}\mathbb{u}\|_{2,q,\Omega}$$
$$\leq C\|(1+\mathbb{A})^{M+N}e^{-t\mathbb{A}}\mathbb{u}\|_{X_q(\Omega)} \leq Ct^{-N-M}\|\mathbb{u}\|_{X_q(\Omega)}$$

for any $\mathbb{u} \in X_q(\Omega)$ and any integers $N \geq 1$, $M \geq 0$, we only to show the case $t \geq 1$. Note that by Corollary 7.5 of [16, Chapter 1] we can write

$$e^{-t\mathbb{A}}\mathbb{u} = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{u}d\lambda = -\frac{1}{2\pi i t}\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}\frac{d}{d\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{u}d\lambda \tag{5.1}$$

for all $\mathbb{u} \in \mathcal{D}(\mathbb{A}^2)$, because

$$\|\frac{d}{d\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{u}\|_{X_q(\Omega)} \leq \frac{C(\beta)}{1+|\lambda|^2}\|\mathbb{u}\|_{X_q(\Omega)} \quad \text{for any } \mathrm{Re}\lambda \geq \beta \tag{5.2}$$

by Theorem A. Since $\mathcal{D}(\mathbb{A}^2)$ is dense in $X_q(\Omega)$, the equation (5.1) holds in $X_q(\Omega)$.

First we shall consider the case $\mathbb{u} \in Y_{q,b}(\Omega)$. Let $b > b_0$ and let $\psi \in C_0^\infty(R^3)$ such that $\psi(x) = 1$ for $|x| \leq b$ and $= 0$ for $|x| \geq b + 1$. Since we can move the path in the following integral to the imaginary axis by Theorem 4.1, (5.1) and (5.2), we have

$$\partial_x^\alpha \psi e^{-t\mathbb{A}}\mathbb{u} = \frac{1}{2\pi i t}\partial_x^\alpha\{\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}\psi\frac{d}{d\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{u}d\lambda\}$$
$$= \frac{1}{2\pi t}\partial_x^\alpha\{\int_{-\infty}^{\infty} e^{its}\psi\frac{d}{ds}(is+\mathbb{A})^{-1}\mathbb{u}ds\}$$

for any $\mathbb{u} \in Y_{q,b}(\Omega)$ and multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$: $\alpha_2 = (\alpha_{2,1},\alpha_{2,2},\alpha_{2,3})$, $\alpha_1 \leq 1$, $|\alpha_2| \leq 2$ and $\alpha_3 \leq 2$. Taking $\eta(s) \in C^\infty(R)$ so that $\eta(s) = 1$ for $|s| \leq 1/4$ and $= 0$ for $|s| \geq 1/2$ we have

$$\partial_x^\alpha \psi e^{-t\mathbb{A}}\mathbb{u} = \mathbb{J}_0(t)\mathbb{u} + \mathbb{J}_1(t)\mathbb{u} \tag{5.3}$$

where

$$\mathbb{J}_0(t)\mathbb{u} = \frac{1}{2\pi t}\partial_x^\alpha(\psi\int_{-\infty}^{\infty} e^{its}\eta(s)\frac{d}{ds}(is+\mathbb{A})^{-1}\mathbb{u}ds),$$

$$\mathbb{J}_\infty(t)\mathbb{u} = \frac{1}{2\pi t}\partial_x^\alpha(\psi\int_{-\infty}^{\infty} e^{its}(1-\eta(s))\frac{d}{ds}(is+\mathbb{A})^{-1}\mathbb{u}ds).$$

By Theorem A we have

$$(5.4)\qquad \begin{aligned}&\|\partial_x^\alpha(1-\eta(s))(\tfrac{d}{ds})^N(is+\mathbb{A})^{-1}\mathbb{u}\|_{q,\Omega}\\ &\le (1-\eta(s))\{\|(is+\mathbb{A})^{-N-1}\mathbb{u}\|_{X_q(\Omega)}+\|\mathbb{P}(is+\mathbb{A})^{-N-1}\mathbb{u}\|_{2,q,\Omega}\}\\ &\le C(N)(1+|s|)^{-N}\|\mathbb{u}\|_{X_q(\Omega)},\end{aligned}$$

and hence by the relation $\frac{1}{t}\cdot\frac{d}{d\lambda}e^{t\lambda} = e^{t\lambda}$, we have

$$(5.5)\qquad \|\partial_t^M\mathbb{J}_\infty(t)\mathbb{u}\|_{q,\Omega} \le C(N,M,\alpha)t^{-N}\|\mathbb{u}\|_{X_q(\Omega)}$$

for any integers $N \ge 2$, $M \ge 0$. On the other hand, noting that

$$\begin{aligned}\partial_t^M\mathbb{J}_0(t)\mathbb{f} &= \frac{1}{2\pi}\sum_{n=0}^{M}\binom{M}{n}\partial_t^{M-n}t^{-1}\partial_x^\alpha\{\psi\int_{-\infty}^{\infty} e^{ist}\eta(s)(is)^n\frac{d}{ds}\mathbb{R}(is)\mathbb{f}ds\}\\ &= t^{-(M+1)}\sum_{n=0}^{M}c(n)\partial_x^\alpha\{\psi\int_{-\infty}^{\infty} e^{ist}(\frac{d}{ds})^n\{\eta(s)(is)^n\frac{d}{ds}\mathbb{R}(is)\mathbb{f}\}ds\},\end{aligned}$$

it follows from Theorem 4.1 and Lemma 5.1 that

$$(5.6)\qquad \|\partial_t^M\mathbb{J}_0(t)\mathbb{u}\|_{q,\Omega} \le C(M,b,q)(1+t)^{-(M+3/2)}\|\mathbb{u}\|_{X_q(\Omega)}$$

for any $\mathbb{u} \in Y_{q,b}(\Omega)$, integer $M \ge 0$ and $t \ge 1$. Combining (5.3), (5.5) and (5.6) we have for any $\mathbb{u} \in Y_{q,b}(\Omega)$, integer $M \ge 0$ and $t \ge 1$

$$(5.7)\qquad \|\partial_t^M e^{-t\mathbb{A}}\mathbb{u}\|_{Y_{q,b}(\Omega)} + \|\partial_t^M\mathbb{P}e^{-t\mathbb{A}}\mathbb{u}\|_{2,q,\Omega_b} \le C(1+t)^{-3/2-M}\|\mathbb{u}\|_{Y_{q,b}(\Omega)}.$$

Next we shall consider the case $\mathbb{u} \in X_{q,b}(\Omega)$. Taking $\phi \in C_0^\infty(\Omega_b)$, such that $\int_{\Omega_b}\phi(x)dx = 1$, in view of Remark 1.4, we have

$$(\lambda+\mathbb{A})^{-1}\mathbb{u} = (\lambda+\mathbb{A})^{-1}\mathbb{N}_1\mathbb{u} + \frac{1}{\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{N}_2\mathbb{u} + \frac{1}{\lambda}\mathbb{N}_3\mathbb{u} \text{ for } \mathbb{u} \in X_{q,b}(\Omega)$$

where $\mathbb{N}_j = \mathbb{N}_j(\phi,\Omega_b)$ $(j = 1,2,3)$ be the same symbol as in (1.2). Combining this and (5.1), we have

$$(5.8)\qquad \begin{aligned}e^{-t\mathbb{A}}\mathbb{u} &= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{N}_1\mathbb{u}d\lambda\\ &+ \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{N}_2\mathbb{u}\frac{d\lambda}{\lambda}\end{aligned}$$

$$+ \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} \frac{1}{\lambda} e^{t\lambda} \mathbb{N}_3 \mathbb{u} d\lambda.$$

Putting $\mathbb{T}_1(b,\phi,t)\mathbb{u} = e^{-t\mathbb{A}}\mathbb{N}_1\mathbb{u}$ and $\mathbb{T}_2(b,\phi,t)\mathbb{u} = \int_0^t e^{-s\mathbb{A}}\mathbb{N}_2\mathbb{u}ds + \mathbb{N}_3\mathbb{u}$, since $\frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty}\frac{1}{\lambda}e^{t\lambda}\mathbb{u}d\lambda = \mathbb{u}$ for any $\mathbb{u} \in X_q(\Omega)$, and since by Theorem 7.4 of [16, Chapter 1] we have

$$\int_0^t e^{-s\mathbb{A}}\mathbb{u}ds = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{u}\frac{d\lambda}{\lambda} \text{ for } \mathbb{u} \in \mathcal{D}(\mathbb{A}) \text{ and } t > 0,$$

it follows from (5.1) and (5.8) that the relation (0.6) holds. Moreover, noting that $\mathbb{N}_1\mathbb{u}$, $\mathbb{N}_2\mathbb{u} \in Y_{q,b}(\Omega)$, since by (5.1) and (5.8) we have

$$\partial_t e^{-t\mathbb{A}}\mathbb{u} = \partial_t\{\frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{N}_1\mathbb{u}d\lambda\} + \frac{-1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} e^{t\lambda}(\lambda+\mathbb{A})^{-1}\mathbb{N}_2\mathbb{u}d\lambda, \tag{5.9}$$

it follows from (5.7), (5.8) and (5.9) that the estimates (0.7) and (0.8) hold. This completes the proof of Theorem B.

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