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Partial regularity for electrochemical machining with threshold current

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The purpose of electrochemical machining is to shape a workpiece which plays the role of the anode by placing it into an electrolytic solution and applying a voltage drop between it and the tool, the cathode. During the process material is removed electrolytically at the surface of the workpiece while the tool remains unaltered. Speaking mathematically this means that an electric potential $u :]0,T[\times\Omega \to [0,\infty[$ takes different values, say the values 0 on the anode A and the value 1 on the cathode C (which can be considered as a set in $\mathbf{R}^n - \Omega$), u is harmonic in between, and the normal speed V of the moving boundary of the anode equals the dissolution rate, which is assumed to be a function of the electric current flowing in normal direction through the anode surface (see [LS], [Sch]):

$$\begin{aligned}
\Delta u &= 0 \text{ in } \{u>0\} \cap (]0,T[\times\Omega)\,,\\
V &= f(-\nabla u\cdot\nu) \text{ on } \partial\{u>0\}\cap(]0,T[\times\Omega)\,,\\
u &= 1 \text{ on }]0,T[\times(C\cap\partial\Omega)\,,\\
u &= 0 \text{ on }]0,T[\times(A\cap\partial\Omega) \text{ and}\\
\nabla u\cdot\nu &= 0 \text{ on }]0,T[\times(\partial\Omega - C - A)\,.
\end{aligned}$$

Here ν means the outer normal on $\partial\{u>0\}$ and $\partial\Omega$ respectively, and $f : \mathbf{R} \to [0,\infty[$ is a non-decreasing function vanishing up to some threshold value Q and being positive for values $> Q$. The corresponding steady state problem is

$$\Delta u = 0 \text{ in } \{u > 0\} \cap \Omega,$$
$$-\nabla u \cdot \nu \leq Q \text{ on } \partial\{u > 0\} \cap \Omega,$$
$$u = 1 \text{ on } C \cap \partial\Omega,$$
$$u = 0 \text{ on } A \cap \partial\Omega \text{ and}$$
$$\nabla u \cdot \nu = 0 \text{ on } \partial\Omega - A - C\,.$$

However, since this problem is highly non-unique, and since it is known that for certain initial data the solution of the evolutionary problem converges to a solution satisfying the equality $-\nabla u \cdot \nu = Q$ on $\partial\{u > 0\} \cap \Omega$, we pass to the *critical steady state problem*, where the condition on the free boundary $\partial\{u > 0\}$ is replaced by the equation $-\nabla u \cdot \nu = Q$.

Existence of variational solutions for this problem has been shown by Alt and Caffarelli in [AC]: they consider the minimum problem

$$E(v) := \int_\Omega \left(|\nabla v|^2 + Q^2 \chi_{\{v>0\}}\right) \rightarrow \min_K ,$$

where $Q \in C^{0,\gamma}(\Omega)$ is positive and K is a closed affine subspace of $H^{1,2}(\Omega)$ which prescribes suitable boundary data on $\partial\Omega$ (the Dirichlet data should be non-negative in order to obtain non-negative solutions). Since the functional $E(v)$ is but lower semicontinuous, only those sequences $u_\epsilon \rightarrow u$ with respect to whom E is continuous can serve to characterize a local minimum. Therefore the following definition makes sense:

Definition 1

$u \in K$ is called a *local minimum* von E, if there is an $\epsilon_0 > 0$ such that $E(u) \leq E(v)$ for all $v \in K$ which satisfy

$$\|u - v\|_{H^{1,2}(\Omega)} + \|\chi_{\{u>0\}} - \chi_{\{v>0\}}\|_{L^1(\Omega)} \leq \epsilon_0 \ .$$

Existence and regularity of local minima have been proved in [AC]: if u is a local minimum of E then $u \in C^{0,1}_{loc}(\Omega)$ and $u \geq 0$ and thereby u divides Ω into the open set $\{u > 0\}$, where u is harmonic, and the set $\{u = 0\}$.

If the free boundary $\partial\{u > 0\}$ is smooth, then the first variation of the minimum problem yields a strong solution of the critical steady state electrochemical machining problem, however regularity of the free boundary in dimensions $n > 2$ is still open. The same problem appears also in the approach of [HS], where a solution is constructed which satisfies the free boundary condition pointwisely: since they do not prove any regularity of the free boundary, it is not clear whether their solution is in $C^1(\overline{\{u > 0\}})$, i.e. a strong solution.

What Alt and Caffarelli did show is the following "flatness-implies-regularity-result" for variational solutions: flatness of $\partial\{u > 0\}$ in a point x_0 implies regularity of $\partial\{u > 0\}$ in a neighborhood of x_0 ([AC]). This corresponds to an analogous result of DeGiorgi for minimal surfaces, however the methods of proof are different.

In the following we are going to derive a partial regularity result for $\partial\{u > 0\}$ by methods which again have their counterparts in the regularity theory of minimal surfaces.

The most important one is a new monotonicity formula for local minima of E which corresponds to a monotonicity formula of Miranda ([Mi]) for mimimal surfaces:

Theorem 2 (Monotonicity formula)

Let u be a local minimum of E. Then for every $D \subset\subset \Omega$ there exists $\delta > 0$ depending on $(n, \|u\|_{H^{1,\infty}(D)}, \epsilon_0, \mathrm{dist}(D, \partial\Omega))$ such that for any $B_\delta(x_0) \subset D$ satisfying $u(x_0) = 0$ the function $\rho \mapsto p(\rho)$

$$:= \frac{1}{\rho^n} \int\limits_{B_\rho(x_0)} \left(|\nabla u|^2 + Q^2(x_0)\chi_{\{u>0\}}\right) - \frac{1}{\rho} \int\limits_0^\rho \frac{1}{r^{n-1}} \int\limits_{\partial B_r(x_0)} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1} \, dr$$

is well-defined and for any $0 < s < \rho < \delta$ the inequality

$$0 \leq \int\limits_s^\rho t^{-3} \int\limits_{\partial B_1(x_0)} \left[t \int\limits_0^t (\nabla u(r\xi) \cdot \xi)^2 \, dr - \left(\int\limits_0^t \nabla u(r\xi) \cdot \xi \, dr \right)^2 \right] d\mathcal{H}^{n-1}(\xi) \, dt$$

$\leq p(\rho)-p(s)+\dfrac{2n}{\gamma}\rho^\gamma [Q^2]_{\gamma,D}$ holds; here $[Q^2]_{\gamma,D} := \sup\limits_{x\neq y\in D} \dfrac{|Q^2(x)-Q^2(y)|}{|x-y|^\gamma}$.

If Q is constant then p is a non-decreasing function in the interval $(0,\delta)$.

For proof see [We, Theorem 1.2].

Notice that the proof of Theorem 2 holds for $Q=0$ and yields in this case a monotonicity formula for harmonic functions which is not standard.

Furthermore observe that by analogy to the monotonicity formula used in the field of harmonic mappings and liquid crystals ([SU],[HKL]) we could prove that $r^{2-n}\int_{B_r(x_0)}\left(|\nabla u|^2+Q^2\chi_{\{u>0\}}\right)$ is (for constant Q) non-decreasing in r, however the scaling of this monotonicity does not accord with the regularity of u and therefore this monotonicity does not give any new information when passing to blow-up limits.

When considering blow-up sequences $u_k(x) := \frac{u(x_0+\rho_k x)}{\rho_k}$ for $x_0 \in \partial\{u > 0\}$ and $\rho_k \to 0$ as $k\to\infty$, it is known ([AC]) that a subsequence of u_k converges in some sense to a blow-up limit u_0 and that any blow-up limit u_0 is an absolute minimum of $\int\limits_{B_R}\left(|\nabla v|^2 + Q^2(x_0)\chi_{\{v>0\}}\right)$ in the affine space $\{v\in H^{1,2}(B_R) : v-u_0\in H_0^{1,2}(B_R)\}$, and the monotonicity formula of Theorem 2 leads to the fact that *any blow-up limit u_0 is homogeneous of degree 1, that is the graph of u_0 is a cone with vertex at 0* (see [We, Chapter 2]).

Another interesting property is the "non-positive mean curvature of $\partial\{u_0 > 0\}$":

Proposition 3

Let D be an open and bounded set in $\mathbf{R}^n$ satisfying $\bar{D}\cap(\partial\{u_0>0\} - \partial_{red}\{u_0>0\})=\emptyset$. Then $P(\{u_0>0\},D)\leq P(F,D)$ for any $F\subset\{u_0>0\}$ such that $\partial F\cap D$ is a C^2-surface and $\mathrm{supp}(\chi_F-\chi_{\{u_0>0\}})\subset D$. Here P means the perimeter and ∂_{red} means the reduced boundary, i.e. the part of the boundary where the exterior normal of [Fe, 4.5.5] has norm 1.

For proof see [We, Theorem 2.1].

Using the above results in a dimension reduction procedure one obtains the following partial regularity (compare to [We, Chapter 3 and 4]):

Theorem 4

There is a dimension $k^\star \in \mathbf{N} \cup \{\infty\}$ such that for any Ω, Q and K as stated above and for any local minimum $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ of E we have the following:

i) if $n < k^\star$ then the free boundary $\partial\{u > 0\}$ is locally in Ω a $C^{1,\alpha}$-surface.

ii) if $n = k^\star$, then the singular set $\Sigma := (\partial\{u > 0\} - \partial_{red}\{u > 0\}) \cap \Omega$ consists at most of in Ω isolated points and the free boundary is a $C^{1,\alpha}$-surface in the open set $\Omega - \Sigma$.

iii) $\mathcal{H}^s(\Sigma) = 0$ for any $s > n - k^\star$.

Corollary 5

Since $k^\star \geq 3$, it follows that $\mathcal{H}^{n-2}(\Sigma) = 0$ and that in dimension $n = 3$ the singular set Σ consists at most of in Ω isolated points.

The estimate of $k^\star$ from below, i.e. the fact that singularities cannot occur in two dimensions, was already shown in [AC] by sophisticated methods, in our context however the estimate follows from the simple geometric argument, that a function which is harmonic and homogeneous of degree 1 in some domain of $\mathbf{R}^2$ must be linear in this domain.

In case of three dimensions which is the realistic situation with respect to the electrochemical machining problem above, more can be shown with respect to the structure of the isolated singularities: the fact that any blow-up limit u_0 is homogeneous of degree 1 implies that the curvature of the free boundary in radial direction vanishes. Therefore we infer from the non-positive mean curvature of $\partial\{u_0 > 0\}$ and from the smoothness of $\partial\{u_0 > 0\}$ in $\mathbf{R}^n - \{0\}$ that $\{u_0 = 0\}$ has to be a finite union of *convex* sets. A special consequence is that any connected component of $\{u_0 = 0\}$ is Lipschitz continuous.

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