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Nonlinear $m$-sectorial operators and time-dependent Ginzburg-Landau equations

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§0. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. We consider the following problem:

\begin{align*}
(1) \quad & \frac{\partial \Phi}{\partial t} - (\lambda + i\alpha)\Delta \Phi + (\kappa + i\beta)|\Phi|^{p-1}\Phi - \gamma \Phi = 0, \\
& \frac{\partial \Phi}{\partial \nu}(x, t) = 0 \quad (x \in \partial \Omega, \ t \geq 0), \\
& \Phi(x, 0) = \Phi_0(x). 
\end{align*}

Here $\lambda > 0$, $\kappa > 0$, $p > 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constants; $\nu$ is unit outward normal of $\partial \Omega$, $i = \sqrt{-1}$ and $\Phi$ is $\mathbb{C}$-valued. (1) is called the time-dependent Ginzburg-Landau equation when $p = 3$ (see Temam [5]). Introducing the new unknown $u = e^{-\gamma t}\Phi(= v + iw)$, the problem (1) is written as

\begin{align*}
(2) \quad & \frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)e^{(p-1)\gamma t}|u|^{p-1}u = 0, \\
& \frac{\partial u}{\partial \nu}(x, t) = 0 \quad (x \in \partial \Omega, \ t \geq 0), \\
& u(x, 0) = u_0(x) \ (= \Phi_0(x)).
\end{align*}

For the mathematical setting we introduce complex Hilbert space $X = L^2(\Omega; \mathbb{C})$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and define operators $A$, $B$ and time-dependent operators $A(t)$, $B(t)$ as follows:

\begin{align*}
D(A) :&= \{ u \in H^2(\Omega; \mathbb{C}); \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \}, \\
Au :&= -\Delta v - i\Delta w \quad \text{for} \quad u = v + iw \in D(A), \\
D(B) :&= \{ u \in X; \ |u|^{p-1}u \in X \},
\end{align*}
\( Bu := |u|^{p-1}u \) for \( u \in D(B) \),
\( B(t)u := e^{(p-1)\gamma t}Bu \) for \( u \in D(B(t)) := D(B) \), \( t \in [0, T] \),
\( D := D(A) \cap D(B) \),
\( A(t)u := (\lambda + i\alpha)Au + (\kappa + i\beta)B(t)u \) for \( u \in D(A(t)) := D \), \( t \in [0, T] \),
\( D := D(A) \cap D(B) \),
where \( H^2(\Omega; \mathbb{C}) \) is the usual Sobolev Hilbert space and \( T > 0 \) is arbitrary.

Then the problem (2) is regarded as one of initial value problems for standard abstract evolution equations of the form

\[
\frac{du}{dt} + A(t)u = 0, \quad t \in [0, T],
\]
\( u(0) = u_0. \)

To solve (3) we can apply the theory of nonlinear evolution equations developed by Kato [2]. In fact, under some conditions for \( \lambda, \kappa, p, \alpha, \beta, \gamma \) we can show that \( A(t) \) \((t \in [0, T])\) is \( m \)-accretive in \( X \) (see Lemma 10) and \( A(\cdot) \) satisfies the smoothness condition:

\[
\|A(t)u - A(s)u\| \leq C(T)|t - s|(1 + \|u\| + \|A(s)u\|), \quad \text{for } t, s \in [0, T], \quad u \in D,
\]
(see Lemma 11).

**§1. The main theorem and its corollary**

we obtain the following theorem.

**Theorem.** Let \( \lambda > 0, \kappa > 0, p > 1, \; \frac{\beta}{\kappa} \leq \frac{2 \sqrt{p}}{p - 1}, \; \kappa \alpha + \kappa \beta > |\lambda \beta - \alpha \kappa| \). Then for any \( \Phi_0 \in D \), there exists a unique global strong solution \( \Phi = \Phi(x, t), \; (x, t) \in \Omega \times [0, \infty) \) to the problem (1) in \( X \).

Put \( \alpha = \beta = 0 \) in the problem (1). Then we have

**Corollary.** If \( \lambda > 0, \kappa > 0, p > 1 \), then for any \( \Phi_0 \in D \) the problem

\[
\frac{\partial \Phi}{\partial t} - \lambda \Delta \Phi + \kappa |\Phi|^{p-1} \Phi - \gamma \Phi = 0, \quad (x, t) \in \Omega \times [0, \infty),
\]
\[
\frac{\partial \Phi}{\partial \nu}(x, t) = 0, \quad (x, t) \in \partial \Omega \times [0, \infty),
\]
\[
\Phi(x, 0) = \Phi_0(x), \quad x \in \Omega.
\]
has a unique global strong solution \( \Phi = \Phi(x, t) \).

**§2. Proof of theorem**

In this section we shall prove our main Theorem. The proof needs some lemmas. Throughout this section, we assume that \( \lambda > 0, \kappa > 0, p > 1 \). It is well-known that the operator \( A \) is a nonnegative selfadjoint operator in \( X \). So we can easily obtain
Lemma 1. \((\lambda + i\alpha)A\) is \(m\)-accretive in \(X\).

In the next Lemma 2 which implies that \(B\) is a nonlinear sectorial operator, the constant \(\frac{p-1}{2\sqrt{p}}\) is recently determined by [3].

Lemma 2([3]). For any \(u_1, u_2 \in D(B)\) we have

\[
|\text{Im}(Bu_1 - Bu_2, u_1 - u_2)| \leq \frac{p-1}{2\sqrt{p}} \text{Re}(Bu_1 - Bu_2, u_1 - u_2).
\]

Since the operator \(B\) is sectorial like this, the accretiveness of \(B\) is preserved under a little rotation. So we can obtain

Lemma 3. Let \(\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}\). Then \((\kappa + i\beta)B\) is accretive in \(X\) (We can replace \(B\) by \(B(t)\)).

Proof. Let \(u_1, u_2 \in D(B)\). Then it follows from Lemma 2 that

\[
\text{Re}((\kappa + i\beta)(Bu_1 - Bu_2), u_1 - u_2) \\
\geq \kappa \text{Re}(Bu_1 - Bu_2, u_1 - u_2) - |\beta| |\text{Im}(Bu_1 - Bu_2, u_1 - u_2)| \\
\geq \kappa \left( \frac{2\sqrt{p}}{p-1} - \frac{|\beta|}{\kappa} \right) |\text{Im}(Bu_1 - Bu_2, u_1 - u_2)| \geq 0. \quad \Box
\]

Let \(f \in X\) then for almost every \(x \in \Omega\) the equation

\[
z + |z|^{p-1}z = f(x) \quad \text{in} \ C
\]

has a unique solution \(z = u(x)\) such that \(|u(x)| \leq |f(x)|\). Therefore \(u \in D(B)\) and we obtain the following lemma.

Lemma 4. \(B\) is \(m\)-accretive in \(X\) (We can replace \(B\) by \(B(t)\)).

For every \(\varepsilon > 0\) we set

\[
J_\varepsilon = (I + \varepsilon B)^{-1}, \quad B_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon).
\]
Lemma 5. Let \( \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1} \). Then \((\kappa+i\beta)B_{\epsilon}\) is accretive in \(X\).

Proof. Let \(v_1, v_2 \in X\). Then it follows from Lemma 3 that

\[
\text{Re}\left((\kappa+i\beta)(B_{\epsilon}v_1 - B_{\epsilon}v_2), v_1 - v_2\right) = \text{Re}\left((\kappa+i\beta)(B_{\epsilon}v_1 - B_{\epsilon}v_2), J_{\epsilon}v_1 - J_{\epsilon}v_2\right) + \text{Re}\left((\kappa+i\beta)(B_{\epsilon}v_1 - B_{\epsilon}v_2), (v_1 - J_{\epsilon}v_1) - (v_2 - J_{\epsilon}v_2)\right) + \text{Re}\left((\kappa+i\beta)(B_{\epsilon}v_1 - B_{\epsilon}v_2), \varepsilon(B_{\epsilon}v_1 - B_{\epsilon}v_2)\right) \\
\geq \varepsilon \kappa \|B_{\epsilon}v_1 - B_{\epsilon}v_2\|^2 \geq 0. \quad \square
\]

Lemma 6. \(C^1(\Omega; \mathbb{C}) \cap X\) is invariant under \((I + \varepsilon B)^{-1}\).

Proof. For any \(f = g + ih \in C^1(\Omega; \mathbb{C}) \cap X\) we know that the equation

\[
u_{\epsilon}(x) + \varepsilon |u_{\epsilon}(x)|^{p-1} u_{\epsilon}(x) = f(x)
\]

has a unique solution \(u_{\epsilon}(x) = v_{\epsilon}(x) + iw_{\epsilon}(x) \in D(B)\). It remains to show that \(u_{\epsilon}(x) \in C^1(\Omega; \mathbb{C})\). This equation is rewritten in the form:

\[
\begin{cases}
v_{\epsilon}(x) + \varepsilon (v_{\epsilon}(x)^2 + w_{\epsilon}(x)^2)^{\frac{p-1}{2}} v_{\epsilon}(x) = g(x), \\
w_{\epsilon}(x) + \varepsilon (v_{\epsilon}(x)^2 + w_{\epsilon}(x)^2)^{\frac{p-1}{2}} w_{\epsilon}(x) = h(x).
\end{cases}
\]

Put

\[
F(x, v_{\epsilon}, w_{\epsilon}) := v_{\epsilon} + \varepsilon (v_{\epsilon}^2 + w_{\epsilon}^2)^{\frac{p-1}{2}} v_{\epsilon} - g(x),
\]

\[
G(x, v_{\epsilon}, w_{\epsilon}) := w_{\epsilon} + \varepsilon (v_{\epsilon}^2 + w_{\epsilon}^2)^{\frac{p-1}{2}} w_{\epsilon} - h(x).
\]

Then

\[
\frac{\partial(F, G)}{\partial(v_{\epsilon}, w_{\epsilon})} := \begin{vmatrix}
\frac{\partial F}{\partial v_{\epsilon}} & \frac{\partial F}{\partial w_{\epsilon}} \\
\frac{\partial G}{\partial v_{\epsilon}} & \frac{\partial G}{\partial w_{\epsilon}}
\end{vmatrix} = \left\{1 + \varepsilon (v_{\epsilon}^2 + w_{\epsilon}^2)^{\frac{p-1}{2}}\right\}^2 + \left\{1 + \varepsilon (v_{\epsilon}^2 + w_{\epsilon}^2)^{\frac{p-1}{2}}\right\} \\
\times \varepsilon (p-1)(v_{\epsilon}^2 + w_{\epsilon}^2)^{\frac{p-3}{2}} (v_{\epsilon}^2 + w_{\epsilon}^2) \geq 1.
\]

Therefore we can apply the implicit function theorem. \(\square\)
Lemma 7. \( \text{Re}(Au, B_\varepsilon u) \geq 0 \) for \( u \in D(A) \).

Proof. Put
\[
\tilde{D}(A) := \{ f = g + ih \in C^2(\Omega; \mathbb{C}) \cap H^2(\Omega; \mathbb{C}); \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial \Omega \}.
\]

It suffices to show our lemma for \( f = g + ih \in \tilde{D}(A) \). We set
\[
v_\varepsilon + iw_\varepsilon = (I + \varepsilon B)^{-1}(g + ih).
\]

Then we have
\[
\frac{\partial v_\varepsilon}{\partial x_j} = \frac{1}{q} \{ (1 + aw_\varepsilon^2 + b) \frac{\partial g}{\partial x_j} - aw_\varepsilon v_\varepsilon \frac{\partial h}{\partial x_j} \},
\]
\[
\frac{\partial w_\varepsilon}{\partial x_j} = \frac{1}{q} \{ -av_\varepsilon w_\varepsilon \frac{\partial g}{\partial x_j} + (1 + av_\varepsilon^2 + b) \frac{\partial h}{\partial x_j} \},
\]
where
\[
a = \varepsilon(p-1)(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-3}{2}}, \ b = \varepsilon(v_\varepsilon^2 + w_\varepsilon^2)^{\frac{p-1}{2}}, \ q = (b+1)^2 + a(b+1)(v_\varepsilon^2 + w_\varepsilon^2).
\]

It follows from this relation that
\[
\text{Re}(Af, B_\varepsilon f) \geq \frac{1}{\varepsilon} \int_\Omega \frac{1}{q} \{ b^2 + b + ab(v_\varepsilon^2 + w_\varepsilon^2) \}(|\nabla g|^2 + |\nabla h|^2) + a(v_\varepsilon |\nabla g| - w_\varepsilon |\nabla h|)^2 \] dx 
\[
\geq 0. \quad \square
\]
Remark 8. Since $X$ is a (complex) Hilbert space in our case, $B_\varepsilon u$ converges $Bu$ ($u \in D(B)$) in $X$ as $\varepsilon \downarrow 0$. Therefore we also have from Lemma 7 that

$$\text{Re}(Au, Bu) \geq 0 \quad \text{for} \quad u \in D = D(A) \cap D(B).$$

Now we shall prove that the operator

$$A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$$

is $m$-accretive for every $t \in [0, T]$. Following the idea of T. Kato (see Brezis, Crandall and Pazy [1]), for every $f \in X$ we consider the approximate equations:

(5) \hspace{1cm} Au_\varepsilon + \frac{\kappa + i\beta}{\lambda + i\alpha}B_\varepsilon u_\varepsilon + u_\varepsilon = f, \quad \varepsilon > 0.

Since $(\lambda + i\alpha)A + (\kappa + i\beta)B_\varepsilon$ is $m$-accretive in $X$ (for $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$), (5) has a unique solution $u_\varepsilon \in D(A)$.

Lemma 9. Let $u_\varepsilon$ be the solution of (5). If $\lambda \kappa + \alpha \beta > 0$, then $\|B_\varepsilon u_\varepsilon\|$ is bounded for any $\varepsilon > 0$.

Proof. It follows from (5) that

$$\text{Re}(Au_\varepsilon, B_\varepsilon u_\varepsilon) + \frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} \|B_\varepsilon u_\varepsilon\|^2 + \text{Re}(u_\varepsilon, B_\varepsilon u_\varepsilon) = \text{Re}(f, B_\varepsilon u_\varepsilon)$$

Noting that $B_0 = 0$ and

$$\text{Re}(B_\varepsilon u_\varepsilon, u_\varepsilon) = \text{Re}(B(J_\varepsilon u_\varepsilon) - B_0, J_\varepsilon u_\varepsilon - 0) + \text{Re}(B_\varepsilon u_\varepsilon, \varepsilon B_\varepsilon u_\varepsilon) \geq \varepsilon \|B_\varepsilon u_\varepsilon\|^2 \geq 0,$$

we have from lemma 7 that

$$\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} \|B_\varepsilon u_\varepsilon\|^2 \leq \|f\| \|B_\varepsilon u_\varepsilon\|. \quad \square$$

In (5), now it is routine work to prove that there exists a unique $u \in D$ such that

$$u_\varepsilon \rightarrow u \text{ strongly in } X, \quad Au_\varepsilon \rightarrow Au \text{ weakly in } X, \quad Bu_\varepsilon \rightarrow Bu \text{ weakly in } X$$

as $\varepsilon \downarrow 0$. Hence we obtain
Lemma 10. Let $\frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1}$ and $\lambda \kappa + \alpha \beta > 0$. Then $(\lambda + i\alpha)A + (\kappa + i\beta)B$ is $m$-accretive in $X$. The same is true for $A(t) = (\lambda + i\alpha)A + (\kappa + i\beta)B(t)$ for every $t \in [0, T]$.

Lemma 11. Let $\lambda \kappa + \alpha \beta > |\lambda \beta - \alpha \kappa|$. Then there exists a constant $C = C(T) > 0$ such that

$$\|A(t)u - A(s)u\| \leq C|t - s||A(s)u| \quad \text{for } t, s \in [0, T], \ u \in D.$$ 

Proof. Let $t, s \in [0, T]$ and $u \in D$. By the mean value theorem there exists a $C_1(T) > 0$ such that

$$\|A(t)u - A(s)u\| = \|(\kappa + i\beta)(e^{(p-1)\gamma t} - e^{(p-1)\gamma s})Bu\| \leq C_1(T)|t - s||B(s)u||.$$ 

On the other hand we know from Remark 8 that

$$\text{Re}(Au, Bu) \geq 0 \quad \text{for} \quad u \in D.$$ 

From this inequality we see that

$$\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} B(b)u \leq \text{Re}(Au, B(b)u) + \|\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} B(b)u\|^2 \leq \|Au + \frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} B(b)u\| \cdot \|\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} B(b)u\|,$$

and hence

$$\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} \|B(b)u\| \leq \|Au + \frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} B(b)u\| \leq \|Au + \frac{\kappa + i\beta}{\lambda + i\alpha} B(b)u\| + \frac{|\lambda \beta - \alpha \kappa|}{\lambda^2 + \alpha^2} \|B(b)u\|.$$

So we have

$$\|B(b)u\| \leq \frac{|\lambda - i\alpha|}{(\lambda \kappa + \alpha \beta) - |\lambda \beta - \alpha \kappa|} \|(\lambda + i\alpha)Au + (\kappa + i\beta)B(b)u\|.$$ 

Thus we obtain

$$\|A(t)u - A(s)u\| \leq \frac{C_1(T)|\lambda - i\alpha|}{(\lambda \kappa + \alpha \beta) - |\lambda \beta - \alpha \kappa|}|t - s||A(s)u||. \quad \square$$

Now we are in a position to prove our Theorem.
Proof of Theorem (completed). Since the domain $D$ of $A(t)$ is independent of $t \in [0, T]$ and $A(t)$ is $m$-accretive in $X$, by Lemma 11 we can apply Kato's Theorem ([2]). Noting that $T > 0$ is arbitrary, the solution $\Phi(x, t)$ to (1) exists for $(x, t) \in \Omega \times [0, \infty)$.

Added after the conference. In our Theorem, we can weaken the assumption. Namely our Theorem is still true when the condition $\lambda \kappa + \alpha \beta > |\lambda \beta - \alpha \kappa|$ is replaced by $\lambda \kappa + \alpha \beta > 0$ (see [6]).

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