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ABSTRACT QUASILINEAR INTEGRODIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

Hirokazu Oka (岡 裕和)
Department of Mathematics, School of Education
Waseda University, Tokyo 169-50, Japan

Introduction

This is a joint work with Naoki Tanaka at Okayama University.

In this note we are concerned with the abstract quasilinear integrodifferential equations of hyperbolic type

$$\text{(QIE)} \qquad \begin{cases} u'(t) = A(t,u(t))u(t) + \int_0^t B(t,s,u(s))u(s)ds \\ u(0) = u_0 \end{cases}$$

in a pair of Banach spaces (Y, X) such that Y is continuously imbedded in X. Our main purpose is to study the problem of existence and uniqueness of local classical solutions to (QIE) without assuming that Y is dense in X, where by a *classical solution* u to (QIE) on $[0, T]$ we mean that $u \in C([0,T]:Y) \cap C^1([0,T]:X)$ and that u satisfies (QIE).

Our investigation of the problem (QIE) is motivated by the work of DA PRATO AND SINESTRARI [5] stated as follows : they studied the inhomogeneous abstract Cauchy problem

$$(\mathrm{ACP}; u_0, f) \qquad \begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = u_0 \end{cases}$$

for a closed linear operator A in X satisfying the Hille-Yosida condition with the exception of the density of the domain $D(A)$ of A

$$\text{(H–Y)} \qquad \begin{cases} \text{there exist } M \geq 1 \text{ and } \omega \geq 0 \text{ such that } (\omega, \infty) \subset \rho(A) \text{ and} \\ \|(\lambda - A)^{-n}\| \leq M(\lambda - \omega)^{-n} \text{ for all } \lambda > \omega \text{ and } n = 1, 2, \cdots, \end{cases}$$

and proved the following interesting result for $(\mathrm{ACP}; u_0, f)$.

Theorem 0. *Suppose that a closed linear operator A in X satisfies the Hille-Yosida condition* (H–Y) *and let $f \in W^{1,1}(0,T:X)$. Then the problem* $(\mathrm{ACP}; u_0, f)$ *has a unique classical solution $u \in C([0,T]:D(A)) \cap C^1([0,T]:X)$ if and only if $u_0 \in D(A)$ and the compatibility condition that $Au_0 + f(0) \in \overline{D(A)}$ holds.*

Remark. The "only if" part is easy to prove. In fact, let u be a unique classical solution to (ACP;u_0, f). Then we have $u(t) \in D(A)$ for $t \in [0,T]$ and $Au_0 + f(0) = u'(0) = \lim_{h\downarrow 0} h^{-1}(u(h) - u(0)) \in \overline{D(A)}$.

We shall show an advantage of Theorem 0 by giving a concrete example.

Example 1. Let $C[0,1]$ be the Banach space of continuous functions on the closed interval $[0,1]$ and $f \in W^{1,1}(0,T : C[0,1])$. Consider the following partial differential equation with periodic boundary condition :

$$\text{(P)} \qquad \begin{cases} u_t(t,x) + u_x(t,x) = f(t,x), & (t,x) \in [0,T] \times [0,1], \\ u(0,x) = u_0(x), & x \in [0,1], \\ u(t,0) = u(t,1), & t \in [0,T]. \end{cases}$$

We will solve the problem (P) by two different methods. One is the way to solve by using Theorem 0 (the case (A)) and the other is the (C_0)-semigroup theory (the case (B)).

(A) By Theorem 0 :

Let $X = C[0,1]$. Define an operator A in X by

$$\begin{cases} D(A) = \{u \in C^1[0,1] : u(0) = u(1)\} \\ (Au)(x) = -u'(x) \quad \text{for } x \in [0,1]. \end{cases}$$

Then A is a closed linear operator satisfying that $(0,\infty) \subset \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| \leq 1$ for $\lambda > 0$ (see e.g. [6]). Theorem 0 asserts that if $u_0 \in C^1[0,1]$ satisfying $u_0(0) = u_0(1)$ and if the compatibility condition that $-u_0'(0) + f(0,0) = -u_0(1) + f(0,1)$ holds, then there exists a unique classical solution u to the problem (P).

(B) By the (C_0)-semigroup theory :

Let $X_0 := \{u \in C[0,1] : u(0) = u(1)\}$ and define an operator A_0 in X_0 by

$$\begin{cases} D(A_0) = \{u \in C^1[0,1] : u(0) = u(1), u'(0) = u'(1)\} \\ (A_0u)(x) = -u'(x) \quad \text{for } x \in [0,1]. \end{cases}$$

Then A_0 generates a (C_0)-semigroup on X_0. Therefore if $u_0 \in C^1[0,1]$ satisfies $u_0(0) = u_0(1)$ and $u_0'(0) = u_0'(1)$ and if $f(t,0) = f(t,1)$ for all $t \in [0,T]$, then the problem (P) has a unique classical solution.

This example shows that the condition imposed on the initial value u_0 and the inhomogeneous term f in the case of (A) is weaker than that in the case of (B). However if $f \equiv 0$, then both (A) and (B) give the same solvability of the problem (P).

Next we turn to the integrodifferential equation.

Example 2. Let $f \in W^{1,1}(0, T : C[0,1])$. Consider the integrodifferential equation :

$$\begin{cases} u_t(t,x) + u_x(t,x) = \int_0^t b(t,s,x)u_x(s,x)ds, & (t,x) \in [0,T] \times [0,1], \\ u(0,x) = u_0(x), \quad x \in [0,1], \\ u(t,0) = u(t,1), \quad t \in [0,T]. \end{cases}$$

Let X and A be as in Example 1. For each $(t,s) \in \Delta := \{(t,s) : 0 \le s \le t \le T\}$ we define an operator $B(t,s)$ in X by

$$\begin{cases} D(B(t,s)) = D(A) \\ (B(t,s)u)(x) = b(t,s,x)u'(x) \quad \text{for } x \in [0,1]. \end{cases}$$

In the case of (A) we make only the regularity assumption of the function $b(t,s,x)$ with respect to $(t,s) \in \Delta$, while in the case of (B) the condition that $b(t,s,\cdot)u'(\cdot) \in X_0$ for $u \in D(A_0)$ must be satisfied, namely an additional assumption that $b(t,s,0) = b(t,s,1)$ for all $(t,s) \in \Delta$ is required.

This is the reason why we study the integrodifferential equation of the form

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t,s)u(s)ds \\ u(0) = u_0 \end{cases}$$

for a non-densely defined closed linear operator A in X satisfying the Hille-Yosida condition (H–Y). We refer the reader to [22] for some results for this problem.

The quasilinear integrodifferential equation (QIE) will be solved in the following manner : we consider the linearized equation

$$(\mathrm{LIE}^v) \qquad \begin{cases} u'(t) = A(t,v(t))u(t) + \int_0^t B(t,s,v(s))u(s)ds \\ u(0) = u_0 \end{cases}$$

for a function v belonging to some function space. If this problem (LIE^v) has a unique solution u for given v, then it defines a mapping $v \mapsto u$. The fixed points of this mapping are classical solutions to (QIE).

To solve the problem (QIE), the theory of linear integrodifferential equations

(LIE) $$\begin{cases} u'(t) = A(t)u(t) + \int_0^t B(t,s)u(s)ds \\ u(0) = u_0 \end{cases}$$

needs to be developed and it will be done in Section 2. The idea for solving (LIE) is to regard the integral term of (LIE) as an inhomogeneous term of the linear evolution equation

(LE;u_0, f) $$\begin{cases} u'(t) = A(t)u(t) + f(t) \\ u(0) = u_0 \end{cases}$$

and to find the fixed point of the mapping defined in the usual way, by using the estimates of solutions to problem (LE;u_0, f), and is therefore based on the theory of linear evolution equations (LE;u_0, f) established in Section 1.

Our approach to linear evolution equations (LE;u_0, f) are different from [28]. Our main concern is to study the problem of existence and uniqueness of generalized solutions of (LE;u_0, f) which are well-known as DS-limit solutions in the nonlinear semigroup theory (see [15]) and to obtain the estimates of generalized solutions which is very important for our discussion later, but his paper is devoted to the construction of the evolution operator generated by a family $\{A(t) : t \in [0,T]\}$ of non-densely defined operators in X and the representation of solutions in terms of the variation of constants formula in a generalized sense.

Section 3 discusses the quasilinear integrodifferential equations (QIE). By the result obtained in Section 2 we shall construct approximate solutions $\{u_n\}$ of problem (QIE) inductively by defining u_n to be the unique solution of (LIE$^{u_{n-1}}$) and $u_0(t) = u_0$. The convergence of $\{u_n\}$ in $C([0,T] : X)$ will be first proved by using the estimate (see (3.6)) of solutions to integrodifferential equations adding the forcing term f to (LIE^v). By this fact we next show that the limits $\hat{A}(t) := \lim_{n\to\infty} A(t, u_n(t))$ and $\hat{B}(t,s) := \lim_{n\to\infty} B(t,s,u_n(s))$ exist in $L(Y,X)$, and then by Theorem 1.1 and Corollary 1.4, given $v \in C([0,T] : Y)$ we find a unique generalized solution $w := w^v$ to the problem

$$\begin{cases} w'(t) = \hat{A}(t)w(t) + \partial\hat{A}(t)v(t) - \lambda_0(Hv)(t) + (d/dt)(Hv)(t) \\ w(0) = (\hat{A}(0) - \lambda_0)u_0, \end{cases}$$

where $\partial\hat{A}(t)$ is the derivative of $\hat{A}(t)$, $(Hv)(t) := \int_0^t \hat{B}(t,s)v(s)ds$ and $\lambda_0 \in \rho(\hat{A}(t))$. If the mapping $(\Phi v)(t) := (\hat{A}(t) - \lambda_0)^{-1}(w^v(t) - (Hv)(t))$ has a unique fixed point v, then u_n

converges to v in $C([0,T]:Y)$ as $n \to \infty$, since the v satisfies the relation $(\hat{A}(t) - \lambda_0)v(t) + \int_0^t \hat{B}(t,s)v(s)ds = w^v(t)$. In the proof of this claim, the estimate (1.5) of generalized solutions to problem (LE;u_0, f) plays a crucial role again. Finally, we shall give an application of our abstract theory to a quasilinear hyperbolic system of integrodifferential equations from viscoelasticity.

1 Linear Evolution Equations

In this section we study linear evolution equations in a Banach space X with norm $\|\cdot\|$

$$\text{(LE;}u_0, f) \qquad \begin{cases} u'(t) = A(t)u(t) + f(t), \ t \in [0,T] \\ u(0) = u_0. \end{cases}$$

We shall denote by (LE;A, u_0, f) the problem (LE;u_0, f) in the case where one needs to indicate $\{A(t) : t \in [0,T]\}$. Let Y be another Banach space with norm $\|\cdot\|_Y$ which is continuously imbedded in X. We impose the following three conditions on a family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X.

(A_1) $D(A(t)) = Y$ is independent of $t \in [0,T]$.

(A_2) There are constants $M \geq 1$ and $\omega \geq 0$ such that

$$(\omega, \infty) \subset \rho(A(t)) \ \text{ for } t \in [0,T]$$

and

$$\left\| \prod_{j=1}^{k} (\lambda I - A(t_j))^{-1} \right\| \leq M(\lambda - \omega)^{-k} \ \text{ for } \lambda > \omega \tag{1.1}$$

and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and $k = 1, 2, \cdots$.

We write $\{A(t) : t \in [0,T]\} \in S_\sharp(X, M, \omega)$ for such family $\{A(t) : t \in [0,T]\}$.

We obtain a fundamental theorem for linear evolution equations (LE;u_0, f).

Theorem 1.1. *Let $f \in L^1(0,T:X)$ and $u_0 \in \overline{Y}$ (the closure of Y in X). Suppose that a family $\{A(t) : t \in [0,T]\}$ of closed linear operators in X satisfies* (A_1), (A_2) *and*

(A_3) *the map $t \mapsto A(t)$ is continuous and of bounded variation in the $L(Y,X)$ norm.*

Moreover assume that there exists a partition $\Delta_n = \{0 = t_0^n < t_1^n < \cdots < t_{N_n}^n \equiv T_n \leq T\}$ and sequences $\{x_k^n\}$ and $\{z_k^n\}$ in X which satisfy the following :

(i) $\lim_{n\to\infty}|\Delta_n| = \lim_{n\to\infty}(T - T_n) = 0$,

where $|\Delta_n| = \max_{1\le k\le N_n} h_k^n,\ h_k^n = t_k^n - t_{k-1}^n$ *and* $|\Delta_n|\omega < \frac{1}{2}$,

(ii) $\dfrac{x_k^n - x_{k-1}^n}{h_k^n} = A(t_k^n)x_k^n + z_k^n,\ x_0^n = u_0, \quad k = 1, 2, \cdots, N_n,\ n \ge 1$,

(iii) $\lim_{n\to\infty}\|f^n - f\|_{L^1(0,T_n:X)} = 0$, *where* $f^n(t) \equiv z_k^n$ *on* $(t_{k-1}^n, t_k^n],\ k = 1, 2, \cdots, N_n$.

Then there exists a function $u \in C([0,T] : X)$ *such that*

$$\lim_{n\to\infty}\sup_{t\in[0,T_n]}\|u^n(t) - u(t)\| = 0,$$

where

$$u^n(t) = \begin{cases} x_k^n, & t \in (t_{k-1}^n, t_k^n], \quad k = 1, 2, \cdots, N_n, \\ u_0, & t = 0. \end{cases}$$

The following is the key lemma to prove Theorem 1.1.

Lemma 1.2([28, Lemma 1.1]). *Assume that a family* $\{A(t) : t \in [0,T]\}$ *of closed linear operators in* X *satisfies* (A_2). *For each* $t \in [0,T]$ *we define another norm* $\|\cdot\|_t$ *on* X *by*

$$\|z\|_t = \sup\left\{(\lambda-\omega)^m\left\|\prod_{k=1}^{m}(\lambda I - A(t_k))^{-1}z\right\| \;\middle|\; \begin{array}{l}\lambda > \omega \text{ and} \\ t \le t_1 \le \cdots \le t_m \le T, m \ge 0\end{array}\right\}$$

for $z \in X$. *Then we have :*

$$\|z\| \le \|z\|_t \le \|z\|_s \le M\|z\| \quad (z \in X; 0 \le s \le t \le T), \tag{1.2}$$

$$\|(\lambda I - A(t))^{-1}z\|_t \le (\lambda-\omega)^{-1}\|z\|_t \quad (z \in X; \lambda > \omega; t \in [0,T]). \tag{1.3}$$

This lemma asserts the existence of norms $\|\cdot\|_t$ with respect to which the operator $A(t)$ is quasi-dissipative for each $t \in [0,T]$. Theorem 1.1 can be proved by applying the well-known technique in the theory of nonlinear evolution operators.

Remark 1.1. The existence of a partition Δ_n and two sequences $\{x_k^n\}$ and $\{z_k^n\}$ in X satisfying (i) through (iii) was shown in [7, Lemma 4.1].

Definition 1.1. The limit function $u \in C([0,T] : X)$ obtained in Theorem 1.1 is called a *generalized solution* of (LE;u_0, f).

We shall list some estimates of generalized solutions to (LE;u_0, f) which will play a crucial role in studying quasilinear integrodifferential equations (QIE).

Theorem 1.3. *Let* $u_0 \in \overline{Y}, \hat{u}_0 \in Y, f \in L^1(0,T:X)$ *and* $\hat{f} \in BV([0,T]:X)$. *Suppose* $\{A(t): t \in [0,T]\}$ *and* $\{\hat{A}(t): t \in [0,T]\}$ *satisfy* (A_1), (A_2) *and* (A_3). *If* u *and* $\hat{u}$ *are generalized solutions of* (LE;A, u_0, f) *and* (LE;$\hat{A}, \hat{u}_0, \hat{f}$) *respectively, then we have*

$$\begin{aligned} \|u(t) - \hat{u}(t)\| \ \le \ & Mexp(2\omega T)\Big(\|u_0 - \hat{u}_0\| + C(\hat{A}, \hat{u}_0, \hat{f}) \int_0^t \|A(s) - \hat{A}(s)\|_{Y,X} ds \\ & + \int_0^t \|f(s) - \hat{f}(s)\| ds\Big) \end{aligned} \quad (1.4)$$

for $t \in [0,T]$, *where* $C(\hat{A}, \hat{u}_0, \hat{f})$ *is a constant depending on* $\{\hat{A}(t)\}, \hat{u}_0$ *and* $\hat{f}$.

Corollary 1.4. *Suppose that* $\{A(t): t \in [0,T]\}$ *satisfies* (A_1), (A_2) *and* (A_3). *Let* $u_0, \hat{u}_0 \in \overline{Y}$ *and* $f, \hat{f} \in L^1(0,T:X)$. *If* u *and* $\hat{u}$ *are generalized solutions of* (LE;u_0, f) *and* (LE;$\hat{u}_0, \hat{f}$) *respectively, then we have*

$$\|u(t) - \hat{u}(t)\| \le Mexp(2\omega T)\left(\|u_0 - \hat{u}_0\| + \int_0^t \|f(s) - \hat{f}(s)\| ds\right) \quad (1.5)$$

for $t \in [0,T]$.

Corollary 1.5. *Suppose that* $\{A(t): t \in [0,T]\}$ *satisfies* (A_1), (A_2) *and* (A_3). *Let* $u_0 \in \overline{Y}$ *and* $f \in L^1(0,T:X)$. *Then the generalized solution* u *of* (LE;u_0, f) *satisfies the estimate*

$$\|u(t) - u_0\| \le \{Mexp(2\omega T) + 1\}\|u_0 - y\| + Mexp(2\omega T) \int_0^t \|f(s) + A(s)y\| ds \quad (1.6)$$

for $t \in [0,T]$ *and* $y \in Y$.

Definition 1.2. Suppose that $\{A(t): t \in [0,T]\}$ satisfies (A_1), (A_2) and (A_3). Let $\{C(t)\}$ be a family of nonlinear continuous operators from X into itself defined for a.e. $t \in [0,T]$ satisfying the condition

(c1) $C(\cdot)x \in L^1(0,T:X)$ for $x \in X$.

Then a function $u \in C([0,T]:X)$ is called a *generalized solution* of the initial-value problem

(LE;u_0)per $\quad \begin{cases} u'(t) = A(t)u(t) + C(t)u(t), \ t \in [0,T] \\ u(0) = u_0 \end{cases}$

if u is a generalized solution of (LE;$u_0, C(\cdot)u(\cdot)$).

The next proposition will be proved by using Theorem 1.1, Corollary 1.4 and Banach's fixed point theorem.

Proposition 1.6. *Suppose that a family* $\{A(t) : t \in [0,T]\}$ *satisfies* (A_1), (A_2) *and* (A_3), *and that a family* $\{C(t)\}$ *of nonlinear continuous operators from* X *into itself satisfies* (c1) *and*

(c2) *there is a function* $\phi \in L^1(0,T)$ *such that*

$$\|C(t)x - C(t)y\| \leq \phi(t)\|x-y\| \quad \text{for } x, y \in X \text{ and a.e. } t \in [0,T]. \tag{1.7}$$

If $u_0 \in \overline{Y}$, *then there exists a unique generalized solution of* $(\mathrm{LE};u_0)_{\mathrm{per}}$.

We turn to the problem of existence and uniqueness of classical solutions to $(\mathrm{LE};u_0,f)$.

Theorem 1.7. *Let* $f \in W^{1,1}(0,T:X)$. *Suppose that* $\{A(t) : t \in [0,T]\}$ *satisfies* (A_1), (A_2) *and*

(A_4) $A(\cdot)y \in C^1([0,T]:X)$ *for* $y \in Y$.

If $u_0 \in Y$ *satisfies the compatibility condition that* $A(0)u_0 + f(0) \in \overline{Y}$, *then there exists a unique classical solution* $u \in C([0,T]:Y) \cap C^1([0,T]:X)$ *to the problem* $(\mathrm{LE};u_0,f)$.

For later use we prepare some estimates of the classical solution to $(\mathrm{LE};u_0,f)$.

Theorem 1.8. *Suppose that the assumptions of Theorem 1.7 are satisfied. The classical solution* u *of* $(\mathrm{LE};u_0,f)$ *satisfies the following estimates :*

$$\begin{aligned}\|(A(t)-\lambda_0)u(t)+f(t)\| \;\leq\; & Mexp(2\omega T)\Big(\|(A(0)-\lambda_0)u_0+f(0)\| \\ & + \int_0^t \|\dot{A}(s)u(s) - \lambda_0 f(s) + \dot{f}(s)\|ds\Big);\end{aligned} \tag{1.8}$$

$$\begin{aligned}\|u(t)-u_0\|_Y \;\leq\; & c_1\{Mexp(2\omega T)+1\}\|(A(0)-\lambda_0)u_0+f(0)-y\| \\ & + c_1 Mexp(2\omega T)\Big(\int_0^t \|\dot{A}(s)u(s) - \lambda_0 f(s) + \dot{f}(s) + A(s)y\|ds\Big) \\ & + c_1\Big(\int_0^t \|\dot{f}(s) + \dot{A}(s)u_0\|ds\Big)\end{aligned} \tag{1.9}$$

for $y \in Y$ *and* $t \in [0,T]$, *where* $c_1 := \sup_{t\in[0,T]} \|(A(t)-\lambda_0)^{-1}\|_{Y,X}$.

2 Linear Integrodifferential Equations

In this section we state the result (see [23]) on linear integrodifferential equations

$$\text{(LIE)} \qquad \begin{cases} u'(t) = A(t)u(t) + \int_0^t B(t,s)u(s)ds + f(t), \quad t \in [0,T] \\ u(0) = u_0. \end{cases}$$

Here $\{A(t) : t \in [0,T]\}$ is a given family of closed linear operators satisfying conditions (A_1),(A_2) and (A_4), and $\{B(t,s) : (t,s) \in \Delta\}$ where $\Delta = \{(t,s) : 0 \le s \le t \le T\}$ is a family in $L(Y,X)$ satisfying the following two conditions.

(B_1) For $y \in Y$, $B(t,s)y$ is continuous on Δ, differentiable with respect to t and $(\partial/\partial t)B(t,s)y$ is continuous on Δ.

(B_2) For $y \in Y$, $B(t,s)y$ is differentiable with respect to s and $(\partial/\partial s)B(t,s)y$ is continuous on Δ.

Theorem 2.1. *Let $f \in W^{1,1}(0,T : X)$ and suppose that $u_0 \in Y$ satisfies the compatibility condition that $A(0)u_0 + f(0) \in \overline{Y}$. Then the problem* (LIE) *has a unique classical solution $u \in C([0,T] : Y) \cap C^1([0,T] : X)$ satisfying*

$$\|u(t)\| \le K\left(\|u_0\| + \int_0^t \|f(s)\|ds\right) \tag{2.1}$$

for $t \in [0,T]$, where K is a constant depending on M, ω and T.

3 Quasilinear Integrodifferential Equations

This section is devoted to the study of quasilinear integrodifferential equations

$$\text{(QIE)} \qquad \begin{cases} u'(t) = A(t,u(t))u(t) + \int_0^t B(t,s,u(s))u(s)ds \\ u(0) = u_0. \end{cases}$$

We make the following hypotheses on the operators $A(t,w)$ appearing in (QIE).

There are a bounded open subset W of Y and a real number $T_0 > 0$ such that $A(t,w)$ is a closed linear operator in X defined for each $(t,w) \in [0,T_0] \times W$, and that the following conditions are satisfied:

(a_1) $D(A(t,w)) = Y$ for $(t,w) \in [0,T_0] \times W$;

(a_2) for each $\rho > 0$ there are constants $M_\rho \geq 1$ and $\omega_\rho \geq 0$ such that

$$\{A(t, v(t)) : t \in [0, T_0]\} \in S_\sharp(X, M_\rho, \omega_\rho)$$

for every $v \in D_\rho$. Here the set D_ρ is defined by

$$D_\rho = \{v \in C([0, T_0] : W) : \|v(t) - v(s)\| \leq \rho|t - s| \text{ for } t, s \in [0, T_0]\} \text{ for } \rho > 0;$$

(a_3) there is a function $F : [0, T_0] \times W \times X \to L(Y, X)$ satisfying two conditions (f_1) and (f_2) below such that if $v \in C([0, T_0] : W) \cap C^1([0, T_0] : X)$ and $y \in Y$, then $A(t, v(t))y$ is differentiable and

$$(d/dt)A(t, v(t))y = F(t, v(t), v'(t))y \text{ for } t \in [0, T_0];$$

(f_1) for $w \in W, p \in X$ and $y \in Y, F(\cdot, w, p)y$ is continuous on $[0, T_0]$;

(f_2) for each $\rho > 0$, there are a constant $\mu_{F,\rho} > 0$ and a nondecreasing function $\sigma_{F,\rho}(\cdot)$ on $[0, \infty)$ with the property that $\lim_{\delta \downarrow 0} \sigma_{F,\rho}(\delta) = 0$ such that

$$\|F(t, w_1, v_1) - F(t, w_2, v_2)\|_{Y,X} \leq \sigma_{F,\rho}(\|w_1 - w_2\|) + \mu_{F,\rho}\|v_1 - v_2\|$$

for $t \in [0, T_0]$, $w_1, w_2 \in W$ and $v_1, v_2 \in B_X(\rho) = \{x \in X : \|x\| \leq \rho\}$;

(a_4) there is a constant $\mu_A > 0$ such that

$$\|A(t, w_1) - A(t, w_2)\|_{Y,X} \leq \mu_A\|w_1 - w_2\| \text{ for } t \in [0, T_0] \text{ and } w_1, w_2 \in W.$$

We also impose the following on a family $\{B(t, s, w) : (t, s) \in \Delta_0, w \in W\}$ in $L(Y, X)$, where $\Delta_0 = \{(t, s) : 0 \leq s \leq t \leq T_0\}$.

(b_1) For $y \in Y$ and $w \in W, B(t, s, w)y$ is continuous on Δ_0, differentiable with respect to t, and $(\partial/\partial t)B(t, s, w)y$ is continuous on Δ_0;

(b_2) there exist constants $\mu_B > 0$ and $\mu'_B > 0$ such that

$$\|B(t, s, w_1) - B(t, s, w_2)\|_{Y,X} \leq \mu_B\|w_1 - w_2\|;$$

$$\|(\partial/\partial t)B(t, s, w_1) - (\partial/\partial t)B(t, s, w_2)\|_{Y,X} \leq \mu'_B\|w_1 - w_2\|$$

for $(t, s) \in \Delta_0$ and $w_1, w_2 \in W$;

(b_3) there is a function $G : \Delta_0 \times W \times X \to L(Y, X)$ satisfying two conditions (g_1) and (g_2) below such that if $v \in C([0, T_0] : W) \cap C^1([0, T_0] : X)$ and $y \in Y, B(t, s, v(s))y$ is differentiable with respect to s and

$$(\partial/\partial s)B(t, s, v(s))y = G(t, s, v(s), v'(s))y \ \text{ for } (t, s) \in \Delta_0;$$

(g_1) $G : \Delta_0 \times W \times X \to L(Y, X)$ is strongly continuous;

(g_2) for $\rho > 0$ there exists a constant $\lambda_{G,\rho} > 0$ such that

$$\|G(t, s, w, p)\|_{Y,X} \leq \lambda_{G,\rho} \text{ for } (t, s, w, p) \in \Delta_0 \times W \times B_X(\rho).$$

Remark 3.1.

(a_5) Condition (a_3) implies that for each $w \in W$, $A(\cdot, w)$ is continuous in the $L(Y, X)$ norm on $[0, T_0]$. This fact, the boundedness of W in Y and condition (a_4) immediately show an existence of $\lambda_A > 0$ satisfying

$$\|A(t, w)\|_{Y,X} \leq \lambda_A \ \text{ for } (t, w) \in [0, T_0] \times W. \tag{3.1}$$

(f_3) By (f_1) and (f_2), for each $\rho > 0$ there is a constant $\lambda_{F,\rho} > 0$ such that

$$\|F(t, w, p)\|_{Y,X} \leq \lambda_{F,\rho} \ \text{ for } (t, w, p) \in [0, T_0] \times W \times B_X(\rho). \tag{3.2}$$

(b_4) Since W is bounded in Y, conditions (b_1) and (b_2) imply that there exist constants $\lambda_B > 0$ and $\lambda'_B > 0$ such that

$$\|B(t, s, w)\|_{Y,X} \leq \lambda_B, \tag{3.3}$$

$$\|(\partial/\partial t)B(t, s, w)\|_{Y,X} \leq \lambda'_B \tag{3.4}$$

for $(t, s, w) \in \Delta_0 \times W$.

Our main result is stated as follows.

Theorem 3.1. *If $u_0 \in W$ satisfies the compatibility condition that $A(0, u_0)u_0 \in \overline{Y}$, then there is a $T \in (0, T_0]$ such that the quasilinear integrodifferential equation* (QIE) *has a unique classical solution u on $[0, T]$.*

Proof of Theorem 3.1.

We shall only state the outline of the proof. See [24] for the details.

Since W is open in Y, for any initial value $u_0 \in W$ of (QIE) satisfying the compatibility condition that $A(0, u_0)u_0 \in \overline{Y}$, we can choose an $r_0 > 0$ so that

$$B_Y(u_0, r_0) := \{w \in Y : \|w - u_0\|_Y \le r_0\} \subset W$$

and then we put

$$\rho_0 = (\lambda_A + \lambda_B T_0)(\|u_0\|_Y + r_0). \tag{3.5}$$

For $\tau \in (0, T_0]$ let E_τ be the set of functions v satisfying

$$\begin{cases} v \in C([0,\tau] : Y) \cap C^1([0,\tau] : X), v(t) \in B_Y(u_0, r_0) \text{ for all } t \in [0,\tau], \\ v(0) = u_0 \text{ and } \|v'(t)\| \le \rho_0 \text{ for all } t \in [0,\tau]. \end{cases}$$

For each $v \in E_\tau$, we write for simplicity

$$A^v(t) = A(t, v(t)) \text{ for } t \in [0,\tau], \text{ and}$$

$$B^v(t,s) = B(t,s,v(s)) \text{ for } (t,s) \in \Delta_\tau := \{(t,s) : 0 \le s \le t \le \tau\}.$$

From conditions (a_1) through (a_4) and (b_1) through (b_3), we obtain the following result for the linearized equation (LIE^v) for $v \in E_\tau$.

Proposition 3.2. *For any $u_0 \in W$ satisfying $A(0, u_0)u_0 \in \overline{Y}$ and $v \in E_\tau$, the linear integrodifferential equation*

$$\text{(LIE}^v) \qquad \begin{cases} u'(t) = A^v(t)u(t) + \int_0^t B^v(t,s)u(s)ds, \quad t \in [0,\tau] \\ u(0) = u_0 \end{cases}$$

has a unique classical solution $u \in C([0,\tau] : Y) \cap C^1([0,\tau] : X)$.

Proposition 3.2 enables us to define a map $\Phi : E_\tau \to C([0,\tau] : Y)$ by $\Phi v = u$.

Then there is a $\tau \in (0, T_0]$ such that $\Phi E_\tau \subset E_\tau$. The claim that $(\Phi v)(t) \in B_Y(u_0, r_0)$ for all $v \in E_\tau$ and $t \in [0,\tau]$ can be proved by using the estimate (see (1.9)) of the classical solution to the problem (LE;$u_0, H^v(\Phi v)$) for $v \in E_\tau$, where for $v \in E_\tau$ we define an operator $H^v : C([0,\tau] : Y) \cap C^1([0,\tau] : X) \to C^1([0,\tau] : X)$ by $(H^v w)(t) := \int_0^t B^v(t,s)w(s)ds$.

In what follows, let $\tau \in (0, T_0]$ be an arbitrary but fixed positive number satisfying $\Phi(E_\tau) \subset E_\tau$. We make E_τ into a metric space by the distance function

$$d(v,w) := \sup_{t \in [0,\tau]} \|v(t) - w(t)\|$$

for $v, w \in E_\tau$.

An application of Theorem 2.1 (cf. [23, Theorem 2.3]) gives the next result.

Proposition 3.3. *Let $v \in E_\tau$, $x \in X$ and $f \in L^1(0, \tau : X)$. Suppose that the problem*

$$\begin{cases} u'(t) = A^v(t)u(t) + \int_0^t B^v(t,s)u(s)ds + f(t), \quad t \in [0,\tau] \\ u(0) = x \end{cases}$$

has a classical solution u^v. Then we have

$$\|u^v(t)\| \le C\left(\|x\| + \int_0^t \|f(s)\|ds\right) \tag{3.6}$$

for $t \in [0, \tau]$, where C is a constant independent of x, f and $v \in E_\tau$.

By (3.6) we obtain

Lemma 3.4. *We have*

$$d(\Phi^n v, \Phi^n w) \le \frac{(C\rho_0 T_0)^n}{n!} d(v,w) \quad \textit{for } v, w \in E_\tau \textit{ and } n = 1, 2, \cdots. \tag{3.7}$$

We now define a sequence $\{u_n\}$ in E_τ by

$$u_0(t) = u_0 \text{ for } t \in [0,\tau] \text{ and } u_n = \Phi u_{n-1} \text{ for } n = 1, 2, \cdots. \tag{3.8}$$

As a direct consequence of Lemma 3.4, we have

Corollary 3.5. *The sequence $\{u_n(t)\}$ converges in X uniformly on $[0, \tau]$.*

For brevity in notation, we write

$$A_n(t) = A(t, u_n(t)),\ S_n(t) = A(t, u_n(t)) - \lambda_0 I \ \text{ for } t \in [0,\tau], \text{ and}$$
$$B_n(t,s) = B(t,s,u_n(s)) \ \text{ for } (t,s) \in \Delta_\tau.$$

Corollary 3.5 and condition (a_4) together imply that

$$\lim_{n\to\infty} A_n(t) := \hat{A}(t) \text{ in } L(Y,X) \ \text{ and}$$
$$\lim_{n\to\infty} S_n(t)^{-1} := Q(t) \text{ in } L(X,Y)$$

exist uniformly in $[0, \tau]$, and then we see that $\hat{A}(\cdot) \in C([0,\tau] : L(Y,X))$ with $\|\hat{A}(t)\|_{Y,X} \le \lambda_A$ and that $Q(\cdot) \in C([0,\tau] : L(X,Y))$. Putting $\hat{S}(t) = \hat{A}(t) - \lambda_0 I$, we have $\hat{S}(t)Q(t) = I$ on

X and $Q(t)\hat{S}(t) = I$ on Y ; hence $\hat{S}(\cdot)^{-1} \in C([0,\tau] : L(X,Y))$. Condition (f_2) shows $\|F(t,u_n(t),p) - F(t,u_m(t),p)\|_{Y,X} \le \sigma_{F,\rho_0}(d(u_n,u_m))$ for $t \in [0,\tau]$ and $p \in B_X(\rho_0)$, which enables us to define $\hat{F}(\cdot,\cdot) : [0,\tau] \times B_X(\rho_0) \to L(Y,X)$ by

$$\hat{F}(t,p) = \lim_{n\to\infty} F(t,u_n(t),p)$$

for $t \in [0,\tau]$ and $p \in B_X(\rho_0)$. Here the convergence in the $L(Y,X)$ norm is uniform for $(t,p) \in [0,\tau] \times B_X(\rho_0)$. We then see that the function $\hat{F}(\cdot,\cdot)$ has the following properties (f_4) and (f_5) which immediately follow from (f_1) together with (3.2) and (f_2):

(f_4) If $p \in C([0,T] : X)$ for some $T \in (0,\tau]$ and $p(t) \in B_X(\rho_0)$ for $t \in [0,T]$, then $\hat{F}(\cdot,p(\cdot)) \in C([0,T] : L(Y,X))$ and $\|\hat{F}(t,p(t))\|_{Y,X} \le \lambda_{F,\rho_0}$ for $t \in [0,T]$;

(f_5) $\|\hat{F}(t,p_1) - \hat{F}(t,p_2)\|_{Y,X} \le \mu_{F,\rho_0}\|p_1 - p_2\|$ for $t \in [0,\tau]$ and $p_1,p_2 \in B_X(\rho_0)$.

Also by Corollary 3.5 and condition (b_2) we have

$$\lim_{n\to\infty} B_n(t,s) := \hat{B}(t,s) \ \text{ in } L(Y,X);$$
$$\lim_{n\to\infty} (\partial/\partial t)B_n(t,s) := \partial\hat{B}(t,s) \ \text{ in } L(Y,X)$$

uniformly on Δ_τ. It is obvious that both $\hat{B}(t,s)y$ and $\partial\hat{B}(t,s)y$ are continuous on Δ_τ in X for $y \in Y$, and so $(\partial/\partial t)\hat{B}(t,s)y = \partial\hat{B}(t,s)y$ for $y \in Y$ and $(t,s) \in \Delta_\tau$. Moreover we obtain $\|\hat{B}(t,s)\|_{Y,X} \le \lambda_B$ and $\|\partial\hat{B}(t,s)\|_{Y,X} \le \lambda'_B$ for $(t,s) \in \Delta_\tau$.

Let $T \in (0,\tau]$ and $E_Y = C([0,T] : B_Y(u_0,r_0))$. E_Y is a complete metric space by the distance function

$$d_Y(v,w) := \sup_{t\in[0,T]} \|v(t) - w(t)\|_Y$$

for $v,w \in E_Y$. Define two operators $D,H : E_Y \to C([0,T] : X)$ by

$$(Dv)(t) = \hat{A}(t)v(t) + \int_0^t \hat{B}(t,s)v(s)ds;$$
$$(Hv)(t) = \int_0^t \hat{B}(t,s)v(s)ds$$

respectively. For $v \in E_Y$ we have $\hat{F}(\cdot,(Dv)(\cdot)) \in C([0,T] : L(Y,X))$ (note that $\|(Dv)(t)\| \le (\lambda_A + \lambda_B T_0)\|v(t)\|_Y \le \rho_0$) and $Hv \in C^1([0,T] : X)$. Since the family $\{\hat{A}(t) : t \in [0,T]\}$

satisfies (A_1), (A_2) and (A_3) in Theorem 1.1 with $\{A(t)\}$ replaced by $\{\hat{A}(t)\}$, Theorem 1.1 and Corollary 1.4 assert that for $v \in E_Y$ there exists a unique generalized solution $u^v(\cdot) \in C([0,T]:X)$ to the problem

$$(\text{LE};\hat{A}, u_1, (Wv)(\cdot)) \qquad \begin{cases} u'(t) = \hat{A}(t)u(t) + (Wv)(t) \\ u(0) = u_1 := (A(0,u_0) - \lambda_0)u_0, \end{cases}$$

where $(Wv)(t) := \hat{F}(t,(Dv)(t))v(t) - \lambda_0(Hv)(t) + (d/dt)(Hv)(t)$ for $v \in E_Y$.

Define an operator $\Psi : E_Y \to C([0,T]:Y)$ by

$$(\Psi v)(t) = \hat{S}(t)^{-1}(u^v(t) - (Hv)(t)).$$

Then there is a $T \in (0,\tau]$ such that $\Psi(E_Y) \subset E_Y$. This assertion can be proved by using the estimate (see (1.6)) of the generalized solution to the problem (LE;$\hat{A}, u_1, Wv$) for $v \in E_Y$.

In what follows we fix $T \in (0,\tau]$ so that $\Psi(E_Y) \subset E_Y$.

The use of the estimate (see (1.5)) of the difference between generalized solutions to (LE;u_1, Wv) and (LE;u_1, Ww) for $v, w \in E_Y$ gives the following.

Lemma 3.6. *There is a unique fixed point $\overline{u} \in E_Y$ of Ψ.*

For any $\varepsilon > 0$ take $u_1^\varepsilon \in Y$ and a function $\hat{f}^\varepsilon \in C^1([0,T]:X)$ such that $\|u_1 - u_1^\varepsilon\| < \varepsilon$ and $\|(W\overline{u})(\cdot) - \hat{f}^\varepsilon(\cdot)\|_{L^1(0,T:X)} < \varepsilon$. We then use the estimate (see (1.4)) of the difference between the generalized solution to (LE;$A_{n-1}, u_1, W_{n-1}u_n$) and the generalized solution to (LE;$\hat{A}, u_1^\varepsilon, \hat{f}^\varepsilon$) to find constants $C_1, C_2(\varepsilon)$(depending on ε), $C_3 > 0$ and a null sequence $\{\delta_n\}$ such that

$$\|u_n(t) - \overline{u}(t)\|_Y \le C_1\varepsilon + C_2(\varepsilon)\delta_n + C_3\int_0^t (\|u_n(s) - \overline{u}(s)\|_Y + \|u_{n-1}(s) - \overline{u}(s)\|_Y)ds,$$

where $(W_n v)(t) := F(t, u_n(t), u_n'(t))v(t) - \lambda_0(H_n v)(t) + (d/dt)(H_n v)(t)$ and $(H_n v)(t) := \int_0^t B_n(t,s)v(s)ds$ for $v \in E_Y$. Then by standard arguments we have

Lemma 3.7. $\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - \overline{u}(t)\|_Y = 0$.

The End of Proof of Theorem 3.1.

Since $u_n'(t) = A(t, u_{n-1}(t))u_n(t) + \int_0^t B(t,s,u_{n-1}(s))u_n(s)ds$ converges to $A(t,\overline{u}(t))\overline{u}(t) + \int_0^t B(t,s,\overline{u}(s))\overline{u}(s)ds$ uniformly in $[0,T]$ by Lemma 3.7, we conclude that $\overline{u}$ is a classical solution of (QIE).

To prove the uniqueness of classical solutions of (QIE), let $u_i (i = 1, 2)$ be classical solutions of (QIE) and set $w = u_1 - u_2$. Then, an easy computation yields

$$\begin{aligned} w'(t) &= A(t, u_1(t))w(t) + \{A(t, u_1(t)) - A(t, u_2(t))\}u_2(t) \\ &\quad + \int_0^t B(t, s, u_1(s))w(s)ds + \int_0^t \{B(t, s, u_1(s)) - B(t, s, u_2(s))\}u_2(s)ds. \end{aligned}$$

By (3.6) we have

$$\begin{aligned} \|w(t)\| &\le C \int_0^t \Big[\|A(s, u_1(s)) - A(s, u_2(s))\|_{Y,X} \|u_2(s)\|_Y \\ &\quad + \int_0^s \|B(s, r, u_1(r)) - B(s, r, u_2(r))\|_{Y,X} \|u_2(r)\|_Y dr \Big] ds \\ &\le C\rho_0 \int_0^t \|w(s)\| ds, \end{aligned}$$

and Gronwall's inequality therefore asserts $u_1 = u_2$. $\square$

4 An Application

We shall give an application of our results obtained in the previous section to a quasilinear hyperbolic system of integrodifferential equations from viscoelasticity :

(QHS)

$$\begin{cases} \partial_t v_1(t,x) = \partial_x v_2(t,x) \quad \text{for } (t,x) \in [0,T] \times [0,1] \\ \partial_t v_2(t,x) = a(t,x,v_1(t,x),v_2(t,x)))\partial_x v_1(t,x) + \int_0^t b(t,s,x,v_1(s,x),v_2(s,x))\partial_x v_1(s,x)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in [0,T] \times [0,1] \\ v_1(t,0) = v_1(t,1), \quad v_2(t,0) = v_2(t,1) \quad \text{for } t \in [0,T] \\ v_1(0,x) = \varphi_1(x), \quad v_2(0,x) = \varphi_2(x) \quad \text{for } x \in [0,1], \end{cases}$$

where the function $a(t, \xi_0, \xi_1, \xi_2)$ is of class C^1 with the property that $a \ge a_0$ (> 0) on $[0, T_0] \times [0,1] \times \mathbf{R} \times \mathbf{R}$ and the function $b(t, s, \xi_0, \xi_1, \xi_2)$ defined on $\Delta_0 \times [0,1] \times \mathbf{R} \times \mathbf{R}$ is of class C^1.

The (QHS) can be rewritten as follows :

$$\begin{aligned} \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix}_t &= \begin{pmatrix} 0 & 1 \\ a(t,x,v_1(t,x),v_2(t,x)) & 0 \end{pmatrix} \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix}_x \\ &\quad + \int_0^t \begin{pmatrix} 0 & 0 \\ b(t,s,x,v_1(s,x),v_2(s,x)) & 0 \end{pmatrix} \begin{pmatrix} v_1(s,x) \\ v_2(s,x) \end{pmatrix}_x ds. \end{aligned}$$

Let $X = C[0,1] \times C[0,1]$ where $C[0,1]$ is the Banach space of all continuous functions on $[0,1]$ with maximum norm $\|\cdot\|_{C[0,1]}$. The space X equipped with norm $\|\cdot\|$ defined by

$\|v\| = \|v_1\|_{C[0,1]} \vee \|v_2\|_{C[0,1]}$ for $v = \binom{v_1}{v_2} \in X$ is a Banach space. As another Banach space which is continuously imbedded in X we take

$$\begin{cases} Y = \left\{ w = \binom{w_1}{w_2} \in C^1[0,1] \times C^1[0,1] : w_1(0) = w_1(1),\ w_2(0) = w_2(1) \right\}, \\ \|w\|_Y = \|w_1\|_{C^1[0,1]} \vee \|w_2\|_{C^1[0,1]} \quad \text{for } w = \binom{w_1}{w_2} \in Y, \end{cases}$$

where $\|w_i\|_{C^1[0,1]} = \|w_i\|_{C[0,1]} + \|w_i'\|_{C[0,1]}$.
Let $\varphi = \binom{\varphi_1}{\varphi_2} \in Y$. Take an $R > 0$ such that $\|\varphi\|_Y < R$ and set $W = \{w \in Y : \|w\|_Y < R\}$.

We now define $P(t,w) \in L(X)$ for $t \in [0, T_0]$ and $w = \binom{w_1}{w_2} \in W$ by

$$(P(t,w)v)(x) = \mathcal{P}(t,w)(x)v(x) \quad \text{for } v \in X,$$

where $\mathcal{P}(t,w)(x) = \dfrac{\sqrt{1 + a(t,x,w_1(x),w_2(x))}}{2\sqrt{a(t,x,w_1(x),w_2(x))}} \begin{pmatrix} \sqrt{a(t,x,w_1(x),w_2(x))} & 1 \\ -\sqrt{a(t,x,w_1(x),w_2(x))} & 1 \end{pmatrix}$
for $(t,w) \in [0,T_0] \times W$, and then introduce another norm $\|\cdot\|_{(t,w)}$ on X depending upon $(t,w) \in [0,T_0] \times W$ by

$$\|v\|_{(t,w)} = \|P(t,w)v\| \quad \text{for } v \in X.$$

Define a family $\{A(t,w) : (t,w) \in [0,T_0] \times W\}$ of closed linear operators in X and a family $\{B(t,s,w) : (t,s,w) \in \Delta_0 \times W\}$ in $L(Y,X)$ by

$$(A(t,w)v)(x) = \begin{pmatrix} 0 & 1 \\ a(t,x,w_1(x),w_2(x)) & 0 \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}_x \quad \text{for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(A(t,w)) = Y;$$
$$(B(t,s,w)v)(x) = \begin{pmatrix} 0 & 0 \\ b(t,s,x,w_1(x),w_2(x)) & 0 \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}_x \text{ for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(B(t,s,w)) = Y.$$

Then the norm $\|\cdot\|_{(t,w)}$ is equivalent to the original norm $\|\cdot\|$ on X and there is a positive constant ω depending on $R > 0$ such that $A(t,w) \in G_\sharp(X,1,\omega)$ with respect to the norm $\|\cdot\|_{(t,w)}$ (see [28, Lemma 3.5]). This fact implies (a_2). It is easy to see that all the other conditions in Theorem 3.1 are satisfied with

$$(F(t,w,p)v)(x) = \begin{pmatrix} 0 & 1 \\ f(t,w,p)(x) & 0 \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}_x \quad \text{for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in Y;$$
$$(G(t,s,w,p)v)(x) = \begin{pmatrix} 0 & 0 \\ g(t,s,w,p)(x) & 0 \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}_x \text{ for } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in Y,$$

where

$$f(t,w,p)(x) \;:=\; (\partial/\partial t)a(t,x,w_1(x),w_2(x)) + p_1(x)(\partial/\partial\xi_1)a(t,x,w_1(x),w_2(x))$$

$$+p_2(x)(\partial/\partial\xi_2)a(t,x,w_1(x),w_2(x))$$

and

$$\begin{aligned} g(t,s,w,p)(x) \;:=\;& (\partial/\partial s)b(t,s,x,w_1(x),w_2(x)) + p_1(x)(\partial/\partial\xi_1)b(t,s,x,w_1(x),w_2(x)) \\ & +p_2(x)(\partial/\partial\xi_2)b(t,s,x,w_1(x),w_2(x)) \end{aligned}$$

for $w = \binom{w_1}{w_2} \in Y$ and $p = \binom{p_1}{p_2} \in X$.

It is shown that if $\varphi_1 \in C^1[0,1], \varphi_1(0) = \varphi_1(1), \varphi_2 \in C^1[0,1], \varphi_2(0) = \varphi_2(1), \varphi_2'(0) = \varphi_2'(1)$ and $a(0,0,\varphi_1(0),\varphi_2(0))\varphi_1'(0) = a(0,1,\varphi_1(1),\varphi_2(1))\varphi_1'(1)$, there exists a $T \in (0,T_0]$ such that the problem (QHS) has a unique classical solution $v = \binom{v_1}{v_2} \in C^1([0,T] : C[0,1]) \times C^1([0,T] : C[0,1])$.

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