数理解析研究所講究録
966巻 1996年 110-115



# GLOBAL SOLUTIONS TO THE SEMILINEAR WAVE EQUATION FOR LARGE SPACE DIMENSIONS

VLADIMIR GEORGIEV

Consider the semilinear wave equation

$$\square u=F(u), \tag{1}$$

where $F(u)=O(|u|^{\lambda})$ near $|u|=0$ and $\lambda>1$. Here and below $\square$ denotes the d'Alembertian on $\mathbf{R}^{n+1}$.

For this semilinear wave equation W.Strauss proposed in [17] the conjecture that the existence of global solution of the corresponding Cauchy problem with small initial data depends essentially on a critical value $\lambda_0(n)$ for the non linearity, namely $\lambda_0(n)$ is the positive root of the equation

$$(n-1)\lambda^{2}-(n+1)\lambda-2=0. \tag{2}$$

More precisely, for the subcritical case ( $1<\lambda<\lambda_0(n)$) the conjecture asserts that the solution with small initial data blows up in finite time, while an existence result is expected for the supercritical case ( $\lambda>\lambda_0(n)$) .

Here below we shall make a brief review of the results concerning this conjecture.

The case $n=3$ was studied by F.John in the pioneer work [6]. The critical value for this case is $\lambda_0(3)=1+\sqrt{2}$ .

For $n=2$ a proof of the conjecture was given by R.Glassey ([4], [5]). A blow-up result for arbitrary space dimensions when $1<\lambda<\lambda_0(n)$ was established by T.Sideris [16].

The critical values $\lambda=\lambda_0(n)$ were studied by J.Schaeffer in [15] for $n=2,3$. A simplified proof was found by H.Takamura [24].

Another interesting effect is the influence of the decay rate of the initial data on the existence of global solutions. In this case the solution might blow-up in finite time when the initial data decay very slowly at infinity even in the supercritical case when $\lambda > \lambda_0(n)$. For the case $n = 3$ this effect was established by F.Asakura [3] for the supercritical case. The critical cases for $n = 2, 3$ were studied by K.Kubota [13] , K.Tsutaya [25], [26], [27], R.Agemi and H.Takamura [2]. For the case $n \geq 4$ and supercritical non linearity the blow-up result for slowly decaying initial data is due to H.Takamura [23].

On the other hand, the existence part of the conjecture of W.Strauss for $n > 3$ is also very actively studied in the recent years.

Y. Zhou [28] has found a complete answer for $n = 4$ by using suitable weighted Sobolev estimates and the method developed by S.Klainerman [7], [8], [9] for proving the existence of small amplitude solutions.

The existence of a global solution for the case $\lambda = (n + 3)/(n - 1)$ was established by W.Strauss [19] by the aid of the conformal methods and the classical Strichartz inequality [20], [21], [22].

Another partial answer was given by R.Agemi, K.Kubota, H. Takamura in [1] for a special class of integral non linearity in (1). The approach in this work follows the approach of F.John.

A complete proof of the conjecture of W.Strauss for spherically symmetric initial data was found by H.Kubo [12] (see also [10], [11]).

By using different estimates H.Lindblad and C.Sogge [14] obtained a similar result as well as the existence of solutions in the supercritical case, non spherically symmetric initial data and space dimensions $n \leq 8$.

Our purpose in this talk shall be the announce of a result concerning the Cauchy problem

$$\begin{gathered} \Box u = F(u), \\ u(0,x) = \varepsilon f \quad , \quad \partial_t u(0,x) = \varepsilon g, \end{gathered} \tag{3}$$

where $f, g$ are compactly supported smooth functions such that

$$\operatorname{supp} f \cup \operatorname{supp} g \subseteq \{|x| \leq R\}, \tag{4}$$

while $\varepsilon$ is a sufficiently small positive number. For the nonlinear function $F(u)$ we shall assume that $F(u) \in C^0$ near $u = 0$ and for some $\lambda > 1$ satisfies

$$\begin{gathered} |F(u)| \leq C|u|^\lambda \quad , \\ |F(u) - F(v)| \leq C|u - v|(|u|^{\lambda-1} + |v|^{\lambda-1}) \end{gathered} \tag{5}$$

near $u, v = 0$ .

Our goal shall be to examine the existence of global solution to (3) for

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1}, \tag{6}$$

where $\lambda_0(n)$ is the positive root of (2). For this case we have the following

**Theorem 1** *Suppose the assumptions (4) , (5) and (6) are fulfilled with $\lambda_0(n)$ being the positive root of the equation*

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0. \tag{7}$$

*Then there exists $\varepsilon_0 > 0$ so that for $0 < \varepsilon < \varepsilon_0$ the Cauchy problem (3) admits a global solution.*

The solution belongs to a Banach space of type

$$u \in L^q_{\alpha,\beta}(\mathbf{R}^{n+1}_+),$$

where $L^q_{\alpha,\beta}(\mathbf{R}^{n+1}_+)$ denotes the Banach space of all measurable functions with finite norm

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbf{R}^{n+1}_+)}.$$

Here and below $\tau_\pm = 1 + |t \pm |x||$ are the weights associated with the characteristic surfaces of the wave equation.

The result of the above theorem shows that the conjecture of W.Strauss is valid for arbitrary space dimensions $n \geq 2$ even in the case of non spherically symmetric initial data.

The main idea to establish the above result is the application of a weighted estimate for the inhomogeneous wave equation

$$\Box u = F, \tag{8}$$

with zero initial data. For simplicity we shall assume that the supports of $u$ and $F$ lie in the light cone, that is

$$\mathrm{supp} F(t,x) \subset \{|x| \leq t + R\}. \tag{9}$$

The key to prove Theorem 1 is the following weighted estimate.

**Theorem 2** *Suppose $1 < p, q < \infty$ satisfy*

$$\frac{1}{q} < \frac{1}{p} \ , \ \frac{1}{q} + \frac{1}{p} \leq 1, \qquad \frac{n-3}{2} < \frac{n}{q} - \frac{1}{p}, \tag{10}$$

*while the parameters $\alpha, \beta, \gamma, \delta$ satisfy*

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \frac{n-1}{2p} - \frac{n+1}{2q} < \beta = \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q} &< \frac{n-1}{2} - \frac{n}{q}, \\ \delta &> 1 - \frac{1}{p}. \end{aligned} \tag{11}$$

*Then the solution u satisfies the estimate*

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbf{R}_+^{n+1})} \leq C\|\tau_+^\gamma \tau_-^\delta F\|_{L^p(\mathbf{R}_+^{n+1})}, \tag{12}$$

*where* $\tau_\pm = 1 + |t \pm |x||$ *and* $\mathbf{R}_+^{n+1} = \{(t,x) \in \mathbf{R}^{n+1} : t \geq 0\}$.

This estimate can be considered as a generalization of the Strichartz estimate and the estimates used by F.John in [6].

The author is grateful to Professors R.Agemi, K.Kubota, Y.Shibata, H.Takamura, K.Tsutaya, and H. Kubo for the important discussions and the support during the preparation of the work.

# References

[1] R.Agemi, K.Kubota, H. Takamura *On certain integral equations related to nonlinear wave equations* , Hokkaido University preprint series in Mathematics, 1991 and Hokkaido Math. Journal, 23(2), (1994) p.241 - 276.

[2] R.Agemi, H. Takamura , *The life-span of classical solutions of nonlinear wave equations in two space dimensions*, Hokkaido Math. J. 23, (1992), p.517 - 542.

[3] F.Asakura , *Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions*, Comm. Part. Diff. Eq., 11(13), (1986) , p. 1459-1487.

[4] R.Glassey , *Finite time blow-up of solutions of nonlinear wave equations*, Math. Zeit., 177, (1981) , p.323 - 340.

[5] R.Glassey , *Existence in the large for* $\Box u = F(u)$ *in two space dimensions* , Math. Z. 178, (1981) , p. 233-261.

[6] F. John, *Blow - up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., 28, (1979), p. 235 - 268.

[7] S.Klainerman, *Weighted* $L^\infty$ *and* $L^1$ *estimates for solutions to the classical wave equation in three space dimensions*, Comm. Pure Appl. Math., 37, (1984), p. 269 - 288.

[8] S. Klainerman , *Uniform decay estimates and the Lorentz invariance of the classical wave equation*, Comm. Pure Appl. Math. , 37 , (1985), p.321-332.

[9] S. Klainerman , *Remarks on global Sobolev inequalities in Minkowski space*, Comm. Pure Appl. Math. 40, (1987), p.111-117.

[10] H.Kubo , K.Kubota , *Asymptotic behaviour of radial solutions to semilinear wave equation in odd space dimensions* , Hokkaido Math. J. 24(1), 1995, p. 9 - 15.

[11] H.Kubo , K.Kubota , *Asymptotic behaviors of racially symmetric solutions of* $\Box u = |u|^p$ *for super critical values p in odd space dimensions* , Hokkaido Preprint Series, 251, June 1994.

[12] H.Kubo , *On the critical decay and power for semilinear wave equations in odd space dimensions,* Hokkaido Preprint Series 274, December 1994.

[13] K.Kubota , *Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions* Hokkaido Math. J., 22, (1993) p. 123 - 180.

[14] H.Lindblad, C.Sogge, *About small-power semilinear wave equations*, preprint 1995.

[15] J. Schaeffer, *The equation* $u_{tt} - \Delta u = |u|^p$ *for the critical value of* $p$, Proc. Royal Soc. of Edinburgh, 101 A, (1985), p. 31-44.

[16] T.Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Diff. Eq., 52, (1984), p.378 - 406.

[17] W.Strauss, *Nonlinear scattering theory at low energy*, J. Func. Anal. , 41, (1981), p. 110 - 133.

[18] W.Strauss, *Nonlinear scattering theory at low energy:sequel* , J. Func. Anal. , 43, (1981), p. 281 - 293.

[19] W.Strauss, *Nonlinear wave equations* C.B.M.S. Lecture Notes, no. 73, American Math. Soc., Providence, RI, 1989.

[20] R.Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc., 148, (1970), p.461 - 471.

[21] R.Strichartz , *Restrictions of Fourier transforms to quadratic surfaces and decay of solution of wave equation*, Duke Math. J., 44(3), (1977), p.705 - 714.

[22] R. Strichartz , *A priori estimates for the wave equation and some applications,* Journal Func. Anal., 5, (1970) , p. 218 - 235.

[23] H.Takamura, *Blow-up for semilinear wave equations with slowly decaying data in high dimensions,* Differential and Int. Equations, 8(3), (1995), p. 647 - 661.

[24] H.Takamura, *An elementary proof of the exponential blow-up for semi-linear wave equation* Math. Meth. Appl. Sci., 17 , (1994), p.239-249.

[25] K. Tsutaya, *A global existence theorem for semilinear wave equations with data of noncompact support in two space dimensions*, Comm. Part. Diff. Eq., 17(11 , 12), (1992), p. 1925 -1954.

[26] K.Tsutaya , *Global existence theorem for semilinear wave equations with non-compact data in two space dimensions*, J. Diff. Eq., 104(2), (1993), p. 332 - 360.

[27] K.Tsutaya , *Global existence and the life span of solutions of semilinear wave equations with data of noncompact support in three space dimensions* to appear in Funkcialaj Ekvacioj

[28] Zhou Yi, *Cauchy problem for semilinear wave equations in four space dimensions with small initial data*, IDMF Preprint (9217), to appear in Journal of P.D.E.

The author was partially supported by Alexander von Humboldt Foundation and the Contract 401 with the Bulgarian Ministry of Culture, Science and Education

Mail Address:
Institute of Mathematics
Bulgarian Academy of Sciences
Sofia 1113, Acad.G.Bonchev bl. 8
Bulgaria
e-mail: VG@BGEARN.BITNET