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Asymptotic behaviors of radially symmetric solutions of $\square u = |u|^p$
for super critical values $p$ in high dimensions

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1. Introduction

We study asymptotic behaviors as $t \to \pm \infty$ of radially symmetric solutions of the nonlinear wave equation

(1.1) \[ u_{tt} - \Delta u = F(u) \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \]

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$ and $n \geq 2$.

Let $p_0(n)$ be the positive root of the quadratic equation in $p$:

(1.2) \[ \Phi(n, p) \equiv \frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1 = 0. \]

Note that $p_0(n)$ is strictly decreasing with respect to $n$ and $p_0(4) = 2$. If $1 < p < p_0(n)$, it is known that the Cauchy problem for (1.1) with initial data prescribed on $t = 0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately, even if they are sufficiently small. (See [6], [8] and [19]). The same is true for $p = p_0(n)$ if $n = 2$ or $n = 3$. (See [18]).

On the other hand, the case where $p > p_0(n)$ seems to be more complicated. When $2 \leq n \leq 4$, it is known that the problem admits a global solution for small initial data. (See [7], [8] and [24]). When $n \geq 5$, for $p \geq (n + 3)/(n - 1)$ a global weak solution of the problem obtained by [13] and [20]. (See also [3], [4], [11] and [12]). Recently, the case where $p$ is between $p_0(n)$ and $(n + 3)/(n - 1)$ is treated by [5] and [14], independently.

Moreover, when $p > p_0(n)$ and either $n = 2$ or $n = 3$, it has been shown that the scattering operator for (1.1) exists on a dense set of a neighborhood of $0$ in the energy space. (See [10],
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[17] and [23]). Namely, let $u_-(x, t)$ be the solution of the homogeneous wave equation

\begin{equation}
    u_{tt} - \Delta u = 0 \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R},
\end{equation}

with small initial data

\begin{equation*}
    u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^n.
\end{equation*}

Then there exists a solution $u(x, t)$ of (1.1) such that $||u(t) - u_-(t)||_\epsilon \to 0$ as $t \to -\infty$, where

\begin{equation}
    ||v(t)||_\epsilon = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x, t)|^2 + |v_t(x, t)|^2) dx \right\}^{1/2},
\end{equation}

and there exists another solution $u_+(x, t)$ of (1.3) such that $||u(t) - u_+(t)||_\epsilon \to 0$ as $t \to \infty$. The analogous results have been obtained also for the high dimensional case, provided $p > p_1(n)$, where $p_1(n)$ is the largest root of the quadratic equation in $p$:

\begin{equation*}
    (n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0.
\end{equation*}

(See [13], [15], [16], and [20]). However here is a gap between $p_0(n)$ and $p_1(n)$. Indeed, since the left-hand-side of the above quadratic equation is rewritten as

\begin{equation*}
    2\{n\Phi(n, p) - 2(1 + \Phi(n, p))/p\},
\end{equation*}

it is easy to see that $p_0(n) < p_1(n)$.

The purpose of this note is to search the asymptotic behaviors of radially symmetric solutions of (1.1), which guarantee the existence of the scattering operator, for $p > p_0(n)$ in high dimensions $n \geq 5$.

2. Statements of main results

Throughout this section, we assume $n \geq 5$ (unless stated otherwise). First we shall consider the Cauchy problem for the homogeneous wave equation:

\begin{equation}
    u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in} \quad \Omega,
\end{equation}

\begin{equation}
    u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for} \quad r > 0,
\end{equation}

where $\Omega = \{(r, t) \in \mathbb{R}^2; \ r > 0\}$ and $u(r, t)$ a real valued function. Then we have
**Theorem 1.** Assume $f \in C^{2}([0, \infty))$ and $g \in C^{1}([0, \infty))$ satisfy

\begin{equation}
|f(r)||r|^{-1} + \sum_{j=0}^{1}(|f^{(j+1)}| + |g^{(j)}(r)|) \leq \varepsilon(r)^{-\kappa-(n+1)/2} \quad \text{for} \quad r > 0,
\end{equation}

where $\varepsilon$ and $\kappa$ are positive numbers and $\langle r \rangle = \sqrt{1 + r^2}$. Here if $n$ is even number, we further assume $\kappa < (n-1)/2$. Then (2.1) admits uniquely a weak solution $u(r,t) \in C^{1}(\Omega)$ such that for $(r,t) \in \Omega$ and $|\alpha| \leq 1$ we have

\begin{equation}
|D_{\Gamma,r}^{\alpha}u(r,t)| \leq C\varepsilon\langle r \rangle^{-1-|\alpha|}\langle r \rangle^{-\kappa} + |\alpha|\Psi(r,|t|),
\end{equation}

where we have set $m = \lceil(n-2)/2\rceil$ and

$$\Psi(r,t) = \langle r + |t| \rangle^{-\chi(n)}(r-t)^{-\kappa},$$

with

$$\chi(n) = \begin{cases} 
1/2 & \text{if } n \text{ is even}, \\
1 & \text{if } n \text{ is odd}, 
\end{cases}$$

and $C$ is a constant depending only on $m$ and $\kappa$.

Next we shall consider the nonlinear wave equation

\begin{equation}
\begin{array}{c}
u_{tt} - \nu_{rr} - \frac{n-1}{r}u_{r} = F(u) \quad \text{in} \quad \Omega,
\end{array}
\end{equation}

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$. Here we assume

\begin{equation}
p_{0}(n) < p < \frac{(n+3)}{(n-1)}.
\end{equation}

We shall introduce a function space $X$, in which we will look for solutions of (2.4), defined by

$$X = \{u(r,t) \in C^{0}(\Omega) : D_{\Gamma}u(r,t) \in C^{0}(\Omega), \|u\| < \infty\},$$

and

$$\|u\| = \sup_{(r,t) \in \Omega} \{(|u(r,t)|r^{m-1}(r) + |D_{\Gamma}u(r,t)||r^{m}\Psi^{-1}(r,|t|)\},$$

where $\Psi$ is the same function as in (2.3). As for the parameter $\kappa$, we assume

\begin{equation}
\frac{1}{2} < \kappa \quad \text{and} \quad \frac{p+1}{p-1} - \frac{n+1}{2} < \kappa \leq q,
\end{equation}
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where we have set

$$q = (1 + \Phi(n,p))/p = \frac{n-1}{2}p - \frac{n+1}{2}$$

with $\Phi(n,p)$ in (1.2). Note that there exist really numbers $\kappa$ satisfying (2.6) for $p > p_0(n)$, because

$$\Phi(n,p) = (p-1)(q - (\frac{p+1}{p-1} - \frac{n+1}{2})) > 0 \quad \text{for} \quad p > p_0(n).$$

We are now in a position to state the main theorem in this note. Let $u_-(r,t)$ be the solution of (2.1) which is obtained in Theorem 1. Note that $u_- \in X$ and

(2.7) \[ ||u_-|| \leq C\varepsilon \quad \text{for any} \quad \varepsilon > 0. \]

Then we have

**Theorem 2. (Main theorem).** Assume conditions (2.2), (2.5) and (2.6) hold. Then there is positive constant $\varepsilon_0$ (depending only on $p$, $n$ and $\kappa$) such that, if $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a weak solution $u(r,t)$ of the nonlinear wave equation (2.4) such that $u \in C^1(\Omega) \cap X$,

(2.7) \[ ||u|| \leq 2||u_-|| \]

and for $(r,t) \in \Omega$ and $|\alpha| \leq 1$ we have

(2.8) \[ |D_{r,t}^\alpha(u(r,t) - u_-(r,t))| \leq C||u||^p r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r,t) \]

and

(2.9) \[ ||u(t) - u_-(t)||_e \leq C||u||^p \langle t \rangle^{-\theta} \quad \text{if} \quad t \leq 0, \]

where $|| \cdot ||$ is defined by (1.4) and we have set

$$\theta = \min\{q, \chi(n)p + pk - 1\},$$

and $C$ is a constant depending only on $p$, $n$ and $\kappa$.

Moreover there exists uniquely a weak solution $u_+(r,t)$ of (2.1) which belongs to $C^1(\Omega) \cap X$, such that for $(r,t) \in \Omega$ and $|\alpha| \leq 1$ we have

(2.8) \[ |D_{r,t}^\alpha(u(r,t) - u_+(r,t))| \leq C||u||^p r^{1-m-|\alpha|} \langle r \rangle^{-1+|\alpha|} \Psi(r,-t) \]
and

\[(2.9)_+ \quad \|u(t) - u_+(t)\|_e \leq C\|u\|^{p}(t)^{-\theta} \quad \text{if} \quad t \geq 0.\]

**Remarks.**

1) If \(n\) is odd, in Theorems 1 and 2, one can replace \(u \in C^1(\Omega)\) by \(u \in C^2(\Omega)\). Moreover in (2.6) we can replace \(\kappa > 1/2\) by \(\kappa > 0\). In this case, we interpret (2.9)\(_\pm\) as follows. When \(\kappa > 1/2p\), (2.9)\(_\pm\) is still valid. When \(0 < \kappa \leq 1/2p\), it holds with \(\theta = \kappa\). (See [9].)

2) For \(n \geq 2\), consider the following Cauchy problem

\[
(2.10) \quad \begin{cases}
    u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) & \text{in} \quad r > 0, \quad t > 0, \\
    u(r, 0) = 0, \quad u_t(r, 0) = g(r) & \text{for} \quad r > 0.
\end{cases}
\]

It is known that, if \(g(r) \geq Mr^{-\mu}\) for \(r \geq 1\) with some positive constants \(M, \mu\) and \(\mu < (p + 1)/(p - 1)\), then (2.10) does not admit global solutions. (See [1], [2], [21] and [22]). Therefore condition (2.6) is partially necessary to obtain Theorem 2.

3) One can also show that the Cauchy problem for the nonlinear wave equation (2.4) admits a unique global solution, provided the hypotheses of Theorems 1 and 2 are fulfilled.

In the proof of Theorem 1, the following lemma plays a key role. Moreover Theorem 2 is obtained by considering the associated integral equation with the differential equation (2.4). So the lemma below is very essential in our work.

**Lemma 3.** Let \(g \in C^0((0, \infty))\) and

\[g(r) = O(r^{-m-1}) \quad \text{as} \quad r \downarrow 0.\]

For \(r > 0\) and \(t \geq 0\) we define a function \(\Theta(g)\) as follows.

(1) \(n\) is odd: \(n = 2m + 3 \quad (m = 1, 2, \cdots)\).

\[\Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda)K(\lambda, r, t)d\lambda,\]

where we have set

\[K(\lambda, r, t) = r^{2-n}\lambda^{2m+1}H_m(\lambda, r, t),\]

\[H_m(\lambda, r, t) = \left(\frac{\partial}{\partial \lambda} \frac{-1}{2\lambda}\right)^m(r^2 - (\lambda - t)^2)^{(n-3)/2}.\]
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(2) $n$ is even: $n = 2m + 2$ ($m = 1, 2, \cdots$).

$$\Theta(g)(r, t) = \int_{[t-r]}^{t+r} g(\lambda)K_1(\lambda, r, t)d\lambda + \int_{0}^{\max(t-r, 0)} g(\lambda)K_2(\lambda, r, t)d\lambda,$$

where we have set

$$K_1(\lambda, r, t) = r^{2-n}\lambda^{2m+1} \int_{\lambda}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

$$K_2(\lambda, r, t) = r^{2-n}\lambda^{2m+1} \int_{t-r}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho,$$

and

$$H_m(\rho, r, t) = \left(\frac{\partial}{\partial \rho} \frac{-1}{2\rho}\right)^m (r^2 - (\rho - t)^2)^{(n-3)/2}.$$ 

And we extend $\Theta(g)(r, t)$ as an odd function with respect to $t$. Then $\Theta(g) \in C^0(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|\Theta(g)(r, t)| \leq C_B r^{-m} \text{ for } (r, t) \in B.$$

Moreover, if we set $u(x, t) = \Theta(g)(|x|, t)$, then $u(\cdot, t) \in C^0(\mathbb{R}; L^1_{loc}(\mathbb{R}^n))$ and $u$ is a weak solution of the Cauchy problem

$$\begin{cases}
u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) = 0, \quad u_t(x, 0) = c_n g(|x|) & \text{for } x \in \mathbb{R}^n
\end{cases}$$

in the sense of distribution, where

$$c_n = \begin{cases}
2 \Gamma\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd}, \\
\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right) & \text{if } n \text{ is even}.
\end{cases}$$

Furthermore, if $g \in C^1((0, \infty))$ and for $j = 0, 1$

$$g^{(j)}(r) = O(r^{-m-j}) \text{ as } r \downarrow 0,$$

then $\Theta(g) \in C^1(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

$$|D^\alpha_{r,t}\Theta(g)(r, t)| \leq C_B r^{1-m-|\alpha|} \text{ for } (r, t) \in B \text{ and } |\alpha| \leq 1.$$
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