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Asymptotic decay toward the planar rarefaction waves of solutions for viscous conservation laws in several space dimensions

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Abstract

This paper concerns the asymptotic decay rate toward the planar rarefaction waves of the solutions for the scalar viscous conservation laws in two or more space dimensions. This is proved by a result on the decay rate of solutions for one dimensional scalar viscous conservation laws and by using an L^2-energy method with a weight of time.

1 Introduction and main result

In this paper, we present the asymptotic decay rate, toward the planar rarefaction waves, of the solutions for scalar viscous conservation laws in two or more space dimensions. Since the proof of the result for the case in more than two dimension will be identical to that for the case in two dimension, we only discuss the equation of the following form:

$$u_t + (f(u))_x + (g(u))_y = u_{xx} + u_{yy}, \tag{1}$$

$$u(0,x,y) = u_0(x,y), \tag{2}$$

where $u = u(t,x,y)$ is a scalar function of time $t \geq 0$ and position $(x,y) \in R^2$. We assume that nonlinear flux functions f and g are smooth and also assume that f is convex i.e., for a fixed constant $\alpha > 0$,

$$f''(u) \geq \alpha. \tag{3}$$

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The initial condition satisfies

$$u_0(x,y) \to u_\pm \quad \text{as } x \to \pm\infty, \tag{4}$$

where $u_\pm$ are constants satisfying $u_- < u_+$.

A planar rarefaction wave is a weak solution of the following problem

$$r_t + (f(r))_x = 0, \tag{5}$$

$$r(0,x) = r_0(x), \tag{6}$$

where $r_0(x)$ is given by

$$r_0(x) = \begin{cases} u_-, & \text{for } x < 0, \\ u_+, & \text{for } x > 0. \end{cases} \tag{7}$$

Then $r(t,x)$ is given explicitly,

$$r(t,x) = \begin{cases} u_-, & \text{for } x < a(u_-)t, \\ a^{-1}(x/t), & \text{for } a(u_-)t < x < a(u_+)t, \\ u_+, & \text{for } a(u_+)t < x, \end{cases} \tag{8}$$

where $a = a(u)$ is defined by

$$a(u) = f'(u). \tag{9}$$

Note

$$a'(u) \geq \alpha > 0. \tag{10}$$

The stability of rarefaction waves was originally considered by Il'in and Oleinik [3], and has recently been studied by many authors [8, 5, 6, 7, 9, 10].

Harabetian [1] first studied the asymptotic decay rate toward rarefaction waves of the solutions of the scalar viscous conservation laws in one space dimension. Hattori and Nishihara [2] showed a more precise result on the decay rate toward rarefaction waves of solutions to the one dimensional Burgers equation instead of the scalar viscous conservation law in one space dimension. Xin [11] first proved the asymptotic stability of planar rarefaction waves for the several dimensional scalar viscous conservation laws, but the paper [11] did not refer to the decay rate of solutions to (1)-(2) toward rarefaction waves.

In this paper we give the asymptotic convergence rate toward the planar rarefaction wave $r(t,x)$ of the solution $u(t,x,y)$ for (1)-(2) in $L^\infty(R_y; L^2(R_x))$. To state our result, following Matsumura and Nishihara [6], we introduce the smooth rarefaction wave, which is a smooth solution of the following problem:

$$w_t + (f(w))_x = 0, \tag{11}$$

$$w(0,x) = w_0(x), \tag{12}$$

where $w_0(x)$ is defined by

$$w_0(x) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2}\kappa \int_0^x (1+\xi^2)^{-1} d\xi, \tag{13}$$

where

$$\kappa = \left(\int_0^\infty (1+\xi^2)^{-1} d\xi \right)^{-1}.$$

We here introduce notations used throughout this paper. For $1 \leq p \leq \infty$, $L^p(R^N)$ is the usual Lebesgue space on R^N. For positive integers m, $W^{m,p}(R^N)$ is the space of all functions whose weak derivatives up to m-th order belong to $L^p(R^N)$. $H^m(R^N)$ denotes $W^{m,2}(R^N)$.

Now we are in position to state our main theorem.

Theorem 1 *Suppose that $u_0 - w_0 \in (H^2 \cap L^1)(R^2)$. Then, there exist positive constants δ_0 and δ_0' such that if*

$$\|u_0 - w_0\|_{H^2(R^2)} \leq \delta_0 \quad and \quad |u_+ - u_-| \leq \delta_0', \tag{14}$$

then the problem (1)-(2) has a smooth unique global solution $u(t,x,y)$ satisfying

$$\sup_{y \in R} \|u(t) - r(t)\|_{L^2(R_x)} \leq Ct^{-1/4} \log(2+t) \quad for\, t > 0, \tag{15}$$

where C is a positive constant depending on u_0 and $|u_+ - u_-|$.

It is possible to say that our method to prove Theorem 1 is valid for more space dimensional case, for the proof is identical.

The rest of the paper is organized as follows. In Section 2, we show that the original planar rarefacion wave $r(t,x)$ in (8) are approximated by the smooth rarefaction wave $w(t,x)$ in (11) in $L^2(R)$ at the asymptotic rate $O(t^{-1/4})$ as $t \to \infty$. In Section 3, it is shown by using an L^2-energy method with polynomial and logarithmic weight of time that the asymptotic behavior of the smooth rarefaction wave $w(t,x)$ in $L^2(R)$ is described by the solution of the viscous scalar conservation law in one space dimension, that is, they converge to each other in $L^2(R)$ at the asymptotic rate $O(t^{-1/4}\log t)$ as $t \to \infty$. Finally, by making use of the result in Sections 2 and 3, we give the proof of Theorem 1. L^2-energy method with weight of time also plays an crucial role here.

2 Convergence of $w(t,x) - r(t,x)$

Throughout this paper, C denote generic positive constants.

The smooth rarefaction wave has the following properties.

Lemma 1 ([5, 6, 7]) *(i)* $u_- < w(t,x) < u_+$, $w_x(t,x) > 0$ *for* $(t,x) \in [0,\infty) \times R$.
(ii) For all p *with* $1 \leq p \leq \infty$ *there is a constant* C_p *such that*

$$\|w_x(t)\|_{L^p(R)} \leq C_p \min(d, d^{1/p} t^{-1+1/p}), \tag{16}$$
$$\|w_{xx}(t)\|_{L^p(R)} \leq C_p \min(d, d^{-(p-1)/2p} t^{-(1+(p-1)/2p)}), \tag{17}$$

for $t > 0$, *where*

$$d = u_+ - u_-. \tag{18}$$

Furthermore we need the following lemma throughout this paper.

Lemma 2 *(i) For all* p *with* $1 \leq p \leq \infty$, *there is a constant* $C_{p,d}$ *such that*

$$\|w_{xxx}(t)\|_{L^p(R)} \leq C_{p,d}(1+t)^{-(1+(2p-1)/2p)}, \tag{19}$$

for $t \geq 0$.
(ii) For all p *with* $1 < p \leq \infty$, *there is a constant* $C_{p,d}$ *such that*

$$\|w(t) - r(t)\|_{L^p(R)} \leq C_{p,d} t^{-(p-1)/2p}, \tag{20}$$

for $t > 0$.

For the proof of Lemma 2, see [4].

3 Approximation of $w(t,x)$ by a solution of one dimentional scalar viscous conservation law

In this section we study the convergence rate between the smooth rarefaction wave $w(t,x)$ and a solution $U(t,x)$ of the scalar viscous conservation law in one space dimension:

$$\begin{cases} U_t + (f(U))_x = U_{xx}, \\ U(0,x) = U_0(x). \end{cases} \tag{21}$$

Our aim in this section is to obtain a detailed asymptotic behavior of $U(t,x)$ in large time to be able to get the convergence rate toward $r(t,x)$ of solutions to the two dimensional scalar viscous conseravation laws.

To do this, we decompose the solution as

$$U(t,x) = w(t,x) + v(t,x).$$

Then, the problem (21) is redeced to

$$v_t + (a(w)v)_x + (F(w,v)v^2)_x = v_{xx} + w_{xx}, \tag{22}$$

$$v(0,x) = v_0(x) \equiv U_0(x) - w_0(x), \tag{23}$$

where

$$F(w,v) = \frac{f(w+v) - f(w) - f'(w)v}{v^2}. \tag{24}$$

Note that F is a smooth and bounded function of (w,v). Thus, the problem we consider from now on becomes (22)-(23). We begin by showing the local existence result.

Lemma 3 (local existence) *Suppose that $v_0 \in H^2(R) \cap L^1(R)$. Then there is a positive constant T_0 depending on $\|v_0\|_{H^2(R)\cap L^1(R)}$ and $d = u_+ - u_-$ such that the problem (22)-(23) has a unique solution $v(t,x)$ satisfying*

$$\begin{aligned} v &\in C^0([0,T_0); H^2(R)) \cap C^1([0,T_0); L^2(R)) \\ &\quad \cap L^2([0,T_0); H^3(R)) \cap C^0([0,T_0); L^1(R)), \\ t^{1/2} v_x &\in C^0([0,T_0); L^1(R)). \end{aligned} \tag{25}$$

Lemma 3 is proved in the standard way, so we omit the proof.

Remark: It should be noted that $\|v_x(t)\|_{L^1(R)}$ is integrable for the time variable even in the neighborhood of $t = 0$. The reason why that holds is $v(t,x)$ is obtaind as a fixed point of a mapping

$$\Psi(v)(t) = G(t) * v_0 - \int_0^t \partial_x G(t-s) * [a(w)v + F(w,v)v^2 - w_x](s)ds,$$

where $G(t,x)$ is the Gauss kernel in one space dimension and $*$ denotes the convolution with respect to the space variable. It then follows from the expression of Ψ and tha lack of the space-integrability of $\partial_x v_0(x)$ that $v_x(t,\cdot)$ has the order $O(t^{-1/2})$ in $L^1(R)$, which shows the claim mentioned above.

Next, we state a priori estimate of v.

Lemma 4 (a priori estimate) *Suppose that $v(t,x)$ is a solution of (22)-(23) belonging to the class as in (25) with $0 \le t \le T$.*

(i) There holds

$$\|v(t)\|_{L^1(R)} \le \|v_0\|_{L^1(R)} + C_d \log(1+t), \tag{26}$$

where C_d is a constant depending on the size of $d = u_+ - u_-$.

(ii) There exists a constant $\delta_1 > 0$ such that if

$$N(T) \equiv \sup_{0 \le t \le T} \|v(t)\|_{H^2(R)} \le \delta_1, \tag{27}$$

and $d \le \delta_1$, then $v(t,x)$ satisfies

$$\|\partial_x^k v(t)\|^2_{L^2(R)} + \int_0^t \int_R w_x |\partial_x^k v(s,x)|^2 ds dx$$
$$+ \int_0^t \|\partial_x^k v_x(s)\|^2_{L^2(R)} ds \le C(\|v_0\|^2_{H^k(R)} + \omega(d)), \quad \textit{for } 0 \le t \le T,\ k = 0,1,2, \quad (28)$$

where $\omega(d)$ is a constant satisfying

$$\omega(d) \to 0 \quad \textit{as } d \to 0.$$

Proof of Lemma 4. The $H^2(R)$-bound of v (28) can be proved by a method as in [11] as well as by applying Lemma 1, so we omit the proof of it.

Here we only show (26). Let $j_\delta(\lambda)$ be the usual smoothing kernel in R^1, i.e.,

$$j_\delta(\lambda) = \delta^{-1} j(\lambda/\delta), \tag{29}$$

where j is a smooth function which has a compact support and satisfies $\int_R j(\lambda)d\lambda = 1$. Let ϕ_δ be the convolution of the sign function and j_δ, i.e.,

$$\phi_\delta(\lambda) = (j_\delta * \mathrm{sign})(\lambda), \tag{30}$$

and put

$$\Phi_\delta(\lambda) = \int_0^\lambda \phi_\delta(\xi)d\xi. \tag{31}$$

Note

$$\phi'_\delta(\lambda) = 2j_\delta(\lambda). \tag{32}$$

Multiplying (22) by $\phi_\delta(v)$ and integrating it with respect to t and x, we have

$$\begin{aligned} &\int_R \Phi_\delta(v)dx + \int_0^t \int_R \phi_\delta(v)\{a(w)v + F(w,v)v^2\}_x dx ds \\ &= \int_R \Phi_\delta(v_0)dx + \int_0^t \int_R \phi_\delta(v) v_{xx} dx ds + \int_0^t \int_R \phi_\delta(v) w_{xx} dx ds. \end{aligned} \tag{33}$$

Claim: There holds

$$H(t) \equiv \int_0^t \int_R \phi_\delta(v)\{a(w)v + F(w,v)v^2\}_x dx ds \to 0 \tag{34}$$

as $\delta \to 0$ for each t.

To see this, integrating by parts and making use of (32), we get

$$\begin{aligned} H(t) &= -\int_0^t \int_R 2j_\delta(v)v_x\{a(w)v + F(w,v)v^2\}dxds \\ &= -2\int_0^t \int_R \left(\int_0^v j_\delta(\xi)\xi d\xi\right)_x \{a(w) + F(w,v)v\}dxds \\ &= 2\int_0^t \int_R \int_0^v j_\delta(\xi)\xi d\xi \cdot \{a'(w)w_x + F_w w_x v + F_v v_x v + F v_x\}dxds. \end{aligned} \tag{35}$$

It follows from the definition of j_δ

$$\left|\int_0^v j_\delta(\xi)\xi d\xi\right| \le \delta \int_0^\infty j(\xi)\xi d\xi \le C\delta.$$

Then, the integrand in (35) does not exceed

$$C\delta(w_x + w_x|v| + |vv_x| + |v_x|),$$

which tends to 0 as $\delta \to 0$ for almost every (t,x), and integrable on $(0,t)\times R$ for each t in view of Remark under Lemma 3. Thus the Lebesgue dominated convergence theorem implies (34). The claim has been verified.

Noting

$$\int_0^t\int_R \phi_\delta(v)v_{xx}dxds = -\int_0^t\int_R \phi_\delta'(v)v_x^2dxds \le 0,$$

and making use of Lemma 1, we let $\delta \to 0$ in (33) and obtain the desired estimate (26). The proof of Lemma 4 is complete.

Combining Lemmas 3 and 4, we obtain the global existence result.

Theorem 2 (global existence) *Suppose that $v_0 \in H^2(R)\cap L^1(R)$. Then, if*

$$\max(\|v_0\|_{H^2(R)}, d) < \delta_1,$$

then the problem (22)-(23) has a unique global solution $v(t,x)$ satisfying

$$\begin{aligned} v &\in C^0([0,\infty);H^2(R))\cap C^1([0,\infty);L^2(R)) \\ &\quad \cap C^0([0,\infty);L^1(R)), \\ v_x &\in L^2([0,\infty);H^2(R)), \\ t^{1/2}v_x &\in C^0([0,\infty);L^1(R)), \end{aligned}$$

and the estimates (26) and (28) hold for any $T>0$.

Our main result in this section is the following decay estimate of $v(t,x)$, which states the convergence rate of $U(t,\cdot)$ toward the smooth rarefaction wave $w(t,\cdot)$ in $H^2(R)$.

Theorem 3 (decay estimate) *Let $v(t,x)$ be the solution of (22)-(23) obtained in Theorem 2. Then, for any $\varepsilon > 0$ there exists a constant $C>0$ such that the following decay estimates hold for $v(t,x)$:*

$$\begin{aligned} &(1+t)^{k+1/2+\varepsilon}\|\partial_x^k v(t)\|_{L^2(R)}^2 + \int_0^t (1+s)^{k+1/2+\varepsilon}\int_R w_x(s,x)|\partial_x^k v(s,x)|^2dxds \\ &+\int_0^t (1+s)^{k+1/2+\varepsilon}\|\partial_x^k v_x(s)\|_{L^2(R)}^2ds \le C\gamma_k^2(1+t)^\varepsilon \rho_k(t)^2, \end{aligned} \tag{36}$$

where $\gamma_k = \|v_0\|_{(H^k\cap L^1)(R)} + C_d$ and $\rho_k(t) = \log^{2^{k+1}-1}(2+t)$ for $k=0,1,2$.

Corollary 1 (Convergence of $U-r$) *There exists a constant $C>0$ depending on U_0 and d such that*

$$\|U(t)-r(t)\|_{L^2(R)} \le Ct^{-1/4}\log(2+t) \quad \textit{for } t>0. \tag{37}$$

Proof of Corollary 1. Combining Theorem 3 with Lemma 2, we can obtain (37). The proof of Corollary 1 is complete.

Proof of Theorem 3. We first show (36) with $k=0$. When $N(t)$ is small ($N(t)$ is defined by (27)), by applying the L^2-energy method as in [11], we find

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(R)} + \frac{1}{2}\alpha\int_R w_x v^2 dx + \|v_x(t)\|^2_{L^2(R)} \le C\left|\int_R v w_{xx} dx\right|. \tag{38}$$

By Lemma 1, the right hand side of (38) is estimated as follows:

$$\left|\int_R v w_{xx} dx\right| \le C\|v(t)\|_{L^\infty(R)}\|w_{xx}(t)\|_{L^1(R)} \le C_d\|v(t)\|_{L^\infty(R)}(1+t)^{-1}. \tag{39}$$

Multiplying (38) by $(1+t)^{1/2+\varepsilon}$ and taking into account of (39), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{1/2+\varepsilon}\|v(t)\|^2_{L^2(R)} + \frac{1}{2}\alpha(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx \\
&\quad +(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)} \\
&\le C(1+t)^{-1/2+\varepsilon}\|v(t)\|^2_{L^2(R)} + C_d(1+t)^{-1/2+\varepsilon}\|v(t)\|_{L^\infty(R)} \\
&\le C(\|v(t)\|_{L^1(R)} + C_d)(1+t)^{-1/2+\varepsilon}\|v(t)\|_{L^\infty(R)}.
\end{aligned} \tag{40}$$

Making use of (26) and Sobolev inequality

$$\|f\|_{L^\infty(R)} \le C\|f\|^{1/3}_{L^1(R)}\|f_x\|^{2/3}_{L^2(R)}, \tag{41}$$

we compute in (40):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{1/2+\varepsilon}\|v(t)\|^2_{L^2(R)} + \frac{1}{2}\alpha(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx \\
&\quad +(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)} \\
&\le C(\|v(t)\|_{L^1(R)} + C_d)(1+t)^{-1/2+\varepsilon}\|v(t)\|^{1/3}_{L^1(R)}\|v_x(t)\|^{2/3}_{L^2(R)} \\
&\le C(\|v_0\|_{L^1(R)} + C_d)^{4/3}(1+t)^{-1/2+\varepsilon}\log^{4/3}(2+t)\|v_x(t)\|^{2/3}_{L^2(R)} \\
&\le C(\|v_0\|_{L^1(R)} + C_d)^2(1+t)^{-1+\varepsilon}\log^2(2+t) \\
&\quad +\frac{1}{2}(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)},
\end{aligned}$$

that is,

$$\frac{d}{dt}(1+t)^{1/2+\varepsilon}\|v(t)\|_{L^2(R)}^2+\alpha(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx$$
$$+(1+t)^{1/2+\varepsilon}\|v_x(t)\|_{L^2(R)}^2 \le C(\|v_0\|_{L^1(R)}+C_d)^2(1+t)^{-1+\varepsilon}\log^2(2+t). \qquad (42)$$

Integrating (42) with respect to time from 0 to t, we get (36) with $k=0$. In particular, we obtain

$$\|v(t)\|_{L^2(R)} \le C\gamma_0(1+t)^{-1/4}\log(2+t). \qquad (43)$$

Next we derive (36) with $k=1$. Making the L^2-energy equality on v_x and multiplying it by $(1+t)^{3/2+\varepsilon}$, we have

$$\frac{1}{2}\frac{d}{dt}(1+t)^{3/2+\varepsilon}\|v_x(t)\|_{L^2(R)}^2+(1+t)^{3/2+\varepsilon}\int_R v_x(a(w)v)_{xx}dx$$
$$+(1+t)^{3/2+\varepsilon}\int_R v_x(F(w,v)v^2)_{xx}dx+(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|_{L^2(R)}^2$$
$$=\frac{1}{2}(3/2+\varepsilon)(1+t)^{3/2+\varepsilon}\|v_x(t)\|_{L^2(R)}^2+(1+t)^{3/2+\varepsilon}\int_R v_x w_{xxx}dx. \qquad (44)$$

We first study the second term in the left hand side of (44). Integration by parts gives

$$(1+t)^{3/2+\varepsilon}\int_R v_x(a(w)v)_{xx}dx$$
$$=-(1+t)^{3/2+\varepsilon}\int_R v_{xx}a'(w)w_x v dx-(1+t)^{3/2+\varepsilon}\int_R a(w)v_x v_{xx}dx. \qquad (45)$$

The first term of the right hand side in (45) is estimated by making use of Lemma 1 as follows:

$$\left|(1+t)^{3/2+\varepsilon}\int_R v_{xx}a'(w)w_x v dx\right|$$
$$\le \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|_{L^2(R)}^2+C(1+t)^{3/2+\varepsilon}\|w_x(t)\|_{L^\infty(R)}\int_R w_x v^2 dx$$
$$\le \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|_{L^2(R)}^2+C_d(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx. \qquad (46)$$

On the other hand, integrating by parts, we find that the second term of the right hand side in (45) is estimated as

$$-(1+t)^{3/2+\varepsilon}\int_R a(w)v_x v_{xx}dx$$
$$=\frac{1}{2}(1+t)^{3/2+\varepsilon}\int_R a'(w)w_x v_x^2 dx \ge \frac{1}{2}\alpha(1+t)^{3/2+\varepsilon}\int_R w_x v_x^2 dx. \qquad (47)$$

Secondly we estimate the third term in the left hand side of (44). Integration by parts gives

$$
\begin{aligned}
&\left|(1+t)^{3/2+\varepsilon}\int_R v_x(F(w,v)v^2)_{xx}dx\right| \\
&= \left|-(1+t)^{3/2+\varepsilon}\int_R v_{xx}(F(w,v)v^2)_x dx\right| \\
&= \left|-(1+t)^{3/2+\varepsilon}\int_R v_{xx}(F_w w_x v^2 + F_v v_x v^2 + F\cdot 2vv_x)dx\right| \\
&\le \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + C(1+t)^{3/2+\varepsilon}\int_R w_x^2 v^4 dx \\
&\quad + C(1+t)^{3/2+\varepsilon}\int_R v^4 v_x^2 dx + C(1+t)^{3/2+\varepsilon}\int_R v^2 v_x^2 dx \\
&\equiv \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + J_1 + J_2 + J_3. \qquad (48)
\end{aligned}
$$

We estimate the last three terms. For J_1, from the Lemma 1, (41), and (28), we compute:

$$
\begin{aligned}
J_1 &\le C(1+t)^{3/2+\varepsilon}\|w_x(t)\|_{L^\infty(R)}\|v(t)\|^2_{L^\infty(R)}\int_R w_x v^2 dx \\
&\le C(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx. \qquad (49)
\end{aligned}
$$

For J_2, as in J_1, we estimate:

$$
\begin{aligned}
J_2 &\le C(1+t)^{3/2+\varepsilon}\sup_{0\le s\le t}\|v(s)\|^2_{L^\infty(R)}\|v(t)\|^2_{L^2(R)}\|v_x(t)\|^2_{L^\infty(R)} \\
&\le C(1+t)^{3/2+\varepsilon}\|v(t)\|^2_{L^2(R)}\cdot\|v_x(t)\|_{L^2(R)}\|v_{xx}(t)\|_{L^2(R)}. \qquad (50)
\end{aligned}
$$

Furthermore, we apply (43) in the right hand side of (50). Then,

$$
\begin{aligned}
J_2 &\le C(1+t)^{1+\varepsilon}\log^2(2+t)\cdot\|v_x(t)\|_{L^2(R)}\|v_{xx}(t)\|_{L^2(R)} \\
&= C(1+t)^{1+\varepsilon-(3/2+\varepsilon)/2}\log^2(2+t)\|v_x(t)\|_{L^2(R)}\cdot(1+t)^{(3/2+\varepsilon)/2}\|v_{xx}(t)\|_{L^2(R)} \\
&\le C(1+t)^{1/2+\varepsilon}\log^4(2+t)\|v_x(t)\|^2_{L^2(R)} + \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}. \qquad (51)
\end{aligned}
$$

J_3 is majorized by the same bound as that of J_2, that is,

$$
J_3 \le C(1+t)^{1/2+\varepsilon}\log^4(2+t)\|v_x(t)\|^2_{L^2(R)} + \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}. \qquad (52)
$$

Collecting (49), (51) and (52), we arrive at the estimate

$$
\begin{aligned}
&\left|(1+t)^{3/2+\varepsilon}\int_R v_x(F(w,v)v^2)_{xx}dx\right| \\
&\le C(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx + C(1+t)^{1/2+\varepsilon}\log^4(2+t)\|v_x(t)\|^2_{L^2(R)} \\
&\quad + \frac{1}{4}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}. \qquad (53)
\end{aligned}
$$

Finally, we estimate the second term of the right hand side in (44). Integration by parts and Lemma 1 gives

$$\begin{aligned}
&\left|(1+t)^{3/2+\varepsilon}\int_R v_x w_{xxx}dx\right| \\
&= \left|-(1+t)^{3/2+\varepsilon}\int_R v_{xx}w_{xx}dx\right| \\
&\le \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + C(1+t)^{3/2+\varepsilon}\|w_{xx}(t)\|^2_{L^2(R)} \\
&\le \frac{1}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + C_d(1+t)^{-1+\varepsilon}. \qquad (54)
\end{aligned}$$

Collecting all the estimates (46)-(54), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{3/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)} + \frac{\alpha}{2}(1+t)^{3/2+\varepsilon}\int_R w_x v_x^2 dx \\
&\quad + \frac{3}{8}(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} \le \frac{1}{2}(3/2+\varepsilon)(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)} \\
&\quad + C(1+t)^{1/2+\varepsilon}\int_R w_x v^2 dx + C(1+t)^{1/2+\varepsilon}\log^4(2+t)\|v_x(t)\|^2_{L^2(R)} + C_d(1+t)^{-1+\varepsilon}.
\end{aligned}$$

Integrating (55) with respect to t and making use of (36), we get

$$\begin{aligned}
&(1+t)^{3/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)} + \int_0^t (1+s)^{3/2+\varepsilon}\int_R w_x v_x^2 dxds + \int_0^t (1+s)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} \\
&\le \|v_{0x}\|^2_{L^2(R)} + \gamma_0^2(1+t)^{\varepsilon}\log^2(2+t) + \gamma_0^2(1+t)^{\varepsilon}\log^6(2+t) + C_d(1+t)^{\varepsilon}.
\end{aligned}$$

Clearing up the above inequality, we arrive at (36) with $k=1$. In particular, we have

$$\|v_x(t)\|_{L^2(R)} \le \gamma_1(1+t)^{-3/4}\log^3(2+t). \qquad (55)$$

Finally we show (36) with $k=2$. Similar procedure to derive (44) also gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{5/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + (1+t)^{5/2+\varepsilon}\int_R v_{xx}(a(w)v)_{xxx}dx \\
&\quad + (1+t)^{5/2+\varepsilon}\int_R v_{xx}(F(w,v)v^2)_{xxx}dx + (1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} \\
&= \frac{1}{2}(5/2+\varepsilon)(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + (1+t)^{5/2+\varepsilon}\int_R v_{xx}w_{xxxx}dx. \qquad (56)
\end{aligned}$$

First we study the second term of the left hand side in (56). Integration by parts gives

$$(1+t)^{5/2+\varepsilon}\int_R v_{xx}(a(w)v)_{xxx}dx$$

$$
\begin{aligned}
&= -(1+t)^{5/2+\varepsilon}\int_R v_{xxx}a''(w)w_x^2 v\,dx \\
&\quad -(1+t)^{5/2+\varepsilon}\int_R v_{xxx}a'(w)w_{xx}v\,dx \\
&\quad -2(1+t)^{5/2+\varepsilon}\int_R v_{xxx}a'(w)w_x v_x\,dx \\
&\quad -(1+t)^{5/2+\varepsilon}\int_R v_{xxx}a(w)v_{xx}\,dx \\
&\equiv K_1+K_2+K_3+K_4.
\end{aligned}
\tag{57}
$$

K_1 is estimated by using Lemma 1 as follows:

$$
\begin{aligned}
|K_1| &\le C(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}\|w_x^2 v(t)\|_{L^2(R)} \\
&\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 + C(1+t)^{5/2+\varepsilon}\|w_x(t)\|_{L^\infty(R)}^3\int_R w_x v^2\,dx \\
&\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 + C(1+t)^{-1/2+\varepsilon}\int_R w_x v^2\,dx.
\end{aligned}
\tag{58}
$$

Computing K_2 as above,

$$
|K_2| \le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 + C(1+t)^{5/2+\varepsilon}\int_R w_{xx}^2 v^2\,dx.
$$

Applying (43) as well as (41) and Lemma 1 to the last integral,

$$
\begin{aligned}
\int_R w_{xx}^2 v^2\,dx &\le \|w_{xx}(t)\|_{L^2(R)}^2\|v(t)\|_{L^2(R)}\|v_x(t)\|_{L^2(R)} \\
&\le \gamma_1^2(1+t)^{-5/2}\cdot(1+t)^{-1/4}\log(2+t)\cdot(1+t)^{-3/4}\log^3(2+t) \\
&\le \gamma_1^2(1+t)^{-7/2}\log^4(2+t).
\end{aligned}
$$

Hence,

$$
|K_2| \le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 + \gamma_1^2(1+t)^{-7/2}\log^4(2+t).
\tag{59}
$$

For K_3, in a similar way,

$$
\begin{aligned}
|K_3| &\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 \\
&\quad + C(1+t)^{5/2+\varepsilon}\|w_x(t)\|_{L^\infty(R)}^2\|v_x(t)\|_{L^2(R)}^2 \\
&\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}^2 + C(1+t)^{1/2+\varepsilon}\|v_x(t)\|_{L^2(R)}^2.
\end{aligned}
\tag{60}
$$

For K_4, we integrate by parts to get

$$
K_4 \ge \frac{1}{2}\alpha(1+t)^{5/2+\varepsilon}\int_R w_x v_{xx}^2\,dx.
\tag{61}
$$

Next we estimate the third term of the left hand side in (56). Estimating it as before, we obtain

$$\begin{aligned}
&\left|(1+t)^{5/2+\varepsilon}\int_R v_{xx}(F(w,v)v^2)_{xxx}dx\right| \\
&\leq (1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|_{L^2(R)}\|(F(w,v)v^2)_{xx}\|_{L^2(R)} \\
&\leq \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|w_x^2v^2(t)\|^2_{L^2(R)} \\
&\quad +C(1+t)^{5/2+\varepsilon}\|w_{xx}v^2(t)\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|w_xvv_x(t)\|^2_{L^2(R)} \\
&\quad +C(1+t)^{5/2+\varepsilon}\|v_x^2v^2\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|v_{xx}v^2\|^2_{L^2(R)} \\
&\quad +C(1+t)^{5/2+\varepsilon}\|vv_x^2\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|v_x^2\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|vv_{xx}\|^2_{L^2(R)} \\
&\equiv \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} + K_5 + K_6 + K_7 + K_8 + K_9 + K_{10} + K_{11} + K_{12}.
\end{aligned}$$

The estimates of K_5-K_{12} can be done in a similar way for the estimates of the second term of the left hand side in (56). We present only the results of them:

$$K_5 \leq C\|v_x(t)\|^2_{L^2(R)}, \tag{62}$$
$$K_6 \leq C(1+t)^{\varepsilon}\|v_x(t)\|^2_{L^2(R)}, \tag{63}$$
$$K_7 \leq C\|v_x(t)\|^2_{L^2(R)}, \tag{64}$$
$$K_8 \leq C\|v_x(t)\|^2_{L^2(R)} + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}, \tag{65}$$
$$K_9 \leq C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}, \tag{66}$$
$$K_{10} \leq \gamma_1^2(1+t)^{-1+\varepsilon} + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}, \tag{67}$$
$$K_{11} \leq C(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)}\log^{12}(2+t) + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}, \tag{68}$$
$$K_{12} \leq \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}\log^4(2+t). \tag{69}$$

Collecting (62)-(69), we get the estimate

$$\begin{aligned}
&\left|(1+t)^{5/2+\varepsilon}\int_R v_{xx}(F(w,v)v^2)_{xxx}dx\right| \\
&\leq C(1+t)^{\varepsilon}\|v_x(t)\|^2_{L^2(R)} + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + \gamma_1^2(1+t)^{-1+\varepsilon} \\
&\quad +C(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)}\log^{12}(2+t) \\
&\quad +C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}\log^4(2+t)
\end{aligned} \tag{70}$$

Finally we estimate the second term of the right hand side of (56). Making use of Lemma 2,

$$\left|(1+t)^{5/2+\varepsilon}\int_R v_{xx}w_{xxxx}dx\right|$$

$$
\begin{aligned}
&\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} + C(1+t)^{5/2+\varepsilon}\|w_{xxx}(t)\|^2_{L^2(R)} \\
&\le \frac{1}{16}(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} + C_d(1+t)^{-1+\varepsilon}. \qquad (71)
\end{aligned}
$$

Now, collecting the estimates (58)-(61), (70), and (71), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)^{5/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)} + \frac{\alpha}{2}(1+t)^{5/2+\varepsilon}\int_R w_x v_{xx}^2 dx \\
&\quad + C(1+t)^{5/2+\varepsilon}\|v_{xxx}(t)\|^2_{L^2(R)} \\
&\le C(1+t)^{-1/2+\varepsilon}\int_R w_x v^2 dx + C_d(1+t)^{-1+\varepsilon} \\
&\quad + C(1+t)^{1/2+\varepsilon}\|v_x(t)\|^2_{L^2(R)}\log^{12}(2+t) + C(1+t)^{3/2+\varepsilon}\|v_{xx}(t)\|^2_{L^2(R)}\log^4(2+t).
\end{aligned}
$$

Integrating the above inequality with respect to time from 0 to t with the aid of (36) with $k=0,1$, we arrive at (36) with $k=2$. The proof of Theorem 3 is complete.

We close this section to state one more property of $U(t,x)$ obtained by Xin [11], which plays an essential role in the next section.

Lemma 5 **([11])** *Suppose that $U_0(x)$ has the following properties:*

$$
\frac{d}{dx}U_0(x) > 0 \quad \textit{and} \quad \left|\frac{d^2}{dx^2}U_0(x)\right| \le C\frac{d}{dx}U_0(x), \qquad (72)
$$

for any $x \in R$. Then, $U(t,x)$ satisfies

$$
U_x(t,x) > 0 \quad \textit{for } t>0 \textit{ and } x \in R, \qquad (73)
$$

and

$$
|U_{xx}(t,x)| \le CU_x(t,x) \quad \textit{for } t>0 \textit{ and } x\in R. \qquad (74)
$$

Note that if we choose $v_0(x) \equiv 0$ in (23), then $U_0(x)$ in (21) satisfies (72).

4 convergence of $u(t,x,y) - r(t,x)$

In this section we prove Theorem 1. Throughout this section, $v_0(x) \equiv 0$ is adopted.

First, as in the work of Xin [11], we decompose the solution $u(t,x,y)$ as follows:

$$
u(t,x,y) = U(t,x) + V(t,x,y),
$$

where $U(t,x)$ is the solution of (21) obtained in Section 3. Then, the problem (1)-(2) is reduced to

$$\begin{cases} V_t + (f'(U)V)_x + (F(U,V)V^2)_x + (g'(U)V)_y + (G(U,V)V^2)_y = V_{xx} + V_{yy}, \\ V(0,x,y) = V_0(x,y) \equiv u_0(x,y) - w_0(x), \end{cases} \tag{75}$$

where $F(U,V)$ is in (24) and $G(U,V)$ is defined by

$$G(U,V) = \frac{g(U+V) - g(U) - g'(U)V}{V^2}. \tag{76}$$

From now on, we study the problem (75).

Our main purpose in this section is to derive the decay estimate for V.

Throughout this section, we use the notation ∂^k as in the meaning

$$\partial^k = \sum_{i+j=k} \partial_x^i \partial_y^j. \tag{77}$$

For the problem (75), Xin [11] showed the following global existence result.

Theorem 4 (global existence [11]) *There exists a constant δ_2 such that if $\|V_0\|_{H^2(R^2)} \le \delta_2$, then the problem (75) has a unique global solution $V(t,x,y)$ satisfying*

$$\begin{aligned} \|V(t)\|_{H^2(R^2)} &+ \int_0^t \iint_{R^2} U_x V^2(s,x) dx dy ds \\ &+ \int_0^t \|\partial^1 V(t)\|^2_{H^2(R^2)} ds \le C \|V_0\|^2_{H^2(R^2)} \end{aligned} \tag{78}$$

for all $t \ge 0$.

Furthermore, when the integrability of V_0 is imposed, we have

Lemma 6 (L^1-estimate) *Suppose further in Theorem 4 that $V_0 \in L^1(R^2)$. Then, the solution $V(t,x,y)$ also satisfies*

$$\|V(t)\|_{L^1(R^2)} \le \|V_0\|_{L^1(R^2)}. \tag{79}$$

The proof of Lemma 6 can be done in a similar way as in having derived (26).

Our main result in this section is the following decay estimate of V.

Theorem 5 (decay estimate) *Let $V(t,x,y)$ be the solution of (75) obtained in Theorem 4 and Lemma 6. Then, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that the following estimate holds:*

$$\begin{aligned} (1+t)^{k+1+\varepsilon} \|\partial^k V(t)\|^2_{L^2(R^2)} &+ \int_0^t (1+s)^{k+1+\varepsilon} \iint_{R^2} U_x |\partial^k V|^2 dx dy ds \\ &+ \int_0^t (1+s)^{k+1+\varepsilon} \|\partial^{k+1} V(s)\|^2_{L^2(R^2)} ds \le C_d (1+t)^\varepsilon \theta_k(t)^2 \|V_0\|^2_{(H^k \cap L^1)(R^2)}, \end{aligned} \tag{80}$$

where $\theta_0(t) = 1$, $\theta_1(t) = \log^5(2+t)$ and $\theta_2(t) = \log^{33/2}(2+t)$.

Here, admitting Theorem 5 temporarily, we prove Theorem 1.

Proof of Theorem 1. First we write

$$\sup_{y\in R}\|u(t)-r(t)\|_{L^2(R_x)} \le \sup_{y\in R}\|u(t)-U(t)\|_{L^2(R_x)} + \|U(t)-r(t)\|_{L^2(R)}. \tag{81}$$

It follows from Corollary 1 that the second term of the right hand side in (81) does not exceed $Ct^{-1/4}\log(2+t)$. So, we only estimate the first term of the right hand side of (81). Note that the follwing Sobolev inequality holds:

$$\sup_{y\in R}\|f\|_{L^2(R_x)} \le C\|f\|^{1/2}_{L^2(R^2)}\|f_y\|^{1/2}_{L^2(R^2)}, \tag{82}$$

for functions $f=f(x,y)$. Then, it follows from (80) and (82)

$$\begin{aligned}\sup_{y\in R}\|V(t)\|_{L^2(R_x)} &\le C\|V(t)\|^{1/2}_{L^2(R^2)}\|V_y(t)\|^{1/2}_{L^2(R^2)}\\ &\le C(1+t)^{-3/4}\log^{5/4}(2+t),\end{aligned}$$

which means

$$\sup_{y\in R}\|u(t)-U(t)\|_{L^2(R_x)} \le C(1+t)^{-3/4}\log^{5/4}(2+t). \tag{83}$$

From the above, we arrive at

$$\sup_{y\in R}\|u(t)-r(t)\|_{L^2(R_x)} \le Ct^{-1/4}\log(2+t),$$

which gives (15). The proof of Theorem 1 is complete.

It remains to prove Theorem 5.

Proof of Theorem 5. By using the L^2-energy method as treated by Xin [11], we get

$$\frac{d}{dt}\|V(t)\|^2_{L^2(R^2)} + \iint_{R^2} U_x V^2\,dx\,dy + C\|\partial^1 V(t)\|^2_{L^2(R^2)} \le 0. \tag{84}$$

Multiplying (84) by $(1+t)^{1+\varepsilon}$, we have

$$\begin{aligned}&\frac{d}{dt}(1+t)^{1+\varepsilon}\|V(t)\|^2_{L^2(R^2)} + (1+t)^{1+\varepsilon}\iint_{R^2} U_x V^2\,dx\,dy\\ &+C(1+t)^{1+\varepsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} \le C(1+t)^{\varepsilon}\|V(t)\|^2_{L^2(R^2)}.\end{aligned} \tag{85}$$

With the aid of (79) and Sobolev inequality

$$\|f\|_{L^\infty(R^2)} \le C\|f\|^{1/2}_{L^1(R^2)}\|\partial^1 f\|^{1/2}_{L^2(R^2)} \quad \text{for } f=f(x,y), \tag{86}$$

we continue the computations:

$$\begin{aligned}
&\frac{d}{dt}(1+t)^{1+\epsilon}\|V(t)\|^2_{L^2(R^2)} + (1+t)^{1+\epsilon}\iint_{R^2} U_x V^2 dxdy \\
&\quad + C(1+t)^{1+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} \\
\le\ & C(1+t)^{\epsilon}\|V_0\|_{L^1(R^2)}\|\partial^1 V(t)\|_{L^2(R^2)} \\
=\ & C(1+t)^{\epsilon-(1+\epsilon)/2}\|V_0\|_{L^1(R^2)}\cdot(1+t)^{(1+\epsilon)/2}\|\partial^1 V(t)\|_{L^2(R^2)} \\
\le\ & Cr^{-1}(1+t)^{-1+\epsilon}\|V_0\|^2_{L^1(R^2)} + r(1+t)^{1+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)}, \qquad (87)
\end{aligned}$$

where r is a positive constant sufficiently small. Integrating (87) with respect to time from 0 to t, we obtain

$$\begin{aligned}
&(1+t)^{1+\epsilon}\|V(t)\|^2_{L^2(R^2)} + \int_0^t (1+s)^{1+\epsilon}\iint_{R^2} U_x V^2 dxdyds \\
&+C\int_0^t (1+s)^{1+\epsilon}\|\partial^1 V(s)\|^2_{L^2(R^2)}ds \le \|V_0\|^2_{L^2(R^2)} + C(1+t)^{\epsilon}\|V_0\|^2_{L^1(R^2)},
\end{aligned}$$

which gives (80) with $k=0$.

Secondly, we derive (80) with $k=1$. L^2-energy method gives

$$\begin{aligned}
&\frac{d}{dt}\|\partial^1 V(t)\|^2_{L^2(R^2)} + \iint_{R^2} U_x|\partial^1 V|^2 dxdy + C\|\partial^2 V(t)\|^2_{L^2(R^2)} \\
\le\ & C\|U_x(t)\|_{L^\infty(R)}\left(\iint_{R^2} U_x V^2 dxdy + \|\partial^1 V(t)\|^2_{L^2(R^2)}\right) \\
&+C\|V(t)\|^2_{L^2(R^2)}\|\partial^1 V(t)\|^2_{L^2(R^2)}. \qquad (88)
\end{aligned}$$

Multipluing (88) by $(1+t)^{2+\epsilon}$, we have

$$\begin{aligned}
&\frac{d}{dt}(1+t)^{2+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} + (1+t)^{2+\epsilon}\iint_{R^2} U_x|\partial^1 V|^2 dxdy \\
&+C(1+t)^{2+\epsilon}\|\partial^2 V(t)\|^2_{L^2(R^2)} \le C(1+t)^{1+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} \\
&+C(1+t)^{2+\epsilon}\|U_x(t)\|_{L^\infty(R)}\left(\iint_{R^2} U_x V^2 dxdy + \|\partial^1 V(t)\|^2_{L^2(R^2)}\right) \\
&+C(1+t)^{2+\epsilon}\|V(t)\|^2_{L^2(R^2)}\|\partial^1 V(t)\|^2_{L^2(R^2)}. \qquad (89)
\end{aligned}$$

It should be noted that the following estimate holds:

$$\|U_x(t)\|_{L^\infty(R)} \le C_d(1+t)^{-1}\log^5(2+t). \qquad (90)$$

This can be obtained by using Lemma 1 and Theorem 3 in the equality

$$U_x = w_x + v_x.$$

Then, integrating (89) with respect to time from 0 to t and making use of (90), we estimate:

$$\begin{aligned}
&(1+t)^{2+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} + \int_0^t (1+s)^{2+\epsilon} \iint_{R^2} U_x|\partial^1 V(s)|^2 dxdyds \\
&\quad + \int_0^t (1+s)^{2+\epsilon}\|\partial^2 V(s)\|^2_{L^2(R^2)} ds \\
&\leq \|\partial^1 V_0\|^2_{L^2(R^2)} + C\int_0^t (1+s)^{1+\epsilon}\|\partial^1 V(s)\|^2_{L^2(R^2)} ds \\
&\quad + C_d \int_0^t (1+s)^{1+\epsilon}\log^5(2+s)\left(\iint_{R^2} U_x V^2 dxdy + \|\partial^1 V(s)\|^2_{L^2(R^2)}\right) ds \\
&\quad + C\int_0^t (1+s)^{2+\epsilon}\|V(s)\|^2_{L^2(R^2)}\|\partial^1 V(s)\|^2_{L^2(R^2)} ds.
\end{aligned}$$

Taking account of (80) with $k=0$ and L^2-bound of $\partial^2 V(t)$, we get

$$\begin{aligned}
&(1+t)^{2+\epsilon}\|\partial^1 V(t)\|^2_{L^2(R^2)} + \int_0^t (1+s)^{2+\epsilon} \iint_{R^2} U_x|\partial^1 V(s)|^2 dxdyds \\
&\quad + \int_0^t (1+s)^{2+\epsilon}\|\partial^2 V(s)\|^2_{L^2(R^2)} ds \\
&\leq \|\partial^1 V_0\|^2_{L^2(R^2)} + C(1+t)^{\epsilon}\|V_0\|^2_{(L^2\cap L^1)(R^2)} + C_d(1+t)^{\epsilon}\log^5(2+t)\|V_0\|^2_{(L^2\cap L^1)(R^2)}.
\end{aligned}$$

Clearing up the right hand side of the above inequality, we get (80) with $k=1$.

Finally, we derive (80) with $k=2$. L^2-energy method gives

$$\begin{aligned}
&\frac{d}{dt}\|\partial^2 V(t)\|^2_{L^2(R^2)} + \iint_{R^2} U_x|\partial^2 V|^2 dxdy + C\|\partial^3 V(t)\|^2_{L^2(R^2)} \\
&\leq C\|(f'(U)+g'(U))_{xx}(t)V(t)\|^2_{L^2(R^2)} + C\|(f'(U)+g'(U))_x(t)\partial^1 V(t)\|^2_{L^2(R^2)} \\
&\quad + C\|\partial^2(F(U,V)V^2)\|^2_{L^2(R^2)} + C\|\partial^2(G(U,V)V^2)\|^2_{L^2(R^2)}. \qquad (91)
\end{aligned}$$

Multiplying (91) by $(1+t)^{3+\epsilon}$ and integrating it with respect to time from 0 to t, we have

$$\begin{aligned}
&(1+t)^{3+\epsilon}\|\partial^2 V(t)\|^2_{L^2(R^2)} + \int_0^t (1+s)^{3+\epsilon} \iint_{R^2} U_x|\partial^2 V|^2 dxdyds \\
&\quad + C\int_0^t (1+s)^{3+\epsilon}\|\partial^3 V(s)\|^2_{L^2(R^2)} ds \\
&\leq \|\partial^2 V_0\|^2_{L^2(R^2)} + C\int_0^t (1+s)^{2+\epsilon}\|\partial^2 V(s)\|^2_{L^2(R^2)} ds \\
&\quad + C\int_0^t (1+s)^{3+\epsilon}\|(f'(U)+g'(U))_{xx}(s)V(s)\|^2_{L^2(R^2)} ds \\
&\quad + C\int_0^t (1+s)^{3+\epsilon}\|(f'(U)+g'(U))_x(t)\partial^1 V(t)\|^2_{L^2(R^2)} ds \\
&\quad + C\int_0^t (1+s)^{3+\epsilon}(\|\partial^2(F(U,V)V^2)\|^2_{L^2(R^2)} + \|\partial^2(G(U,V)V^2)\|^2_{L^2(R^2)}) ds. \qquad (92)
\end{aligned}$$

We here only estimate the term

$$\Lambda(t) \equiv \int_0^t (1+s)^{3+\varepsilon} \|U_{xx}(s)V(s)\|^2_{L^2(R^2)} ds,$$

which arises from the first term of the right hand side in (92). The rest terms in (92) can be treated as before. Note

$$\begin{aligned} \|U_{xx}(s)V(s)\|^2_{L^2(R^2)} &\le \|U_{xx}(s)\|^2_{L^2(R^1_x)} \sup_{x\in R} \int_R V(s,x,y)^2 dy \\ &\le \|U_{xx}(s)\|^2_{L^2(R_x)} \|V(s)\|_{L^2(R^2)} \|V_x(s)\|_{L^2(R^2)}. \end{aligned}$$

It then follows from Lemma 1, (36) with $k = 2$, and (80) with $k = 0, 1$

$$\begin{aligned} \Lambda(t) &\le \int_0^t (1+s)^{3+\varepsilon} \|U_{xx}(s)\|^2_{L^2(R^1_x)} \|V(s)\|_{L^2(R^2)} \|V_x(s)\|_{L^2(R^2)} ds \\ &\le C_d \|V_0\|^2_{(H^1\cap L^1)(R^2)} \int_0^t (1+s)^{-1+\varepsilon} \log^{33/2}(2+s) ds \\ &\le C_d \|V_0\|^2_{(H^1\cap L^1)(R^2)} (1+t)^\varepsilon \log^{33/2}(2+t), \end{aligned}$$

which gives the right hand side of (80) with $k = 2$. The proof of Theorem 5 is complete.

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