Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for supercritical values $p$ in high dimensions (Nonlinear Evolution Equations and Applications)

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Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$
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1. Introduction

We study asymptotic behaviors as $t \to \pm\infty$ of radially symmetric solutions of the nonlinear wave equation

$$u_{tt} - \Delta u = F(u) \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R},$$

where $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$ and $n \geq 2$.

Let $p_0(n)$ be the positive root of the quadratic equation in $p$:

$$\Phi(n, p) \equiv \frac{n-1}{2}p^2 - \frac{n+1}{2}p - 1 = 0.$$  \hspace{1cm} (1.2)

Note that $p_0(n)$ is strictly decreasing with respect to $n$ and $p_0(4) = 2$. If $1 < p < p_0(n)$, it is known that the Cauchy problem for (1.1) with initial data prescribed on $t = 0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately, even if they are sufficiently small. (See [6], [8] and [19]). The same is true for $p = p_0(n)$ if $n = 2$ or $n = 3$. (See [18]).

On the other hand, the case where $p > p_0(n)$ seems to be more complicated. When $2 \leq n \leq 4$, it is known that the problem admits a global solution for small initial data. (See [7], [8] and [24]). When $n \geq 5$, for $p \geq (n + 3)/(n - 1)$ a global weak solution of the problem obtained by [13] and [20]. (See also [3], [4], [11] and [12]). Recently, the case where $p$ is between $p_0(n)$ and $(n + 3)/(n - 1)$ is treated by [5] and [14], independently.

Moreover, when $p > p_0(n)$ and either $n = 2$ or $n = 3$, it has been shown that the scattering operator for (1.1) exists on a dense set of a neighborhood of 0 in the energy space. (See [10],
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[17] and [23]). Namely, let $u_-(x,t)$ be the solution of the homogeneous wave equation

(1.3) \[ u_{tt} - \Delta u = 0 \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \]

with small initial data

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^n. \]

Then there exists a solution $u(x,t)$ of (1.1) such that $\|u(t) - u_-(t)\|_e \to 0$ as $t \to -\infty$, where

(1.4) \[ \|v(t)\|_e = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x,t)|^2 + |v_t(x,t)|^2) dx \right\}^{1/2}, \]

and there exists another solution $u_+(x,t)$ of (1.3) such that $\|u(t) - u_+(t)\|_e \to 0$ as $t \to \infty$. The analogous results have been obtained also for the high dimensional case, provided $p > p_1(n)$, where $p_1(n)$ is the largest root of the quadratic equation in $p$:

\[ (n^2 - n)p^2 - (n^2 + 3n - 2)p + 2 = 0. \]

(See [13], [15], [16], and [20]). However here is a gap between $p_0(n)$ and $p_1(n)$. Indeed, since the left-hand-side of the above quadratic equation is rewritten as

\[ 2\{n\Phi(n,p) - 2(1 + \Phi(n,p))/p\}, \]

it is easy to see that $p_0(n) < p_1(n)$.

The purpose of this note is to search the asymptotic behaviors of radially symmetric solutions of (1.1), which guarantee the existence of the scattering operator, for $p > p_0(n)$ in high dimensions $n \geq 5$.

2. Statements of main results

Throughout this section, we assume $n \geq 5$ (unless stated otherwise). First we shall consider the Cauchy problem for the homogeneous wave equation:

(2.1) \[
\begin{align*}
(2.1)_0 & \quad u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in} \quad \Omega, \\
(2.1)_1 & \quad u(r, 0) = f(r), \quad u_t(r, 0) = g(r) \quad \text{for} \quad r > 0,
\end{align*}
\]

where $\Omega = \{(r,t) \in \mathbb{R}^2 ; r > 0\}$ and $u(r,t)$ a real valued function. Then we have
Theorem 1. Assume \( f \in C^2([0, \infty)) \) and \( g \in C^1([0, \infty)) \) satisfy

\[
|f(r)|\langle r \rangle^{-1} + \sum_{j=0}^{1}(|f^{(j+1)}(r)| + |g^{(j)}(r)|) \leq \epsilon \langle r \rangle^{-\kappa-(n+1)/2} \quad \text{for} \quad r > 0,
\]

where \( \epsilon \) and \( \kappa \) are positive numbers and \( \langle r \rangle = \sqrt{1+r^2} \). Here if \( n \) is even number, we further assume \( \kappa < (n-1)/2 \). Then (2.1) admits uniquely a weak solution \( u(r,t) \in C^1(\Omega) \) such that for \( (r,t) \in \Omega \) and \( |\alpha| \leq 1 \) we have

\[
|D^{\alpha}_{r,t}u(r,t)| \leq C \epsilon \langle r \rangle^{1-|\alpha|} \langle r \rangle^{-1} + |\alpha| \Psi(r,|t|),
\]

where we have set \( m = [(n-2)/2] \) and

\[
\Psi(r,t) = \langle r + |t| \rangle^{-\chi(n)}(r-t)^{-\kappa}
\]

with

\[
\chi(n) = \begin{cases} 
1/2 & \text{if } n \text{ is even,} \\
1 & \text{if } n \text{ is odd,}
\end{cases}
\]

and \( C \) is a constant depending only on \( m \) and \( \kappa \).

Next we shall consider the nonlinear wave equation

\[
 u_{tt} - u_{rr} - \frac{n-1}{r} u_r = F(u) \quad \text{in} \quad \Omega,
\]

where \( F(u) = |u|^p \) or \( F(u) = |u|^{p-1}u \). Here we assume

\[
 p_0(n) < p < \frac{(n+3)}{(n-1)}.
\]

We shall introduce a function space \( X \), in which we will look for solutions of (2.4), defined by

\[
 X = \{ u(r,t) \in C^0(\Omega) : D_r u(r,t) \in C^0(\Omega), \|u\| < \infty \},
\]

and

\[
 \|u\| = \sup_{(r,t) \in \Omega} \{|u(r,t)|r^{m-1}(r) + |D_r u(r,t)|r^m\Psi^{-1}(r,|t|)\},
\]

where \( \Psi \) is the same function as in (2.3). As for the parameter \( \kappa \), we assume

\[
 \frac{1}{2} < \kappa \quad \text{and} \quad \frac{p+1}{p-1} - \frac{n+1}{2} < \kappa \leq p,
\]
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where we have set

$$q = (1 + \Phi(n, p))/p = \frac{n - 1}{2}p - \frac{n + 1}{2}$$

with $\Phi(n, p)$ in (1.2). Note that there exist really numbers $\kappa$ satisfying (2.6) for $p > p_0(n)$, because

$$\Phi(n, p) = (p - 1)\{q - (\frac{p + 1}{p - 1} - \frac{n + 1}{2})\} > 0 \quad \text{for} \quad p > p_0(n).$$

We are now in a position to state the main theorem in this note. Let $u_-(r, t)$ be the solution of (2.1) which is obtained in Theorem 1. Note that $u_\in X$ and

(2.7) \[ ||u_-|| \leq C\varepsilon \quad \text{for any} \quad \varepsilon > 0. \]

Then we have

**Theorem 2. (Main theorem).** Assume conditions (2.2), (2.5) and (2.6) hold. Then there is positive constant $\varepsilon_0$ (depending only on $p$, $n$, and $\kappa$) such that, if $0 < \varepsilon \leq \varepsilon_0$, there exists uniquely a weak solution $u(r, t)$ of the nonlinear wave equation (2.4) such that $u \in C^1(\Omega) \cap X$,

(2.7) \[ ||u|| \leq 2||u_-|| \]

and for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

(2.8) \[ |D_{r,t}^\alpha(u(r,t) - u_-(r, t))| \leq C||u||^p r^{1-m-|\alpha|} |r|^{-1+|\alpha|} \Psi(r, t) \]

and

(2.9) \[ ||u(t) - u_-(t)||_e \leq C||u||^p \langle t \rangle^{-\theta} \quad \text{if} \quad t \leq 0, \]

where $|| \cdot ||$ is defined by (1.4) and we have set

$$\theta = \min\{q, \chi(n)p + p\kappa - 1\},$$

and $C$ is a constant depending only on $p$, $n$ and $\kappa$.

Moreover there exists uniquely a weak solution $u_+(r, t)$ of (2.1) which belongs to $C^1(\Omega) \cap X$, such that for $(r, t) \in \Omega$ and $|\alpha| \leq 1$ we have

(2.8) \[ |D_{r,t}^\alpha(u(r,t) - u_+(r, t))| \leq C||u||^p r^{1-m-|\alpha|} |r|^{-1+|\alpha|} \Psi(r, -t) \]
and

\[ (2.9)_+ \quad \|u(t) - u_+(t)\|_e \leq C\|u\|^p\langle t\rangle^{-\theta} \quad \text{if} \quad t \geq 0. \]

**Remarks.**

1) If \( n \) is odd, in Theorems 1 and 2, one can replace \( u \in C^1(\Omega) \) by \( u \in C^2(\Omega) \). Moreover in (2.6) we can replace \( \kappa > 1/2 \) by \( \kappa > 0 \). In this case, we interpret (2.9)\( \pm \) as follows. When \( \kappa > 1/2p \), (2.9)\( \pm \) is still valid. When \( 0 < \kappa \leq 1/2p \), it holds with \( \theta = \kappa \). (See [9]).

2) For \( n \geq 2 \), consider the following Cauchy problem

\[ (2.10) \quad \left\{ \begin{array}{ll} u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) & \text{in} \quad r > 0, \quad t > 0, \\ u(r,0) = 0, \quad u_t(r,0) = g(r) & \text{for} \quad r > 0. \end{array} \right. \]

It is known that, if \( g(r) \geq Mr^{-\mu} \) for \( r \geq 1 \) with some positive constants \( M, \mu \) and \( \mu < (p+1)/(p-1) \), then (2.10) does not admit global solutions. (See [1], [2], [21] and [22]). Therefore condition (2.6) is partially necessary to obtain Theorem 2.

3) One can also show that the Cauchy problem for the nonlinear wave equation (2.4) admits a unique global solution, provided the hypotheses of Theorems 1 and 2 are fulfilled.

In the proof of Theorem 1, the following lemma plays a key role. Moreover Theorem 2 is obtained by considering the associated integral equation with the differential equation (2.4). So the lemma below is very essential in our work.

**Lemma 3.** Let \( g \in C^0((0, \infty)) \) and

\[ g(r) = O(r^{-m-1}) \quad \text{as} \quad r \downarrow 0. \]

For \( r > 0 \) and \( t \geq 0 \) we define a function \( \Theta(g) \) as follows.

\[ (1) \quad \text{n is odd} : n = 2m + 3 \quad (m = 1, 2, \cdots). \]

\[ \Theta(g)(r,t) = \int_{[t-r]}^{t+r} g(\lambda)K(\lambda, r, t)d\lambda, \]

where we have set

\[ K(\lambda, r, t) = r^{2-n}\lambda^{2m+1}H_m(\lambda, r, t), \]

\[ H_m(\lambda, r, t) = \left( \frac{\partial}{\partial \lambda} \frac{1}{2\lambda} \right)^m (r^2 - (\lambda - t)^2)^{(n-3)/2}. \]
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(2) $n$ is even: $n = 2m + 2$ ($m = 1, 2, \cdots$).

\[ \Theta(g)(r, t) = \int_{|t-r|}^{t+r} g(\lambda) K_1(\lambda, r, t) d\lambda + \int_0^{\max(t-r,0)} g(\lambda) K_2(\lambda, r, t) d\lambda, \]

where we have set

\[ K_1(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_\lambda^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho, \]

\[ K_2(\lambda, r, t) = r^{2-n} \lambda^{2m+1} \int_{t-r}^{t+r} \frac{H_m(\rho, r, t)}{\sqrt{\rho^2 - \lambda^2}} d\rho, \]

and

\[ H_m(\rho, r, t) = \left( \frac{\partial}{\partial \rho} \frac{1}{2\rho} \right)^m (r^2 - (\rho - t)^2)^{(n-3)/2}. \]

And we extend $\Theta(g)(r, t)$ as an odd function with respect to $t$. Then $\Theta(g) \in C^0(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

\[ |\Theta(g)(r, t)| \leq C_B r^{-m} \quad \text{for} \quad (r, t) \in B. \]

Moreover, if we set $u(x, t) = \Theta(g)(|x|, t)$, then $u(\cdot, t) \in C^0(\mathbb{R}; L^2_{loc}(\mathbb{R}^n))$ and $u$ is a weak solution of the Cauchy problem

\[ \begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = 0, \quad u_t(x, 0) = c_n g(|x|) & \text{for} \quad x \in \mathbb{R}^n \end{cases} \]

in the sense of distribution, where

\[ c_n = \begin{cases} \frac{2}{\sqrt{\pi}} \Gamma(\frac{n-1}{2}) & \text{if} \quad n \text{ is odd}, \\ \frac{2}{\Gamma(\frac{n-1}{2})} & \text{if} \quad n \text{ is even}. \end{cases} \]

Furthermore, if $g \in C^1((0, \infty))$ and for $j = 0, 1$

\[ g^{(j)}(r) = O(r^{-m-j}) \quad \text{as} \quad r \downarrow 0, \]

then $\Theta(g) \in C^1(\Omega)$ and for each bounded subset $B \subset \Omega$ we have

\[ |D^\alpha_{r,t} \Theta(g)(r, t)| \leq C_B r^{1-m-|\alpha|} \quad \text{for} \quad (r, t) \in B \text{ and } |\alpha| \leq 1. \]
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References


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