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Operator Matrices and Systems of Evolution Equations

KLAUS-J. ENGEL

Abstract. Many initial value problems like Volterra equations, delay equations or wave equations can be reduced to an abstract Cauchy problem governed by an operator matrix. We introduce a new class of unbounded operator matrices corresponding to these equations and study the spectral theory, compute the adjoint and analyze the generator property of its elements. The abstract results are illustrated by the above mentioned evolution equations.

1. Introduction

Systems of linear evolution equations as well as linear initial value problems with more than one initial data lead in a natural way to an abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = \mathcal{A}u(t), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

involving an operator matrix $\mathcal{A}$ defined on a product of Banach spaces. The main problem is to establish the *well-posedness* of (ACP), i.e., to prove the existence of a unique solution of (ACP) for sufficiently many initial values u_0 and the continuous dependence upon them. In an abstract framework this is equivalent to the question if $\mathcal{A}$ is the *generator* of some kind of semigroup on the underlying Banach space. Note, that these generators in general are characterized by spectral conditions.
Once the well-posedness of (ACP) is verified one is also interested in the qualitative behavior of its solution, e.g. in the regularity, decay or positivity. Once again, these properties correspond to spectral conditions of the generator $\mathcal{A}$.
This demonstrates the importance of a detailed spectral analysis of $\mathcal{A}$, where we have to keep in mind that in our situation $\mathcal{A}$ is given by an *unbounded operator matrix.*
While at this point most authors treat these matrices by ad-hoc methods we continue in this paper our investigations towards a "matrix theory" for unbounded operator matrices hereby extending the results of [Nag89], [Nag90]. In particular, we introduce a new class of unbounded operator matrices and study its operator theoretical properties. This abstract approach is, in our opinion, justified by the systematic use of highly intuitive matrix methods and by the diversity of the applications to concrete examples.

To start with we introduce the following hypothesis, where we use the notation $X \hookrightarrow E$ to indicate that X is continuously embedded in E. Moreover, $[D(A)]$ denotes the domain of A equipped with the graph norm.

Hypothesis (H). *Let E and F be Banach spaces and assume that*
(i) $A : D(A) \subseteq E \to E$ *and* $D : D(D) \subseteq F \to F$ *are densely defined, invertible linear operators.*
(ii) X *and* Y *are Banach spaces such that* $[D(A)] \hookrightarrow X \hookrightarrow E$ *and* $[D(D)] \hookrightarrow Y \hookrightarrow F$.
(iii) $K \in \mathcal{L}(Y, X)$, $L \in \mathcal{L}(X, Y)$ *are bounded linear operators.*

Given operators satisfying these assumptions we define an operator matrix $\mathcal{A}$ on $\mathcal{E} := E \times F$ in the following way. Here and in the sequel we use the notation $\mathcal{X} := X \times Y$.

Definition 1.1. *If A, D, K, L satisfy Hypothesis (H) we consider*

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad D(\mathcal{A}_0) := D(A) \times D(D),$$
$$\mathcal{K} := \begin{pmatrix} 0 & K \\ L & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X})$$

and define the operator $\mathcal{A}$ on $\mathcal{E} = E \times F$ by

$$(1.1) \qquad \mathcal{A} := \mathcal{A}_0(Id + \mathcal{K}), \qquad D(\mathcal{A}) : = \{x \in \mathcal{X} : (Id + \mathcal{K})x \in D(\mathcal{A}_0)\}$$
$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y : \begin{matrix} x + Ky \in D(A) \\ Lx + y \in D(D) \end{matrix} \right\}.$$

If the matrix $\mathcal{A}$ defined by (1.1) satisfies

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times D(D) : x + Ky \in D(A) \right\}$$

*it is called **one-sided K-coupled**.*

Remark 1.2. Accordingly, it is possible to define one-sided L-coupled operator matrices. However, since every one-sided L-coupled matrix on $E \times F$ is isometrically similar to a one-sided K-coupled matrix on $F \times E$ we consider here only one-sided K-coupled matrices. Note that $\mathcal{A}$ is one-sided K-coupled if $LX \subseteq D(D)$.

Before we show in several concrete examples that the abstract situation of Hypothesis (H) is satisfied quite frequently we give a condition which ensures that $\mathcal{A}$ is densely defined.

Proposition 1.3. *Let $\mathcal{A}$ be defined by (1.1). If $KY \subseteq D(A)$ or $LX \subseteq D(D)$ then $\mathcal{A}$ is densely defined.*

Proof. Fix some $\binom{u}{v} \in \mathcal{E}$ and $\varepsilon > 0$ and assume $LX \subseteq D(D)$. Then, by the denseness of $D(A) \times D(D)$ in $\mathcal{E}$, there exists $y \in D(D)$ and $d \in E$ such that

$$\left\| \binom{0}{v} - \binom{0}{y} \right\| < \tfrac{\varepsilon}{2}, \qquad \left\| \binom{d}{0} \right\| < \tfrac{\varepsilon}{2} \qquad \text{and} \qquad (u + d) + Ky \in D(A).$$

Since $x := u + d \in D(A) - Ky \subseteq X$ we obtain from the assumption $LX \subseteq D(D)$ that $Lx + y \in D(D)$. Finally,

$$\left\| \binom{u}{v} - \binom{x}{y} \right\| \leq \left\| \binom{0}{v} - \binom{0}{y} \right\| + \left\| \binom{d}{0} \right\| < \varepsilon,$$

i.e., $\mathcal{A}$ is densely defined. The case $KY \subseteq D(A)$ follows similarly, hence the proof is complete. $\square$

In case $D(\mathcal{A}) = D(A) \times D(D)$ we call $\mathcal{A}$ an operator matrix with ***diagonal domain.*** These matrices are thoroughly studied in [Nag89] and already have important applications as shown in our first example.

Example 1.4. (Volterra Equation) We demonstrate how the *Volterra integro–differential equations*

$$\text{(VE)} \qquad \begin{cases} \dot{u}(t) = Au(t) + \displaystyle\int_0^t C(t-r)u(r)\,dr + f(t), & t \geq 0, \\ u(0) = u_0 \end{cases}$$

can be treated within our framework. To this end let E be a Banach space, $A : D(A) \subseteq E \to E$ be a linear operator on E and $F := \mathrm{F}(\mathbb{R}_+, E)$ be a translation invariant space of E-valued functions such that the shift semigroup $\big(S(t)\big)_{t\geq 0}$ defined by $(S(t)f)(s) := f(s+t)$ is strongly continuous. Denoting its generator by $\frac{d}{ds}$ we finally assume that the Dirac measure in zero $\delta_0 : [D(\frac{d}{ds})] \subset F \to E$ and the operator $C : [D(A)] \subseteq E \to F$ are bounded.
Now we define the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ C & \frac{d}{ds} \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times D(\tfrac{d}{ds})$$

on the product space $\mathcal{E} := E \times \mathrm{F}(\mathbb{R}_+, E)$. Then it is shown in [Mil74], [DGS88], [DS85], [CG80], [NS93] that (VE) is equivalent to the abstract Cauchy problem (ACP) associated to $\mathcal{A}$ on $\mathcal{E}$ with initial data $\mathfrak{u}_0 = \binom{u_0}{f}$.
To show how to represent the above matrix as in (1.1) define $X := [D(A)]$, $Y := [D(\frac{d}{ds})]$ and for $\lambda \in \rho(A) \cap \rho(\frac{d}{ds})$ consider

$$K_\lambda := -R(\lambda, A)\,\delta_0 \in \mathcal{L}(Y, X), \qquad L_\lambda := -R(\lambda, \tfrac{d}{ds})\,C \in \mathcal{L}(X, Y).$$

This obviously gives

$$\mathcal{A} - \lambda = \begin{pmatrix} A - \lambda & 0 \\ 0 & \frac{d}{ds} - \lambda \end{pmatrix} \begin{pmatrix} Id & K_\lambda \\ L_\lambda & Id \end{pmatrix}.$$

We will continue this example in 2.7. The generator property of $\mathcal{A}$ will be discussed in Examples 3.7 and 3.9.

In our next example as well as in the sequel it will be convenient to use the following notation.
Assume that E, F are Banach spaces, $\mathrm{F}(I, F)$ is a Banach space of F-valued functions defined on an interval $I \subseteq \mathbb{R}$, $T \in \mathcal{L}(E, F)$ is a bounded linear operator and $f : I \to \mathbb{C}$ is a complex-valued function such that $f(\cdot)y \in \mathrm{F}(I, F)$ for all $y \in F$. We then define the linear operator $f \otimes T : E \to \mathrm{F}(I, F)$ by

$$((f \otimes T)x)(s) := f(s) \cdot Tx$$

for all $x \in E$, $s \in I$.

Example 1.5. (Delay Equation) For an operator $C \in \mathcal{L}\left(\mathrm{W}^{1,p}([-1,0],\mathbb{C}^n),\mathbb{C}^n\right)$ where $1 \leq p < \infty$ we consider the *delay equation*

$$\text{(DE)} \qquad \begin{cases} \dot{u}(t) = Cu_t & \text{for } t \geq 0, \\ u(s) = \varphi(s) & \text{for } s \in [-1,0]. \end{cases}$$

Here $u_t : [-1,0] \to \mathbb{C}^n$ is defined by $u_t(s) := u(t+s)$ and $\varphi : [-1,0] \to \mathbb{C}^n$ is a given "history"-function.
One approach to treat (DE) with semigroup methods is to introduce the variable $v := u_t(0)$. Then one can show, cf. [Kap86] or [KpZ86], that (DE) is equivalent to the Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} u_t \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{ds} & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} u_t \\ v(t) \end{pmatrix}, \quad t \geq 0,$$
$$u_0 = \varphi, \quad v(0) = \varphi(0)$$

on $\mathrm{L}^p([-1,0],\mathbb{C}^n) \times \mathbb{C}^n$. Since $v = u_t(0)$ we have to choose for $\mathcal{A} := \left(\begin{smallmatrix} \frac{d}{ds} & 0 \\ C & 0 \end{smallmatrix}\right)$ the domain

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ y \end{pmatrix} \in \mathrm{W}^{1,p}([-1,0],\mathbb{C}^n) \times \mathbb{C}^n : f(0) = y \right\}.$$

In order to show how $\mathcal{A}$ can be represented as in (1.1) we introduce the Banach spaces $E := \mathrm{L}^p([-1,0],\mathbb{C}^n)$, $X := \mathrm{W}^{1,p}([-1,0],\mathbb{C}^n)$ and $F := Y := \mathbb{C}^n$. Moreover, for $\lambda \neq 0$ let the operators A_λ, D_λ, K_λ and L_λ be defined by

$$\begin{aligned} &A_\lambda \subset \tfrac{d}{ds} - \lambda, && D(A_\lambda) := \{f \in X : f(0) = 0\}, \\ &D_\lambda \in \mathcal{L}(F), && D_\lambda := -\lambda\, Id, \\ &L_\lambda \in \mathcal{L}(X,Y), && L_\lambda := -\tfrac{1}{\lambda} C, \\ &K_\lambda \in \mathcal{L}(Y,X), && K_\lambda := -\varepsilon_\lambda \otimes Id, \end{aligned}$$

where $\varepsilon_\lambda(s) := e^{\lambda s}$, i.e., $(K_\lambda y)(s) := -e^{\lambda s} y$ for $y \in Y$ and $s \in [-1,0]$. Then for $\lambda \neq 0$ we obtain

$$\text{(1.2)} \qquad \begin{aligned} \mathcal{A}_\lambda :&= \begin{pmatrix} A_\lambda & 0 \\ 0 & D_\lambda \end{pmatrix}\begin{pmatrix} Id & K_\lambda \\ L_\lambda & Id \end{pmatrix} \\ &= \mathcal{A} - \lambda. \end{aligned}$$

For a continuation of this example see 2.9. The generator property of $\mathcal{A}$ will be discussed in Example 3.13.

The following example treats a partial-functional differential equation.

Example 1.6. (Wave Equation with Viscoelastic Damping) In [BF87], [BF89] the authors study a mathematical model for the longitudinal motion of a uniform bar of length 1 with fixed ends and viscoelastic damping of Boltzmann type. This model is described by the partial differential equation

$$\text{(PDE)} \quad \ddot{u}(t,x) = \tfrac{d}{dx}\left(\alpha \tfrac{d}{dx} u(t,x) + \int_{-r}^{0} k(s) \tfrac{d}{dx} u(t+s,x)\, ds\right) + f(t,x), \;\; x \in (0,1), \;\; t > 0,$$

with initial conditions

$$\text{(IC)} \qquad \begin{cases} u(0,\cdot) = u_0, & \dot{u}(0,\cdot) = u_1, \\ \frac{d}{dx}u(s,\cdot) = \varphi(s,\cdot), & -r \le s < 0 \end{cases}$$

and the boundary conditions

$$\text{(BC)} \qquad u(t,0) = 0 = u(t,1), \qquad t > 0.$$

Here $u(t,x)$ is the longitudinal displacement at position x along the bar at time t, α is a positive material constant and f the applied body force. The function k appearing in the "history" integral term of (PDE) is negative and can be considered as a damping parameter. It satisfies some physically reasonable assumptions where the most important ones for us are

$$(1.3) \qquad k \in \mathrm{H}^1[-r,0], \qquad k(-r) = 0 \qquad \text{and} \qquad \alpha + \int_{-r}^{0} k(s)\,ds > 0.$$

By introducing the spaces

$$\begin{aligned} &E_1 := \mathrm{L}^2[0,1], && E_2 := \left\{ f \in \mathrm{L}^2[0,1] : \textstyle\int_0^1 f(x)\,dx = 0 \right\}, \\ &E := E_1 \times E_2, && F := \mathrm{L}^2([-r,0], E_2), \\ &\mathcal{E} := E \times F \end{aligned}$$

it is shown that (PDE), (IC), (BC) can be expressed as a Cauchy problem

$$\begin{aligned} \dot{\mathfrak{u}}(t) &= \mathcal{A}\mathfrak{u}(t) + \mathcal{F}(t), \quad t \ge 0, \\ \mathfrak{u}(0) &= \begin{pmatrix} u_1 \\ \frac{d}{dx}u_0 \\ \varphi \end{pmatrix} \end{aligned}$$

on the product space $\mathcal{E}$. Here

$$\mathfrak{u}(t) := \begin{pmatrix} \dot{u}(t) \\ u_x(t) \\ u_x(t+\cdot) \end{pmatrix}, \qquad \mathcal{F}(t) := \begin{pmatrix} f(t,\cdot) \\ 0 \\ 0 \end{pmatrix}$$

and $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} g \\ h \\ w \end{pmatrix} \in \mathcal{E} : \begin{array}{c} g \in \mathrm{H}_0^1[0,1], \\ (\alpha h + \int_{-r}^{0} k(s) w(s)\,ds\,) \in \mathrm{H}^1[0,1] \\ w \in \mathrm{H}^1([-r,0], E_2),\ w(0) = h \end{array} \right\},$$

$$\mathcal{A}\begin{pmatrix} g \\ h \\ w \end{pmatrix} := \begin{pmatrix} \frac{d}{dx}\left(\alpha h + \int_{-r}^{0} k(s) w(s)\,ds \right) \\ \frac{d}{dx} g \\ \frac{d}{ds} w \end{pmatrix}.$$

See [BF87] and [BF89] for the details.

We claim that $\mathcal{A}$ can be represented as an operator matrix in the sense of (1.1). To prove this assertion we take $X := E$, $Y := F$ and define the operators

$$(1.4)\qquad \begin{aligned} &A := \begin{pmatrix} 0 & \alpha\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, && D(A) := \mathrm{H}^1_0[0,1] \times \Big(\mathrm{H}^1[0,1] \cap E_2\Big), \\ &D_m := \tfrac{d}{ds}, && D(\tfrac{d}{ds}) := \mathrm{H}^1\big([-r,0], E_2\big), \\ &D \subset D_m, && D(D) := \left\{w \in D(\tfrac{d}{ds}) : w(0) = 0\right\}, \\ &L \in \mathcal{L}(X,Y), && L := (0, -\mathbb{1} \otimes Id), \\ &K \in \mathcal{L}(Y,X), && Kw := \begin{pmatrix} 0 \\ \frac{1}{\alpha}\int_{-r}^0 k(s) w(s,\cdot)\, ds \end{pmatrix}, \end{aligned}$$

where the boundedness of K can be easily verified by using Hölder's inequality and the fact that $k \in \mathrm{L}^2[-r,0]$. Then it follows from $\mathbb{1} \in \ker(D_m)$ and

$$L\begin{pmatrix} g \\ h \end{pmatrix} + w \in D(D) \qquad \Longleftrightarrow \qquad w \in D(D_m),\ h - w(0) = 0$$

for $\binom{g}{h} \in X$ and $w \in Y$ that

$$(1.5)\qquad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Id & K \\ L & Id \end{pmatrix}.$$

Since it is not difficult to verify that A and D are invertible on E and F, respectively, all assumptions of Definition 1.1 are satisfied and hence our claim is proved.
We will return to this example in 2.8. In Example 3.15 we will show that $\mathcal{A}$ generates a semigroup on $\mathcal{E}$.

2. Spectral Theory of Operator Matrices

It is well known that for many classes of operators the growth of the solution $u(\cdot)$ of (ACP) is determined by the location of the spectrum of $\mathcal{A}$, cf. [Nag86, A,B,C-III,IV]. Therefore we compute in this section the spectrum of operator matrices $\mathcal{A}$ defined by (1.1). However, we will restrict ourself to the most important case, namely to one-sided K-coupled operator matrices and refer the reader to [Eng95] for a treatment of the general case. Moreover, we give formulas for the resolvent as well as for the adjoint of $\mathcal{A}$.
First we state the following simple lemma which is easily verified. Here we set again $\mathcal{X} := X \times Y$ for two Banach spaces X and Y.

Lemma 2.1. *Let $\mathcal{K} := \left(\begin{smallmatrix} 0 & K \\ L & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathcal{X})$. Then we have*

$$(2.1)\qquad Id + \mathcal{K} = \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & Id - LK \end{pmatrix} \begin{pmatrix} Id & K \\ 0 & Id \end{pmatrix}$$

$$(2.2)\qquad = \begin{pmatrix} Id & K \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id - KL & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix}.$$

In particular, $Id + \mathcal{K} \in \mathcal{L}(\mathcal{X})$ is invertible if and only if $Id - KL \in \mathcal{L}(X)$ is invertible if and only if $Id - LK \in \mathcal{L}(Y)$ is invertible.

As we will see in the next proposition one-sided K-coupled operator matrices allow a factorization which turns out to be extremely useful. As a preparation we need the following result.

Lemma 2.2. *If $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Id & K \\ L & Id \end{pmatrix}$ is one-sided K-coupled, then $L(D(A)) \subset D(D)$ and $LK(D(D)) \subseteq D(D)$. Moreover, the operators $DLA^{-1} : E \to F$ and $DLKD^{-1} : F \to F$ are well-defined and bounded.*

Proof. By hypothesis we have $U = V$ where

$$U := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in X \times Y : \begin{matrix} u + Kv \in D(A) \\ Lu + v \in D(D) \end{matrix} \right\}, \quad V := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times D(D) : x + Ky \in D(A) \right\}.$$

If we assume that there exists $x \in D(A)$ such that $Lx \notin D(D)$ then we obtain the contradiction $\binom{x}{0} \in V$, $\binom{x}{0} \notin U$, hence $L(D(A)) \subset D(D)$. Now assume that there exists $y \in D(D)$ such that $LKy \notin D(D)$. Then we again obtain a contradiction $\binom{-Ky}{y} \in V$, $\binom{-Ky}{y} \notin U$, hence $LK(D(D)) \subseteq D(D)$. This also shows that $DLA^{-1} : E \to F$ and $DLKD^{-1} : F \to F$ are well-defined. Since $LA^{-1} \in \mathcal{L}(E, F)$ and D is closed the operator DLA^{-1} is closed, hence bounded by the closed graph theorem. To show that $DLKD^{-1}$ is bounded note that $D : D(D) \subseteq Y \to F$ is closed. Since $LKD^{-1} \in \mathcal{L}(F, Y)$, by the same arguments as before, $DLKD^{-1} \in \mathcal{L}(F)$. $\square$

We now present a factorization for one-sided K-coupled operator matrices which is the key tool for our further investigations.

Proposition 2.3. *For a one-sided K-coupled operator matrix $\mathcal{A}$ on $\mathcal{E}$ we have*

$$\mathcal{A} = \begin{pmatrix} Id & 0 \\ DLA^{-1} & Id - DLKD^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & KD^{-1} \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & D \end{pmatrix}. \tag{2.3}$$

Proof. We denote by $\tilde{\mathcal{A}}$ the right hand side of (2.3). Since $KD^{-1} \in \mathcal{L}(F, E)$ it follows that

$$D(\tilde{\mathcal{A}}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in E \times D(D) : x + Ky \in D(A) \right\},$$

i.e., $D(\mathcal{A}) \subseteq D(\tilde{\mathcal{A}})$. However, since for $\binom{x}{y} \in D(\tilde{\mathcal{A}})$ we have $x \in D(A) - Ky \subseteq X$ we obtain $D(\mathcal{A}) = D(\tilde{\mathcal{A}})$. The assertion now follows easily if one applies $\binom{x}{y} \in D(\mathcal{A})$ to the product on the right hand side of (2.3) and observes that $\binom{x}{y} \in D(\mathcal{A})$ implies that $x \in X$ and $Lx \in D(D)$. $\square$

The main advantage of this factorization over the one given in Definition 1.1 is that the only non-invertible factor is bounded on $\mathcal{E}$ and appears on the left of the product in (2.3). Moreover, all off-diagonal entries are bounded between E and F. This makes it easy to compute the adjoint of $\mathcal{A}$ as well as to characterize the invertibility of $\mathcal{A}$.
Using similar arguments we can show a corresponding decomposition for $\lambda - \mathcal{A}$. Hereby the operators $K_\lambda \in \mathcal{L}(Y, X)$ and $L_\lambda \in \mathcal{L}(X, Y)$ are defined by

$$\begin{aligned} K_\lambda :&= -AR(\lambda, A)K = K - \lambda R(\lambda, A)K \in \mathcal{L}(Y, X), \\ L_\lambda :&= -DR(\lambda, D)L = L - \lambda R(\lambda, D)L \in \mathcal{L}(X, Y). \end{aligned}$$

Proposition 2.4. *For a one-sided K-coupled operator matrix $\mathcal{A}$ we have for all $\lambda \in \rho(A) \cap \rho(D)$*

$$\begin{aligned} &\lambda - \mathcal{A} = \mathcal{B}_\lambda \mathcal{U}_\lambda, \\ &\mathcal{B}_\lambda := \begin{pmatrix} Id & 0 \\ (\lambda - D)L_\lambda R(\lambda, A) & Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D) \end{pmatrix}, \\ &\mathcal{U}_\lambda := \begin{pmatrix} \lambda - A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & K_\lambda R(\lambda, D) \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & \lambda - D \end{pmatrix}. \end{aligned} \tag{2.4}$$

Next we state the main result of this section using the notation

$$\begin{aligned} \Delta_F(\lambda - \mathcal{A}) :&= Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D) \\ &= Id + DLK_\lambda R(\lambda, D) \ \in \mathcal{L}(F) \end{aligned}$$

for $\lambda \in \rho(A) \cap \rho(D)$. Moreover, we denote by $\sigma_p(\mathcal{A})$ the point spectrum of $\mathcal{A}$, while $\sigma_{\mathrm{ess}}(\mathcal{A})$ denotes the essential spectrum of $\mathcal{A}$. More precisely, we define

$$\begin{aligned} &\sigma_p(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda - \mathcal{A} \quad \text{is not injective}\}, \\ &\sigma_{\mathrm{ess}}(\mathcal{A}) := \left\{\lambda \in \mathbb{C} : \dim\left(\ker(\lambda - \mathcal{A})\right) = \infty \quad \text{or} \quad \dim\left(\mathcal{E}/_{\mathrm{Rg}(\lambda - \mathcal{A})}\right) = \infty\right\}. \end{aligned}$$

Theorem 2.5. *Every one-sided K-coupled operator matrix $\mathcal{A}$ is densely defined. Moreover, the following assertions hold true.*

(a) For $\lambda \in \rho(A) \cap \rho(D)$ we have

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}) \quad &\Longleftrightarrow \quad 0 \in \sigma\left(\Delta_F(\lambda - \mathcal{A})\right), \\ \lambda \in \sigma_p(\mathcal{A}) \quad &\Longleftrightarrow \quad 0 \in \sigma_p\left(\Delta_F(\lambda - \mathcal{A})\right), \\ \lambda \in \sigma_{\mathrm{ess}}(\mathcal{A}) \quad &\Longleftrightarrow \quad 0 \in \sigma_{\mathrm{ess}}\left(\Delta_F(\lambda - \mathcal{A})\right). \end{aligned}$$

(b) For $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ of $\mathcal{A}$ is given by

$$\begin{pmatrix} \left(Id - K_\lambda R(\lambda, D)\, \Delta_F(\lambda - \mathcal{A})^{-1} DL\right) R(\lambda, A) & -K_\lambda R(\lambda, D)\, \Delta_F(\lambda - \mathcal{A})^{-1} \\ R(\lambda, D)\, \Delta_F(\lambda - \mathcal{A})^{-1} DLR(\lambda, A) & R(\lambda, D)\, \Delta_F(\lambda - \mathcal{A})^{-1} \end{pmatrix}. \tag{2.5}$$

(c) The adjoint of $\mathcal{A}$ is given by

$$\mathcal{A}' = \begin{pmatrix} Id & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} Id & 0 \\ (KD^{-1})' & Id \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & (DLA^{-1})' \\ 0 & Id - (DLKD^{-1})' \end{pmatrix}. \tag{2.6}$$

Proof. Since $D(\mathcal{A}) = D\left(\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} Id & K \\ 0 & Id \end{smallmatrix}\right)\right)$ the operator $\mathcal{A}$ is densely defined by Proposition 1.3.
(a) Using the fact that $K_\lambda R(\lambda, D) \in \mathcal{L}(F, E)$ we see that $\mathfrak{U}_\lambda$, defined as in (2.4), is invertible. Hence, $\lambda - \mathcal{A}$ is invertible if and only if $\mathfrak{B}_\lambda \in \mathcal{L}(E)$ is invertible which is the case if and only if $\Delta_F(\lambda - \mathcal{A}) \in \mathcal{L}(F)$ is invertible. Moreover, since $\mathfrak{U}_\lambda$ is invertible we have

$$\ker(\lambda - \mathcal{A}) = \ker(\mathfrak{B}_\lambda) \simeq \ker(Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D)),$$
$$\mathrm{Rg}(\lambda - \mathcal{A}) = \mathrm{Rg}(\mathfrak{B}_\lambda) \simeq E \times \mathrm{Rg}(Id - (\lambda - D)L_\lambda K_\lambda R(\lambda, D))$$

and the assertions regarding the essential- and the point-spectrum follow easily.
(b) Formula (2.5) follows by inverting each factor in (2.4) by using the fact that $(\lambda - D)L_\lambda = -DL$.
(c) By (2.4) we have $\mathcal{A} = -\mathfrak{B}_0\mathfrak{U}_0$. Since $\mathfrak{B}_0$ is bounded and every factor defining $\mathfrak{U}$ is invertible formula (2.6) follows from [FL77, 7.Thm.]. □

Remark 2.6. If $\dim(F) < \infty$ then the condition $0 \in \sigma(\Delta_F(\lambda - \mathcal{A}))$ in Theorem 2.5 (a) is equivalent to the *characteristic equation*

$$\det\big(\Delta_F(\lambda - \mathcal{A})\big) = 0.$$

To illustrate these abstract results we review the examples given in the introduction. We start by continuing 1.4 associated to the Volterra equation (VE).

Example 2.7. (Volterra Equation) Let

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ C & \frac{d}{ds} \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times D(\tfrac{d}{ds})$$

on $\mathcal{E} := E \times \mathrm{F}(\mathbb{R}_+, E)$ be defined as in Example 1.4. Then, by Remark 1.2 we obtain for $\lambda \in \rho(A) \cap \rho(\frac{d}{ds})$

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}) \quad &\Longleftrightarrow \quad 0 \in \sigma\Big(Id - \delta_0 R(\lambda, \tfrac{d}{ds})CR(\lambda, A)\Big) \\ &\Longleftrightarrow \quad 0 \in \sigma\Big(\lambda - A - \delta_0 R(\lambda, \tfrac{d}{ds})C\Big). \end{aligned}$$

If $\mathrm{Re}(\lambda) > \omega(\frac{d}{ds})$ where $\omega(\cdot)$ denotes the growth bound, cf. [Nag86, A-III.1], we can replace the resolvent of $\frac{d}{ds}$ by the Laplace transform of the shift semigroup and obtain

$$\lambda \in \sigma(\mathcal{A}) \quad \Longleftrightarrow \quad 0 \in \sigma\left(\lambda - A - \int_0^\infty e^{-\lambda t} C(t)\, dt\right),$$

where the integral is understood in the strong sense for elements in $D(A)$. In case E is finite dimensional this last condition reduces to the well known characteristic equation

$$\lambda \in \sigma(\mathcal{A}) \quad \Longleftrightarrow \quad \det\left(\lambda - A - \int_0^\infty e^{-\lambda t} C(t)\, dt\right) = 0.$$

Note, that by Theorem 2.5 (c) we can also easily compute the adjoint of $\mathcal{A}$ and obtain in this way [GS89, Thm.2.1.]. For results concerning the generator property of $\mathcal{A}$ we refer to Examples 3.7 and 3.9.

In our next example we first have to apply a similarity transformation to $\mathcal{A}$ before we can use our results.

Example 2.8. (Wave Equation with Viscoelastic Damping) We continue the discussion of 1.6. Using the condition $\alpha + \int_{-r}^{0} k(s)\,ds > 0$ in (1.3) it is easy to see that $Id - KL \in \mathcal{L}(E)$ is invertible, thus, by Lemma 2.1, the matrix

$$Id + \mathcal{K} := \begin{pmatrix} Id & K \\ L & Id \end{pmatrix} \in \mathcal{L}(\mathcal{E})$$

in (1.5) is invertible as well. Therefore the operator matrix $\mathcal{A}$ associated to the wave equation with viscoelastic damping is similar to

$$\tilde{\mathcal{A}} := \begin{pmatrix} Id & K \\ L & Id \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{2.7}$$

where A and D are defined in (1.4). Since $\tilde{\mathcal{A}}$ has a diagonal domain $D(\tilde{\mathcal{A}}) = D(A) \times D(D)$ we can apply our results as in the previous example to compute $\sigma(\mathcal{A}) = \sigma(\tilde{\mathcal{A}})$. Moreover, since $Id + \mathcal{K}$ is invertible we obtain from [FL77, 7.Thm. and 9.Lem.]

$$\mathcal{A}' = \begin{pmatrix} Id & L' \\ K' & Id \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & D' \end{pmatrix}.$$

In Example 3.15 we will show that $\mathcal{A}$ generates a semigroup.

We now apply our results to the operator $\mathcal{A}$ related to the delay equation considered in 1.5.

Example 2.9. (Delay Equation) From Theorem 2.5 we obtain $\sigma_{\text{ess}}(\mathcal{A}) = \emptyset$ and

$$\begin{aligned} \lambda \in \sigma(\mathcal{A}) \quad &\iff \quad \lambda \in \sigma_p(\mathcal{A}) \\ &\iff \quad \det\left(\lambda - C \circ \varepsilon_\lambda \otimes Id\right) = 0. \end{aligned}$$

Moreover, since $(\mathbb{1} \otimes Id)' = I \in \mathcal{L}(\mathrm{L}^q[-1,0], \mathbb{C}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$, where $Ig := \int_{-1}^{0} g(s)\,ds$ and $(A_{0,p})' = -A_{-1,q}$ for

$$A_{i,r} \subset \tfrac{d}{ds}, \qquad D(A_{i,r}) := \left\{ g \in \mathrm{W}^{1,r}([-1,0], \mathbb{C}^n) : g(i) = 0 \right\},$$

on $\mathrm{L}^r([-1,0], \mathbb{C}^n)$, $r \in \{p,q\}$, $i \in \{-1,0\}$, we obtain

$$\mathcal{A}' = \begin{pmatrix} Id & 0 \\ I & Id \end{pmatrix} \begin{pmatrix} -A_{-1,q} & 0 \\ 0 & -(C \circ \mathbb{1} \otimes Id)' \end{pmatrix} \begin{pmatrix} Id & (C \circ (A_{0,p})^{-1})' \\ 0 & Id \end{pmatrix}.$$

3. Operator Matrices as Semigroup Generators

As we mentioned in the Introduction (ACP) is "well-posed" if and only if the associated operator $\mathcal{A}$ generates a semigroup on $\mathcal{E}$. In this section we therefore study the problem when an operator matrix $\mathcal{A}$ as defined in 1.1 is the generator of a strongly continuous semigroup. As in the previous section we restrict ourself to one-sided coupled matrices and refer to [Eng95] for the general case.
We start by considering triangular operator matrices, i.e., matrices $\mathcal{A}$ defined by (1.1) for $L = 0$. In view of Proposition 2.3 the operator $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A} &:= \begin{pmatrix} A & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & Q \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & D \end{pmatrix}, \\ D(\mathcal{A}) &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in E \times D(D) : x + QDy \in D(A) \right\},\end{aligned} \tag{3.1}$$

i.e.,

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x+QDy) \\ Dy \end{pmatrix},$$

where in (2.3) for simplicity we set $Q := KD^{-1} \in \mathcal{L}(F, E)$. By the factorization given in Proposition 2.3 an arbitrary one-sided K-coupled matrix can be regarded as a multiplicative perturbation of a triangular matrix. This makes it possible to treat the general case by a variety of perturbation results, cf. [Hol92], [DS89] and [PS95].
If $L = 0$, $Q = KD^{-1}$ we obtain from Theorem 2.5 the following result.

Lemma 3.1. *The operator matrix $\mathcal{A}$ defined by (3.1) is closed and densely defined. Moreover, $\rho(A) \cap \rho(D) \subseteq \rho(\mathcal{A})$ and for $\lambda \in \rho(A) \cap \rho(D)$ the resolvent of $\mathcal{A}$ is given by*

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & AR(\lambda, A)QDR(\lambda, D) \\ 0 & R(\lambda, D) \end{pmatrix}. \tag{3.2}$$

We now turn to the problem of characterizing the generator property of $\mathcal{A}$. First we introduce some additional notation.
If A and D generate strongly continuous semigroups $(T(t))_{t\geq 0}$, $(S(t))_{t\geq 0}$, respectively, we define the operator family $(Q(t))_{t\geq 0} \subset \mathcal{L}(F, E)$ by

$$Q(t)y := \int_0^t T(t-r)QS(r)y\,dr$$

for $y \in F$. Clearly, $(Q(t))_{t\geq 0}$ is exponentially bounded and strongly continuous. Therefore, we can apply the Laplace transform $\mathcal{L}$ to $Q(\cdot)$ and by the convolution theorem (see [Doe70, Satz 10.1]) we obtain

$$\mathcal{L}\left(Q(\cdot)y\right)(\lambda) = R(\lambda, A)QR(\lambda, D)y \tag{3.3}$$

for all $y \in F$ and all $\lambda \in \mathbb{C}$ with sufficiently large real part.
Next we show that the operators $Q(t)$ behave nicely with respect to the domains $D(A)$ and $D(D)$.

Lemma 3.2. *For all $t \geq 0$ we have $Q(t)D(D) \subseteq D(A)$.*

Proof. In fact, for $y \in D(D)$ one has

$$
\begin{aligned}
Q(t)y &= \int_0^t T(t-r)QS(r)y\,dr \\
&= -A^{-1}T(t-r)QS(r)y\big|_{r=0}^{t} + A^{-1}\int_0^t T(t-r)QS(r)Dy\,dr \\
&= A^{-1}\left(T(t)Qy - QS(t)y + \int_0^t T(t-r)QS(r)Dy\,dr\right), \qquad (3.4)
\end{aligned}
$$

which is an element of $D(A)$. □

Hence, for all $t \geq 0$ and $y \in D(D^2)$ we can define

$$
(3.5) \qquad \tilde{R}(t)y := AQ(t)Dy = T(t)QDy - QDS(t)y + \int_0^t T(t-r)QS(r)D^2 y\,dr,
$$

which yields an operator from $D(D^2)$ into E.

Theorem 3.3. *Let $\mathcal{A}$ be defined by (3.1). If there exists $w \in \mathbb{R}$ such that $(w, \infty) \subset \rho(A) \cap \rho(D)$ then the following assertions are equivalent.*
(a) $\mathcal{A}$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{E}$.
(b) (i) A and D are generators of strongly continuous semigroups $(T(t))_{t\geq 0}$ on E and $(S(t))_{t\geq 0}$ on F, respectively.
(ii) For all $t \geq 0$ the operators $\tilde{R}(t) : D(D^2) \subseteq F \to E$ are bounded and satisfy $\overline{\lim}_{t\downarrow 0} \|\tilde{R}(t)\| < \infty$.
In case these conditions hold true, the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is given by

$$
\mathcal{T}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix},
$$

where $R(t) \in \mathcal{L}(F, E)$ is the unique bounded extension of $\tilde{R}(t)$.

Proof. (a)⇒(b). By Lemma 3.1 we know that for $\lambda \in (w, \infty)$ the resolvent $R(\lambda, \mathcal{A})$ is given by equation (3.2). Since the diagonal entries of $R(\lambda, \mathcal{A})^n$ equal $R(\lambda, A)^n$, $R(\lambda, D)^n$, respectively, we conclude by the Hille-Yosida theorem that A and D generate strongly continuous semigroups $(T(t))_{t\geq 0}$ on E and $(S(t))_{t\geq 0}$ on F, respectively. In order to determine the off-diagonal entries of $\mathcal{T}(t)$ we use the fact that the Laplace transform of $\mathcal{T}(\cdot)$ is given by $R(\cdot, \mathcal{A})$, i.e.,

$$
\mathcal{L}\left(\mathcal{T}(\cdot)x\right)(\lambda) = R(\lambda, \mathcal{A})x
$$

for all $x \in \mathcal{E}$ and $\mathrm{Re}(\lambda)$ sufficiently large. In particular, if $\mathcal{T}(t) = (T_{ij}(t))_{2\times 2}$ we obtain
(1) $\mathcal{L}\left(T_{21}(t)x\right)(\lambda) = 0$ for all $x \in E$ and
(2) $\mathcal{L}\left(T_{12}(t)y\right)(\lambda) = AR(\lambda, A)QDR(\lambda, D)y$ for all $y \in F$.

By the uniqueness theorem for the Laplace transform (cf. [Doe70, Satz 5.1] or [HN93, Cor.1.4]) we conclude from (1) that $T_{21}(t) = 0$ for all $t \geq 0$. In order to determine $T_{12}(t)$ from (2) we assume that $y \in D(D^2)$. Then, using (3.3) we obtain

$$\begin{aligned}\mathcal{L}\left(A^{-1}T_{12}(\cdot)y\right)(\lambda) &= A^{-1}\,\mathcal{L}\left(T_{12}(\cdot)y\right)(\lambda)\\ &= R(\lambda, A)QR(\lambda, D)Dy\\ &= \mathcal{L}\left(Q(\cdot)Dy\right)(\lambda).\end{aligned}$$

Again by the uniqueness of the Laplace transform we conclude that $A^{-1}T_{12}(t)y = Q(t)Dy$ for all $y \in D(D^2)$. Hence $T_{12}(t)y = \tilde{R}(t)y$ and therefore $\overline{\lim}_{t\downarrow 0} \|\tilde{R}(t)\| < \infty$.
(b)⇒(a) We proceed in two steps. First we show that

$$\mathcal{S}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0$$

defines a strongly continuous semigroup $\left(\mathcal{S}(t)\right)_{t\geq 0}$ on $\mathcal{E}$. In the second step we show that its generator is given by $\mathcal{A}$.
Step 1. Clearly, $\mathcal{S}(0) = Id$. In order to verify the semigroup property it suffices to show that

$$\left[\mathcal{S}(s)\mathcal{S}(t)\right]_{12} y = \left[\mathcal{S}(s+t)\right]_{12} y$$

for all $y \in D(D^2)$ and $s,\ t \geq 0$. Hence, let $y \in D(D^2)$. Then $S(t)y \in D(D^2)$ and we obtain

$$\begin{aligned}\left[\mathcal{S}(s)\mathcal{S}(t)\right]_{12} y &= T(s)\tilde{R}(t)y + \tilde{R}(s)S(t)y\\ &= A\int_0^t T(t+s-r)QS(r)Dy\,dr + A\int_t^{s+t} T(t+s-r)QS(r)Dy\,dr\\ &= \tilde{R}(s+t)y = \left[\mathcal{S}(s+t)\right]_{12} y.\end{aligned}$$

We show next that $\left(\mathcal{S}(t)\right)_{t\geq 0}$ is strongly continuous. For this it suffices to verify that $\left(R(t)\right)_{t\geq 0}$ is strongly continuous in $t = 0$. Let $y \in D(D^2)$. Using (3.5) we obtain

$$R(t)y = \left(T(t)QDy - QDy\right) - Q\left(S(t)Dy - Dy\right) + \int_0^t T(t-r)QS(r)D^2y\,dr,$$

hence $\lim_{t\downarrow 0} R(t)y = 0$ for all $y \in D(D^2)$. Since $\|R(t)\|$ is bounded for $t \downarrow 0$ and $D(D^2)$ is dense in F an easy 2-ε argument shows that $\lim_{t\to 0} R(t)y = 0$ for all $y \in F$. Hence $\left(\mathcal{S}(t)\right)_{t\geq 0}$ defines a strongly continuous semigroup on $\mathcal{E}$.
Step 2. Let $\mathcal{D}$ be the generator of $\left(\mathcal{S}(t)\right)_{t\geq 0}$. In order to show that $\mathcal{D} = \mathcal{A}$ it suffices to prove that $R(\lambda, \mathcal{A}) = R(\lambda, \mathcal{D})$ for λ sufficiently large. Using again the fact that the Laplace transform $\mathcal{L}\left(\mathcal{S}(\cdot)\right)(\lambda)$ coincides with $R(\lambda, \mathcal{D})$ we easily obtain from Lemma 3.1 that $\left[R(\lambda, \mathcal{A})\right]_{ij} = \left[R(\lambda, \mathcal{D})\right]_{ij}$ for $(i,j) \neq (1,2)$.
It remains to show that $\left[R(\lambda, \mathcal{A})\right]_{12} = \left[R(\lambda, \mathcal{D})\right]_{12}$. To this end fix some $y \in D(D^2)$. Then by (3.3) we have

$$\begin{aligned}A^{-1}\left[R(\lambda, \mathcal{D})\right]_{12} y &= A^{-1}\,\mathcal{L}\left(\tilde{R}(\cdot)y\right)(\lambda)\\ &= \mathcal{L}\left(Q(\cdot)Dy\right)(\lambda) = R(\lambda, A)QR(\lambda, D)Dy\\ &= A^{-1}\left[R(\lambda, \mathcal{A})\right]_{12} y.\end{aligned}$$

Since $D(D^2)$ is dense in F this completes the proof. $\square$

Next we study some special cases in which $\mathcal{A}$ is a generator if and only if A and D are generators.

Corollary 3.4. *Let $\mathcal{A}$ satisfy the assumptions of Theorem 3.3. If one of the following conditions (a)–(c) is satisfied then $\mathcal{A}$ is a generator on $\mathcal{E}$ if and only if A and D are generators on E and F, respectively.*
(a) $A^2Q \in \mathcal{L}(F, E)$.
(b) $QD : D(D) \to E$ has a bounded extension $\overline{QD} \in \mathcal{L}(F, E)$ and $A\overline{QD} \in \mathcal{L}(F, E)$.
(c) $QD^2 : D(D^2) \to E$ has a bounded extension in $\mathcal{L}(F, E)$.

Proof. In view of Theorem 3.3 we only have to show that each of the conditions (a)–(c) implies that there exists a constant $M \geq 0$ such that

$$\|\tilde{R}(t)y\| \leq M \cdot \|y\| \tag{3.6}$$

for all $y \in D(D^2)$ and $t \in (0, 1]$.
(a) If $A^2Q \in \mathcal{L}(F, E)$ we obtain using integration by parts

$$\tilde{R}(t)y = AQS(t)y - T(t)AQy + \int_0^t T(t-r)A^2QS(r)y\,dr,$$

which implies (3.6).
(b) In case $B := A\overline{QD} \in \mathcal{L}(F, E)$ we have

$$\tilde{R}(t)y = \int_0^t T(t-r)BS(r)y\,dr,$$

and again (3.6) follows easily.
(c) In this case (3.6) follows immediately from (3.5). $\square$

Example 3.5. If A or D is bounded then $\mathcal{A}$ is a generator if and only if D or A, respectively, is a generator.

Remark 3.6. Let $\mathcal{A}$ be a triangular matrix with diagonal domain, i.e.,

$$\mathcal{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(D), \tag{3.7}$$

where $B \in \mathcal{L}([D(D)], E)$ and define

$$\tilde{R}(t)y := \int_0^t T(t-r)BS(s)y\,dr \tag{3.8}$$

for $y \in D(D^2)$. Then it is not difficult to show that Theorem 3.3 is also valid for $\mathcal{A}$ given by (3.7) if $\tilde{R}(t)$ is defined by (3.8), even if A and D are not invertible. Moreover, from Corollary 3.4 combined with the closed graph theorem we obtain that in this case $\mathcal{A}$ is a generator on $\mathcal{E}$ if and only if A and D are generators on E and F, respectively, provided one of the assumptions
(a)' $B \in \mathcal{L}([D(D)], [D(A)])$,
(c)' $B' \in \mathcal{L}([D(A')], [D(D')])$
is satisfied.

Using this remark we will reconsider the operator matrix $\mathcal{A}$ related to Volterra integro-differential equations. First we study the case where the Dirac measure defines a bounded linear functional.

Example 3.7. (Volterra Equation) If we apply the previous remark to the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ C(\cdot) & \frac{d}{ds} \end{pmatrix}, \quad D(\mathcal{A}) := D(A) \times D(\tfrac{d}{ds})$$

on the space $\mathcal{E} = E \times \mathrm{F}(\mathbb{R}_+, E)$ as considered in Examples 1.4 and 2.7 we obtain the following result due to Chen and Grimmer, cf. [CG80, Thm.4.1].

Proposition 3.8. *If A generates a semigroup on E, $\delta_0 \in \mathcal{L}(\mathrm{F}(\mathbb{R}_+, E), E)$ and C can be written as $C = C_1 A + C_2$ where $C_1 \in \mathcal{L}\left(E, [D(\frac{d}{ds})]\right)$ and $C_2 \in \mathcal{L}(E, \mathrm{F}(\mathbb{R}_+, E))$ then $\mathcal{A}$ is a generator on $\mathcal{E}$.*

Proof. By the bounded perturbation theorem $\mathcal{A}$ is a generator if and only if

$$\tilde{\mathcal{A}} := \begin{pmatrix} A & 0 \\ C_1 A & \frac{d}{ds} \end{pmatrix}, \quad D(\tilde{\mathcal{A}}) := D(A) \times D(\tfrac{d}{ds})$$

is a generator. The assertion then follows from Remark 1.2 and Remark 3.6 (a)'. □

Next we study the case where C is bounded and δ_0 is unbounded.

Example 3.9. (Volterra Equation) In the previous example we already gave conditions which ensure that the operator matrix $\mathcal{A}$ associated to the Volterra integro-differential equation considered in 1.4 generates a semigroup on $\mathcal{E}$. Here we consider the case where C is bounded. More precisely, we can show the following.

Proposition 3.10. *Let A be the generator of a semigroup $(T(t))_{t\geq 0}$ on the Banach space E and $C \in \mathcal{L}(E, \mathrm{L}^p(\mathbb{R}_+, E))$, $1 \leq p < \infty$. Then the operator matrix $\mathcal{A}$ defined on $\mathcal{E} := E \times \mathrm{L}^p(\mathbb{R}_+, E)$ by*

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ C & \frac{d}{ds} \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times \mathrm{W}^{1,p}(\mathbb{R}_+, E)$$

generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$.

Proof. It suffices to show that $\overline{\lim}_{t\downarrow 0} \|\tilde{R}(t)\| < \infty$. If $(S(t))_{t\geq 0}$ denotes the left-shift semigroup on $\mathrm{L}^p(\mathbb{R}_+, E)$ and $g \in D(\frac{d}{ds})$ then we have

$$\tilde{R}(t)g := \int_0^t T(t-r)\delta_0 S(r)g(\cdot)\, dr = \int_0^t T(t-r)g(r)\, dr.$$

However, since there exists $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \in [0,1]$ we obtain for $g \in D(\frac{d}{ds})$ and $t \in [0,1]$ the estimates

$$\begin{aligned} \|\tilde{R}(t)g\| &\leq M \int_0^t \|g(r)\|\, dr \\ &\leq M t^{\frac{1}{q}} \left(\int_0^t \|g(r)\|^p\, dr \right)^{\frac{1}{p}} \\ &\leq M \cdot \|g\|, \end{aligned}$$

where we used Hölders inequality for $\frac{1}{p}+\frac{1}{q}=1$. This proves the desired estimate, hence the proof is complete. □

Next we will consider operator matrices

$$\mathcal{A}=\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Id & K \\ L & Id \end{pmatrix}$$

defined by (1.1) where $K \in \mathcal{L}(F,X)$ and KD has a bounded extension $\overline{KD} \in \mathcal{L}(F,E)$. As it turns out this case can, under some additional assumption, be reduced to the case studied previously. Clearly, also here Remark 1.2 applies, i.e., corresponding results hold true if $L \in \mathcal{L}(E,Y)$ and $\overline{LA} \in \mathcal{L}(E,F)$.

Theorem 3.11. *Let $\mathcal{A}$ be a one-sided K-coupled operator matrix. If $K \in \mathcal{L}(F,X)$ and KD has a bounded extension in $\mathcal{L}(F,E)$ then there exists a bounded matrix $\mathcal{C} \in \mathcal{L}(\mathcal{E})$ such that $\mathcal{A}$ is similar to $\tilde{\mathcal{A}}$ where*

$$\tilde{\mathcal{A}} := \begin{pmatrix} A+KDL & 0 \\ DL & D-DLK \end{pmatrix} + \mathcal{C}, \qquad D(\tilde{\mathcal{A}}) := D(A)\times D(D). \tag{3.9}$$

Proof. Using the facts that $\mathcal{A}$ is one-sided K-coupled and $K \in \mathcal{L}(F,X)$ it follows from Proposition 2.3 that $\mathcal{A}$ is given by

$$\mathcal{A}=\begin{pmatrix} Id & 0 \\ DLA^{-1} & Id-DLKD^{-1} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Id & K \\ 0 & Id \end{pmatrix}. \tag{3.10}$$

Let $\mathcal{T} := \begin{pmatrix} Id & -K \\ 0 & Id \end{pmatrix}$. Then $\mathcal{T}$ is invertible and using Lemma 2.2 we obtain

$$\begin{aligned}\mathcal{T}^{-1}\mathcal{A}\mathcal{T} &= \begin{pmatrix} Id+KDLA^{-1} & K(Id-DLKD^{-1}) \\ DLA^{-1} & Id-DLKD^{-1} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ &= \begin{pmatrix} A+KDL & KD-KDLK \\ DL & D-DLK \end{pmatrix}\end{aligned}$$

Since $KD-KDLK \subseteq KD(Id-LK)$ and $\overline{KD} \in \mathcal{L}(F,E)$, Formula (3.9) follows for $\mathcal{C} := \begin{pmatrix} 0 & \overline{KD}(Id-LK) \\ 0 & 0 \end{pmatrix}$. □

Clearly, we now can apply Remark 3.6 to the unbounded part of $\tilde{\mathcal{A}}$ in (3.9) in order to study the generator property of $\mathcal{A}$. This is especially easy if D is bounded. Note that in this case K is bounded by Hypothesis (H).

Corollary 3.12. *Let $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} Id & K \\ L & Id \end{pmatrix}$ be a one-sided K-coupled operator matrix with $D \in \mathcal{L}(F)$. Then the following assertions are equivalent.*

(a) $\mathcal{A}$ is a generator on $\mathcal{E}$.

(b) $(A+KDL, D(A))$ is a generator on E.

Proof. If $D \in \mathcal{L}(F)$ then by (3.9) $\mathcal{A}$ is, up to a bounded perturbation, similar to

$$\hat{\mathcal{A}} := \begin{pmatrix} A+KDL & 0 \\ DL & 0 \end{pmatrix}, \qquad D(\hat{\mathcal{A}}) = D(A) \times F.$$

Hence, by Remarks 1.2 and 3.6 (a)', $\mathcal{A}$ is a generator on $\mathcal{E}$ if and only if $(A+KDL, D(A))$ is a generator on E. □

We proceed with the discussion of the matrix associated to the delay equation.

Example 3.13. (Delay Equation) In 1.5 and 2.9 we studied the operator matrix $\mathcal{A}$ on $\mathcal{E} := E \times F = \mathrm{L}^p([-1,0], \mathbb{C}^n) \times \mathbb{C}^n$, $1 \leq p < \infty$ defined by

$$\mathcal{A} = \begin{pmatrix} A_0 + Id & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & \mathbb{1} \otimes Id \\ C & Id \end{pmatrix} - Id,$$

for $A_0 \subset \frac{d}{ds}$, $D(A_0) := \{f \in \mathrm{W}^{1,p}([-1,0], \mathbb{C}^n) : f(0) = 0\}$. In this situation we obtain from Corollary 3.12 the equivalence

$$\text{(3.11)} \qquad \mathcal{A} \quad \text{is a generator on } \mathcal{E} \quad \Longleftrightarrow \quad A_0 - (\mathbb{1} \otimes Id)C \quad \text{is a generator on } E.$$

Since A_0 is invertible we can write

$$A_0 - (\mathbb{1} \otimes Id)C = \big(Id - (\mathbb{1} \otimes Id)CA_0^{-1}\big)A_0,$$

where $(\mathbb{1} \otimes Id)CA_0^{-1} \in \mathcal{L}(E)$ and $\mathrm{Rg}\,\big((\mathbb{1} \otimes Id)CA_0^{-1}\big) \subseteq \mathbb{1} \otimes \mathbb{C}^n := \{\mathbb{1} \otimes x : x \in \mathbb{C}^n\}$ for $(\mathbb{1} \otimes x)(s) \equiv x$. Since in [Eng95] we verified that $\mathbb{1} \otimes \mathbb{C}^n$ is a subspace of E satisfying condition (Z) with respect to the nilpotent shift-semigroup generated by A_0 we conclude from [DS89, Thm.5] that

$$\big(Id - (\mathbb{1} \otimes Id)CA_0^{-1}\big)A_0$$

is a generator on E. Using (3.11) we finally obtain that $\mathcal{A}$ is a generator of a strongly continuous semigroup and therefore the delay equation (DE) in Section 2 is well posed.

In our next result we allow two-sided coupling, however we assume that K and L are bounded.

Theorem 3.14. *Let $\mathcal{A}$ be a defined by (1.1) where $K \in \mathcal{L}(F,E)$, $L \in \mathcal{L}(E,F)$ and KD has a bounded extension $\overline{KD} \in \mathcal{L}(F,E)$. If there exists $\lambda \in \rho(D)$ such that $Id + \overline{KD}R(\lambda, D)L$ is invertible, then there is a bounded matrix $\mathcal{C} \in \mathcal{L}(\mathcal{E})$ such that*

$$\text{(3.12)} \qquad \mathcal{A} \simeq \tilde{\mathcal{A}} := \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \mathcal{C}, \qquad D(\tilde{\mathcal{A}}) := D(A) \times D(D).$$

Proof. Let $\lambda \in \rho(D)$ such that $Id + \overline{KD}R(\lambda, D)L$ is invertible. Then by Lemma 2.1 $\left(\begin{smallmatrix} Id & K \\ L_\lambda & Id \end{smallmatrix}\right)$ is invertible where $L_\lambda = -DR(\lambda, D)L$. Hence,

$$\mathcal{S} := \begin{pmatrix} Id & K \\ L_\lambda & Id \end{pmatrix}^{-1} \begin{pmatrix} Id & 0 \\ -\lambda R(\lambda, D)L & Id \end{pmatrix}$$

is invertible with inverse

$$\mathcal{S}^{-1} = \begin{pmatrix} Id & K \\ L & Id + \lambda R(\lambda, D)LK \end{pmatrix}.$$

Then we obtain from

$$\mathcal{A} - \begin{pmatrix} 0 & 0 \\ 0 & \lambda Id \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} Id & K \\ L_\lambda & Id \end{pmatrix}$$

that

$$\begin{aligned} \mathcal{S}^{-1}\left(\mathcal{A} - \begin{pmatrix} 0 & 0 \\ 0 & \lambda Id \end{pmatrix}\right)\mathcal{S} &= \begin{pmatrix} Id & K \\ L & Id + \lambda R(\lambda, D)LK \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - \lambda \end{pmatrix} \begin{pmatrix} Id & 0 \\ -\lambda R(\lambda, D)L & Id \end{pmatrix} \\ &=: \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \tilde{\mathcal{C}}, \end{aligned}$$

where $\tilde{\mathcal{C}} \in \mathcal{L}(\mathcal{E})$. This proves (3.12). □

Note that if $R(\lambda, D)$ exists on some positive half line and converges to zero for $\lambda \to \infty$ then the operator $Id + \overline{KD}R(\lambda, D)L$ is invertible for λ sufficiently large. By the Hille-Yosida theorem this is in particular the case if D is a generator on F.
Since the generator property is invariant under similarity transformations and bounded perturbations the previous theorem allows us to treat the operator matrix of 1.6.

Example 3.15. (Wave Equation with Viscoelastic Damping) We continue the discussion of 1.6 and 2.8. First we show that KD has a bounded extension. Using the notation $K = \left(\begin{smallmatrix} 0 \\ K_2 \end{smallmatrix}\right)$ and (1.3) we obtain for $w \in D(D) = \{v \in \mathrm{H}^1([-r, 0], E_2) : v(0) = 0\}$ that

$$\begin{aligned} \alpha K_2 D\, w &= \int_{-1}^{0} k(s)\, \tfrac{d}{ds}\, w(s, \cdot)\, ds \\ &= \left[k(s)w(s, \cdot)\right]_{-r}^{0} - \int_{-1}^{0} \tfrac{d}{ds}\, k(s)w(s, \cdot)\, ds \\ &= -\int_{-1}^{0} \tfrac{d}{ds}\, k(s)w(s, \cdot)\, ds. \end{aligned}$$

By Hölders inequality this implies $\overline{KD} \in \mathcal{L}(F, E)$. Next we claim that A and D are generators on $E = E_1 \times E_2$ and F respectively. Since A is invertible and similar to the dissipative operator

$$\tilde{A} := \begin{pmatrix} 0 & \sqrt{\alpha}\frac{d}{dx} \\ \sqrt{\alpha}\frac{d}{dx} & 0 \end{pmatrix}, \quad D(\tilde{A}) = D(A)$$

this follows for A from the Lumer-Phillips theorem.

On the other hand it is well known that D is the generator of the nilpotent shift-semigroup $\big(S(t)\big)_{t\geq 0}$. Since L is bounded this in particular implies that

$$\lim_{\lambda\to\infty} \left\|\overline{KD}R(\lambda, D)L\right\| = 0$$

and therefore $Id+\overline{KD}R(\lambda, D)L$ is invertible for λ sufficiently large. Hence, we can apply the previous theorem and obtain that $\mathcal{A}$ is a generator if and only if

$$\hat{\mathcal{A}} := \begin{pmatrix} Id & 0 \\ L & Id \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

is a generator on $\mathcal{E} = E \times F$. Since $\mathrm{Rg}(L) = \mathbb{1} \otimes E_2 := \{\mathbb{1} \otimes x : x \in E_2\}$ for $(\mathbb{1} \otimes x)(s) \equiv x$, the latter, however, follows from [DS89, Thm.5] and the fact that $\mathbb{1} \otimes E_2$ satisfies condition (Z) with respect to the nilpotent shift-semigroup on E_2, cf. [Eng95]. Hence, $\mathcal{A}$ is the generator of a strongly continuous semigroup and therefore the system (PDE), (IC), (BC) in Example 1.6 is well posed.

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Klaus-Jochen Engel, Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Hiroshima 739, JAPAN

e-mail: engel@top2.math.sci.hiroshima-u.ac.jp