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Regularizing effects for a class of Hamilton-Jacobi equations

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The first author talked on the subject of the title. The following is the résumé of the talk.

1 Introduction

We show that there are three regularizing effects: Lipschitz regularization, (local) semi-concavity regularization effect, and $C_{loc}^{1,1}$ regularizing effect for the following class of time-dependent Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + y \cdot \nabla_x u + H(\nabla_y u) = 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

(1)

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

(2)

where $u_0$ is a bounded, continuous function in $\mathbb{R}^N \times \mathbb{R}^N$; $\cdot$ stands for the scalar product in $\mathbb{R}^N \times \mathbb{R}^N$; $H(\cdot)$ is a continuous function from $\mathbb{R}^N$ to $\mathbb{R}$, satisfying the following assumptions.

$H(\cdot)$ is convex, nonnegative, $H(0) = 0$,

(H)

$$\lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty, \quad \text{as} \quad |p| \to \infty.$$

Our study is based on the viscosity solution theory introduced by M.G.Crandall and P.-L.Lions in [2]. We say that there is a Lipschitz regularizing effect of (1) when the solution $u(t, x, y)$ of (1), (2) is Lipschitz continuous in $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ for all $t > 0$, for an arbitrary initial conditions $u_0(x, y)$ assumed to be bounded and continuous.

We say that there is a local semi-concavity regularizing effect of (1), when for an arbitrary continuous initial condition $u_0(x, y)$, the solution $u(t, x, y)$ of (1)-(2) is locally
semi-concave in $(x, y) \in R^N \times R^N$, for all $t > 0$. That is for any $R > 0$, $t > 0$, there exists a number $C_{R,t} > 0$ such that

$$u(t, x, y) - C_{R,t}(|x|^2 + |y|^2)$$

is semi-concave in the ball $B_R(0,0) \in R^N \times R^N$. The local semi-concavity regularizing effect leads the $C^1_{loc}$ regularizing effect of (1), which means here that the solution $u(t, x, y)$ of (1), (2) belongs to $C^1_{loc}(R^N \times R^N)$ for all $t > 0$, if $u_0(x, y)$ is bounded below, continuous and convex in $(x, y) \in R^N \times R^N$. These kinds of regularizations were first shown by J.M.Lasry and P.-L.Lions in [4], and by P.-L. Lions in [6].

All the regularization effects above come from the existence of the function $L(t, x, y)$ defined on $t > 0$, $(x, y) \in R^N \times R^N$ with which the following inf-convolution formula

$$u(t, x, y) = \inf_{(x', y') \in R^N \times R^N} \{ u_0(x', y') + L(t, x-x'-ty', y-y') \}, \quad t > 0, \quad (x, y) \in R^N \times R^N,$n

(3)
yields a viscosity solution of (1), (2), that is the value function of the associated control problem ( (5)-(6) below). More precisely, denoting by $H^*$ the Frenchel transform of $H(p)$

$$H^*(p) = \sup_{q \in R^N} \{ \langle p, q \rangle - H(q) \}, \quad p \in R^N,$n

(4)
we set

$$u(t, x, y) = \inf_{\alpha(\cdot) \in A} \{ \int_0^t H^*(\alpha(s)) ds + u_0(x_\alpha(t), y_\alpha(t)) \}, \quad t > 0, \quad (x, y) \in R^N \times R^N,$n

(5)
where $A$ is the set of all measurable functions $\alpha(\cdot)$ from $[0, t]$ to $R^N$ such that $H(\alpha(\cdot))$ is integrable in $[0, t]$; $(x_\alpha(s), y_\alpha(s))$ $0 \leq s \leq t$ is the solution of the ordinary differential equation

$$\frac{d}{ds}(x_\alpha(s), y_\alpha(s)) = (-y_\alpha(s), \alpha(s)), \quad s \geq 0,$n

(6)
$$(x_\alpha(0), y_\alpha(0)) = (x,y), \quad (x, y) \in R^N \times R^N.$n

It is worth remarking that the existence of the "kernel" $L(t, x, y)$ in (3) comes from the controllability of the system (6), namely (6) has the following property: for any $t > 0,(x', y') \in R^N \times R^N$, there exists $\alpha(\cdot) \in A$ for which the solution $(x_\alpha(s), y_\alpha(s))$ $0 \leq s \leq t$ of (6) satisfies $(x_\alpha(t), y_\alpha(t)) = (x', y').$

Some properties of $L(t, x, y)$: non-negativity, convexity, lower and upper bounds, invariance property etc. will be given in Theorem 2.
Then, we shall investigate the regularity of $L(t, x, y)$ itself and $u(t, x, y)$ given in (3). As for the Lipschitz regularity, $L(t, x, y)$ is locally Lipschitz continuous in $(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N}$ for any $t > 0$ (which we do not give the proof here) and $u(t, x, y)$ is Lipschitz continuous in $(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N}$ for any $t > 0$. These results will be given in Theorems 3, 4. We mention that when the initial function $u_{0}(x, y)$ in (2) is bounded uniformly continuous, since $u(t, x, y)$ given in (3) is Lipschitz continuous (Theorem 4), this is the unique viscosity solution of (1)-(2) in the framework of the bounded uniformly continuous functions in $\mathbb{R}^{N} \times \mathbb{R}^{N}$. (See M.G. Crandall, L.C. Evans and P.L. Lions [3].) As for the local semi-concave regularity and $C_{loc}^{1,1}$ regularity, we shall first show in Theorem 5 that if $H(p) \in C_{loc}^{1,1}(\mathbb{R}^{N})$ and if $H(p)$ satisfies additional conditions (see Theorem 5), then $L(t, x, y) \in C_{loc}^{1,1}(\mathbb{R}^{N} \times \mathbb{R}^{N})$ for any $t > 0$, which leads the local semi-concavity regularizing effect of (1). (Corollary 1) Next, in Theorem 6, we shall show that $u(t, x, y)$ given in (3) with a convex, continuous and bounded below initial function $u_{0}(x, y)$ belongs to $C_{loc}^{1,1}(\mathbb{R}^{N} \times \mathbb{R}^{N})$ for any $t > 0$. (Theorem 6) As an example, for the case of $H(p) = |p|^{2}$, we can compute $L(t, x, y)$ explicitly and watch that there is the $C_{loc}^{1,1}$ regularising effect. This result is analogous to [4].

Moreover, we add the uniqueness problem for (1): first with continuous initial conditions $u_{0}$ (possibly unbounded from above) in the framework of the positive, continuous solutions (see Theorem 7); next with a singular initial condition (see Theorem 8). In particular, the second result gives a characterization of the kernel $L(t, x, y)$ as the unique solution of

$$
\frac{\partial L}{\partial t} = y \cdot \nabla_{x} L + H(\nabla_{y} L), \quad t > 0, \quad (x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N},
$$

$$
\lim_{t \downarrow 0} L(t, x, y) = 0, \quad (x, y) = (0, 0), \quad = \infty, \quad (x, y) \neq (0, 0),
$$

with the condition of $\bar{L} \geq 0$ and $L \in C_{loc}^{0,1}((0, \infty] \times \mathbb{R}^{N} \times \mathbb{R}^{N})$. These will be done in Theorems 5, 6.

In the subsequent arguments, we use the notations $R$, $N$, $\mathbb{R}^{+}$ for the sets of real, natural and positive real numbers respectively. We denote the norm in $\mathbb{R}$, $\mathbb{R}^{N}$ and $\mathbb{R}^{N} \times \mathbb{R}^{N}$ by $|r|$, $|x|$ and $|(x, y)|$ ($r \in \mathbb{R}$, $x, y \in \mathbb{R}^{N}$) respectively without confusion; the distance between two points $(x, y)$ and $(\hat{x}, \hat{y})$ by $|(x, y) - (\hat{x}, \hat{y})|$. For a positive number $R > 0$, we write $B_{R}(0)$, $B_{R}(0, 0)$ for the sets $\{x \in \mathbb{R}^{N} \mid |x| < R\}$, $\{(x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \mid |(x, y)| < R\}$ respectively. We denote by $(x_{\alpha}(s), y_{\alpha}(s))$ the solution of the ordinary differential equation (6); by $A(t; x, y; x', y')$ the set of all measurable functions $\alpha(\cdot)$ from $[0, t]$ to $\mathbb{R}^{N}$ such that $H^{*}(\alpha(\cdot))$ is integrable on $[0, t]$ and $(x_{\alpha}(t), y_{\alpha}(t)) = (x', y')$. As remarked above, from the controllability of the system (6)

$$
A(t; x, y; x', y') \neq \emptyset, \quad t > 0, \quad (x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N}.
$$

In the throughout of this paper, the solution of the equation is in the sense of the viscosity solution, and we refer [2], [6] for its definition.
2 Existence of the kernel for the inf-convolution

We show the existence of the kernel $L(t, x, y)$ for the inf-convolution formula (3).

**Theorem 1** Let the Hamiltonian $H(p)$ in (1) satisfy the assumption (H). Let $u_0(x, y)$ in (2) be bounded and uniformly continuous. Then, there exists a unique viscosity solution of (1)-(2) and it is given by

$$u(t, x, y) = \inf_{(x', y') \in \mathbb{R}^N \times \mathbb{R}^N} \{u_0(x', y') + L(t, x - x' - ty', y - y')\}, \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $L(t, x, y)$ is a real-valued function defined for $t > 0$, $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ by

$$L(t, x, y) = \inf_{\alpha(\cdot) \in A(t, x, y; 0, 0)} \int_0^t H^*(\alpha(s)) ds. \quad (8)$$

**Proof**

We shall rewrite (1) by using the Frenchel transformation (4)

$$\frac{\partial u}{\partial t} + y \cdot \nabla_x u + \sup_{\alpha \in \mathbb{R}^N} \{-\langle \alpha, \nabla_y u \rangle - H^*(\alpha)\} = 0,$$

$$t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (9)$$

It is known (see [6]) that the viscosity solution of (1)-(2) is given by

$$u(t, x, y) = \inf_{\alpha(\cdot) \in A} \{u(t-s, x_{\alpha}(s), y_{\alpha}(s)) + \int_0^s H^*(\alpha(s')) ds'\}, \quad 0 \leq s \leq t, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

$$0 \leq s \leq t, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\mathcal{A}$ is the set of all measurable functions $\alpha(\cdot)$ from $[0, t]$ to $\mathbb{R}^N$ such that $H^*(\alpha(s'))$ is integrable on $s' \in [0, t]$. We shall prove that (10) is equivalent to (3) with $L$ defined in (8). So, let $s = t$ in (10) and we have

$$u(t, x, y) = \inf_{(x', y') \in \mathbb{R}^N \times \mathbb{R}^N} \inf_{\alpha(\cdot) \in A(t, x, y; x', y')} \{u_0(x', y') + \int_0^t H^*(\alpha(s)) ds\}. \quad (11)$$

By comparing (11), to (3) with (8), we see that it is enough to prove

$$\mathcal{A}(t; x, y; x', y') = \mathcal{A}(t; x - x' - ty', y - y'; 0, 0). \quad (12)$$

Let $\alpha_1(s) \in \mathcal{A}(t; x, y; x', y')$, denote by $(x_{\alpha_1}(s), y_{\alpha_1}(s))$ the solution of the O.D.E. (6) with $\alpha = \alpha_1$, and solve the O.D.E.

$$\frac{d}{ds} (\hat{x}_{\alpha_1}(s), \hat{y}_{\alpha_1}(s)) = (-\hat{y}_{\alpha_1}(s), -\alpha_1(s)), \quad 0 \leq s \leq t,$$
$$(\hat{x}_{\alpha_{1}}(0), \hat{y}_{\alpha_{1}}(0)) = (x - x' - ty', y - y').$$

Then, $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (x_{\alpha_{1}}(\mathit{S}) - xy_{\alpha_{1}}') + y')$$, $0 \leq s \leq t$ and we have $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (x_{\alpha_{1}}(S) - xy_{\alpha_{1}}' + y'))$$, $0 \leq s \leq t$ and we have $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (0, 0)$$. We have $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (x_{\alpha_{1}}(S) + x' + ty', y_{\alpha_{1}}(S) + y')$$, $0 \leq s \leq t$ and we have $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (x_{\alpha_{1}}(S) + x' + ty', y_{\alpha_{1}}(S) + y')$$, $0 \leq s \leq t$ and we have $$(\hat{x}_{\alpha_{1}}(S), \hat{y}_{\alpha_{1}}(S)) = (0, 0)$$. We have $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (0, 0)$$, that is $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (0, 0)$$. We have $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (0, 0)$$, that is $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (0, 0)$$. We have $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (x_{\alpha_{2}}(t), y_{\alpha_{2}}(t)) = (0, 0)$$, that is $$(\hat{x}_{\alpha_{2}}(S), \hat{y}_{\alpha_{2}}(S)) = (0, 0)$$.

Therefore, (12) is proved and we have proved that (3) and (10) are equivalent.

It is known that for the bounded, uniformly continuous initial function $u_0(x, y)$, the solution $u(t, x, y)$ of (1),(2) is unique in the framework of bounded, uniformly continuous functions. We shall show below in Theorem 4 that $u(t, x, y)$ defined in (5) is Lipshitz continuous in $(x, y) \in R^N \times R^N$. By admitting temporarily this fact, since $u(t, x, y)$ is clearly bounded, we have proved our assertion.

### 3 Properties of the kernel

In this section, we shall show some properties of the kernel $L(t, x, y)$ of the inf-convolution formula (3), which will be used later in section 4 to study the regularizing effects.

**Theorem 2** The function $L(t, x, y)$ given in (8) has the following properties.

(i) (Non-negativity) 

$$L(t, x, y) > 0, \quad (x, y) \in R^N \times R^N \setminus (0, 0);$$

$$L(t, 0, 0) = 0, \quad (x, y) = (0, 0).$$

(ii) (Inf-convolution) For any $t, s > 0$,

$$L(t + s, x, y) = \inf_{(x', y') \in R^N \times R^N} \{L(t, x', y') + L(s, x - x' - sy', y - y')\}, \quad (x, y) \in R^N \times R^N. \quad (13)$$

(iii) (Convexity) For any $t > 0$, $L(t, x, y)$ is convex in $(x, y) \in R^N \times R^N$: for any $0 \leq k \leq 1$,

$$L(t, kx + (1 - k)x', ky + (1 - k)y') \leq kL(t, x, y) + (1 - k)L(t, x', y'), \quad (14)$$

$$t > 0, \quad (x, y), (x', y') \in R^N \times R^N.$$
(iv) (Scaling invariance) For any $\lambda > 0$,

$$L(t, x, y) = \lambda^{-1}L(\lambda t, \lambda^2 x, \lambda y), \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (15)$$

(v) (Lower estimate)

$$L(t, x, y) \geq \max\{tH^*(\frac{x}{t^2}), \quad tH^*(\frac{y}{t})\} \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (16)$$

(vi) (Upper estimate)

$$L(t, x, y) \leq \frac{t}{2}\{H^*(\frac{4x-ty}{t^2}) + H^*(\frac{-4x+3ty}{t^2})\} \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (17)$$

Proof

(i) is obvious in view of the definition of $L(t, x, y)$ in (8).

(ii) The relationship (13) comes also from the definition of $L$ (8), by noticing

$$A(\mathbf{s}; \mathbf{x}; \mathbf{x}', \mathbf{y}') = A(\mathbf{s}; \mathbf{x}, \mathbf{y}; \mathbf{x}-\mathbf{x}' - s\mathbf{y}', \mathbf{y} - \mathbf{y}') \quad s > 0, \quad (\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \mathbb{R}^N \times \mathbb{R}^N,$$

which was shown in the proof of Theorem 1.

(iii) Let $(x, y) = (x_1, \ldots, x_N, y_1, \ldots, y_N), (x', y') = (x_1', \ldots, x_N', y_1', \ldots, y_N') \in \mathbb{R}^N \times \mathbb{R}^N$; let $\alpha = (\alpha_1, \ldots, \alpha_N) \in A(t; \mathbf{x}, \mathbf{y}; 0, 0), \alpha' = (\alpha_1', \ldots, \alpha_N') \in A(t; \mathbf{x}', \mathbf{y}'; 0, 0)$. We shall denote $(x'', y'') = (kx + (1-k)x', ky + (1-k)y'), \quad \alpha'' = k\alpha + (1-k)\alpha'$, and solve the following O.D.E.

$$\frac{d}{ds}(x''_{\alpha}(s), y''_{\alpha}(s)) = (-y''_{\alpha}(s), -\alpha''(s)), \quad 0 \leq s \leq t,$$

$$(x''_{\alpha}(0), y''_{\alpha}(0)) = (x'', y'').$$

Then, we have $(x''_{\alpha''}(t), y''_{\alpha''}(t)) = (0, 0)$ and $\alpha'' \in A(t; x'', y''; 0, 0)$. Thus, by the convexity of $H^*$,

$$L(t, x'', y'') \leq \int_0^t H^*(\alpha''(s))ds \leq \int_0^t kH^*(\alpha(s))ds + \int_0^t (1-k)H^*(\alpha'(s))ds.$$ 

Since $\alpha, \alpha'$ are arbitrary we have proved

$$L(t, x'', y'') \leq kL(t, x, y) + (1-k)L(t, x', y').$$

(iv) For arbitrary fixed $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad t > 0$, let $\alpha(\cdot) \in A(t; \mathbf{x}, \mathbf{y}; 0, 0)$ and let $\lambda > 0$. Define $\beta(s) = \alpha(\frac{s}{\lambda})$ for $0 \leq s \leq \lambda t$ and solve

$$\frac{d}{ds}(x_\beta(s), y_\beta(s)) = (-y_\beta(s), -\beta(s)), \quad 0 \leq s \leq \lambda t,$$

$$(x_\beta(0), y_\beta(0)) = (\lambda^2 x, \lambda y).$$
Then,

\[ y_{\beta}(s) = \lambda y - \int_{0}^{s} \beta(s')ds' = \lambda y - \lambda \int_{0}^{s} \alpha(s')ds', \]

\[ x_{\beta}(s) = \lambda^2(x) - \lambda xs + \lambda \int_{0}^{s} \int_{0}^{s'} \alpha(s'')ds''ds' = \lambda^2 x - \lambda xs + \lambda^2 \int_{0}^{s} \int_{0}^{s'} \alpha(s'')ds''ds', \]

and we have \((x_{\beta}(\lambda t), y_{\beta}(\lambda t)) = (0, 0)\), \(\beta(\cdot) \in A(\lambda t; \lambda^2 x, \lambda y; 0, 0)\).

Similarly, we can prove that if \(\{\beta(s)\}(0 \leq s \leq \lambda t) \in A(\lambda t; \lambda^2 x, \lambda y; 0, 0)\), then \(\alpha(s) = \beta(\lambda s),\ 0 \leq s \leq t\) belongs to \(A(t; x, y; 0, 0)\).

Now, let \(\alpha(s),\ 0 \leq s \leq t,\ \beta(s),\ 0 \leq s \leq \lambda t\) related by \(\beta(s) = \alpha(\frac{s}{\lambda})\). Since

\[ \int_{0}^{\lambda t} H^{*}(\beta(s))ds = \int_{0}^{\lambda t} H^{*}(\alpha(\frac{s}{\lambda}))ds = \lambda \int_{0}^{t} H^{*}(\alpha(s))ds, \]

from the above argument we have proved

\[ \lambda^{-1}L(\lambda t, \lambda^2 x, \lambda y) = L(t, x, y). \]

(v) For any \(\alpha \in A(t; x, y; 0, 0)\), it is easy to see that the following holds

\[ x = \int_{0}^{t} s\alpha(s)ds, \quad y = \int_{0}^{t} \alpha(s)ds. \]

For \(0 \leq t \leq 1\), since \(H^{*}\) is convex and \(H^{*}(0) = 0\), from Jensen's inequality,

\[ H^{*}(x) = H^{*}(\int_{0}^{t} s\alpha(s)ds) \leq \int_{0}^{t} H^{*}(s\alpha(s))ds \]

\[ \leq \int_{0}^{t} sH^{*}(\alpha(s))ds \leq \int_{0}^{t} H^{*}(\alpha(s))ds, \]

\[ H^{*}(y) = H^{*}(\int_{0}^{t} \alpha(s)ds) \leq \int_{0}^{t} H^{*}(\alpha(s))ds \]

which leads to

\[ \max\{H^{*}(x), H^{*}(y)\} \leq L(t, x, y), \quad 0 \leq t \leq 1, \quad (x, y) \in R^{N} \times R^{N}. \]  

(18)

For \(t > 1\), from the invariance of \(L(t, x, y)\) (iii) and (18),

\[ L(t, x, y) = tL(1, \frac{x}{t^2}, \frac{y}{t}) \geq \max\{tH^{*}(\frac{x}{t^2}), tH^{*}(\frac{y}{t})\}. \]

(vi) For arbitrary \((x, y) \in R^{N} \times R^{N}\), the following control

\[ \alpha = 4x - y, \quad 0 \leq s \leq \frac{1}{2}; \quad = -4x + 3y, \quad \frac{1}{2} \leq s \leq 1, \]
belongs to $A(1; x, y; 0, 0)$. Thus from the definition of $L(t, x, y)$
\[ L(1, x, y) \leq \frac{1}{2}\{H^*(4x - y) + H^*(-4x + 3y)\}. \]
By using the invariance of $L(t, x, y)$,
\[ L(t, x, y) = tL(1, \frac{x}{t^2}, \frac{y}{t}) \leq \frac{t}{2}\{H^*(\frac{4x}{t^2} - \frac{y}{t}) + H^*(\frac{-4x}{t^2} + \frac{3y}{t})\}. \]

4 Regularising effect

In this section, we shall study Lipschitz regularising effect, local semi-concavity regularizing effect and $C^1_{loc}$ regularising effect of the inf-convolution formula (3). We begin with the Lipschitz regularising effect.

**Theorem 3** The function $L(t, x, y)$ defined in (8) is locally Lipschitz continuous in $(x, y) \in R^N \times R^N$: for any $R > 0$, $t > 0$, there exists a constant $M_R > 0$ such that
\[ L(t, x, y) - L(t, \hat{x}, \hat{y}) \leq \sup_{|z| \leq M + 1} \frac{1}{t}H^*(16z)(x, y) - (\hat{x}, \hat{y}), \]
for $(x, y), (\hat{x}, \hat{y}) \in B_R(0, 0)$.

The constant $M_R$ is given by
\[ M_R = \sup\{r \in R^+: \sup_{|z| \leq r} H^*(4z) \leq \sup_{|z| \leq R} H^*(42z)\}. \]

**Proof**

We do not give the proof. It will be done similarly as the proof of the following Theorem.

**Theorem 4** Let $u_0(x, y)$ be bounded and continuous in $R^N \times R^N$, $|u_0(x, y)| \leq M$, $(x, y) \in R^N \times R^N$. Then,
\[ u(t, x, y) = \inf_{(x', y') \in R^N \times R^N} \{u_0(x', y') + L(t, x - x' - ty', y - y')\} \]
for any $t > 0$.

is Lipschitz continuous in $R^N \times R^N$ for any $t > 0$ :
\[ u(t, x, y) - u(t, \hat{x}, \hat{y}) \leq 2t \sup_{|z| \leq G_{M, t} + 1} H^*\left(\frac{4(t + 1)z}{t^2}\right)|(x, y) - (x', y')|, \]
where
\[ G_{M, t} = \sup\{r \in R^+: \sup_{|z| \leq r} H^*(\frac{z}{t^2}), \sup_{|z| \leq r} H^*(\frac{z}{t}) \leq \frac{M}{t}\}. \]
Proof

First, let \( t > 0 \), and assume that \((x, y), (\hat{x}, \hat{y}) \in R^N \times R^N\) satisfy \(|(x, y) - (\hat{x}, \hat{y})| \leq 1\). Since,

\[
u(t, \hat{x}, \hat{y}) = \inf_{(x^{'}, y^{'}) \in R^N \times R^N} \{u_0(x', y') + L(t, \hat{x} - x' - ty', \hat{y} - y') \} \leq M,
\]
denoting

\[
B_0 = \{(x', y') \mid |L(t, \hat{x} - x - ty, \hat{y} - y')| \leq M\},
\]
we can write

\[
u(t, \hat{x}, \hat{y}) = \inf_{(x', y') \in B_0} \{u_0(x', y') + L(t, \hat{x} - x' - ty', \hat{y} - y')\}.
\]

(22)

From the lower bound (16) on \( L(t, x, y) \) which we seeked in Theorem 2, the set \( B_0 \) is contained in the set

\[
\{(x', y') \mid tH^*(\frac{\hat{x} - x' - ty'}{t^2}), tH^*(\frac{\hat{y} - y'}{t}) \leq M\}.
\]

We denote

\[
G_{M,t} = \sup\{r \in R^+ \mid \sup_{|z| \leq r} H^*(\frac{z}{t^2}), \sup_{|z| \leq r} H^*(\frac{z}{t}) \leq \frac{M}{t}\},
\]
and remark that \((x', y') \in B_0\) satisfies

\[
|\hat{x} - x' - ty'|, \quad |\hat{y} - y'| \leq G_{M,t}.
\]

(23)

Next, by using the inf-convolution formula (3), there exists \((x', y') \in R^N \times R^N\) which satisfies (23) and and let \( k = \max_{1 \leq i \leq N} \{ |k_i| \mid \} \), \( w = (k_1, \ldots, k_N), \quad z = (l_1, \ldots, l_N) \). Then, \( 0 < k \leq 1, \)

\[
(x - \hat{x}, y - \hat{y}) = k_1, \ldots, k_N, l_1, \ldots, l_N). \] Then, \( 0 < k \leq 1, \)

\[
|\frac{k_i}{k}|, \quad |\frac{l_i}{k}| \leq 1, \quad 1 \leq i \leq N.
\]

Therefore, by the convexity of \( L(t, x, y) \),

\[
L(t, \hat{x} - x' - ty' + x - \hat{x}, \hat{y} - y' + y - \hat{y})
\]
\[ = L(t, \dot{x} - x' - ty' + kw, \dot{y} - y' + kz) \]
\[ \leq (1 - k)L(t, \dot{x} - x' - ty' + w, \dot{y} - y' + z). \]

Inserting this inequality into (24), and by using the upper estimate on \( L(t, x, y) \) and (23),
\[
\begin{align*}
    u(t, x, y) - u(t, \hat{x}, \hat{y}) & \leq kL(t, \dot{x} - x' - ty' + w, \dot{y} - y' + z) \\
    & \leq t \{ H^*\left(\frac{4(\dot{x} - x' - ty' + w)}{t^2} - t(\dot{y} - y' + z)\right) \\
    & \quad + H^*\left(-\frac{4(\dot{x} - x' - ty' + w) - 3t(\dot{y} - y' + z)}{t^2}\right)\} \frac{|(x, y) - (\hat{x}, \hat{y})|}{2},
\end{align*}
\]
and we have
\[
\begin{align*}
    u(t, x, y) - u(t, \hat{x}, \hat{y}) & \leq t \sup_{|\beta| \leq C_{M,r+1}} H^*\left(\frac{4(\dot{t} + 1)z}{t^2}\right),
\end{align*}
\]
for any \((x, y), (\hat{x}, \hat{y}) \in \mathbb{R}^N \times \mathbb{R}^N\) such that \(|(x, y) - (\hat{x}, \hat{y})| \leq 1\). We therefore deduce (20) from the above inequality.

Next, we study \( C^{1,1}_{loc} \) regularity of the kernel \( L(t, x, y) \) of the inf-convolution formula (5).

**Theorem 5** Let the Hamiltonian \( H(p) \) in (1) belong to \( C^{2,1}_{loc}(\mathbb{R}^N) \) and assume that its Frenchel transformation \( H^*(p) \) is strictly convex. Then, \( L(t, x, y) \) defined in (8) belongs to \( C^{1,1}_{loc} (\mathbb{R}^N \times \mathbb{R}^N) \) for \( t > 0 \).

**Proof**

First, we shall see that if \( \hat{\alpha}(\cdot) \in A(t; x, y; 0, 0) \) satisfies
\[
\int_0^t \nabla H^*(\hat{\alpha}(s)) \beta(s) ds = 0 \tag{25}
\]
holds for any measurable function \( \beta(s) \) from \([0, t]\) to \( \mathbb{R}^N \) such that
\[
\int_0^t \beta(s) ds = 0, \quad \int_0^t s\beta(s) ds = 0, \tag{26}
\]
then,
\[
L(t, x, y) = \int_0^t H^*(\hat{\alpha}(s)) ds. \tag{27}
\]
In fact, if (25) holds, from the strict convexity of \( H^*(p) \):
\[
H^*(\hat{\alpha}(s) + \beta(s)) > H^*(\hat{\alpha}(s)) + \nabla H^*(\hat{\alpha}(s)) \cdot \beta(s), \quad a.e.s \in [0, t]
\]
implies that \( \hat{\alpha}(\cdot) \) is a global minimizer of the functional \( \int_0^t H^*(\alpha(s)) ds \) among \( \alpha(\cdot) \in A(t; x, y; 0, 0) \). Moreover, it is the unique minimizer, because if there is another local
minimizer, say $\gamma \in A(t; x, y; 0, 0)$, since $\text{meas}\{s \in [0, t] \mid \hat{\alpha}(s) \neq \gamma(s)\} > 0$, the strict convexity of $H^*(p)$ leads
\[
\text{meas}\{s \in [0, t] \mid H^*(\frac{\hat{\alpha}(s) + \gamma(s)}{2}) < \frac{1}{2} H^*(\frac{\hat{\alpha}(s)}{2}) + \frac{1}{2} H^*(\frac{\gamma(s)}{2})\} > 0,
\]
which contradicts to (25), for $\beta(s) = \hat{\alpha}(s) - \frac{1}{2}(\hat{\alpha}(s) + \gamma(s))$ satisfies (26). Thus, $\hat{\alpha}(s) = \gamma(s)$, $a.e. s \in [0, t]$.

Next, from the implicit function theorem and also the strict convexity of $H$, for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, there exists $(a, b) \in \mathbb{R}^N \times \mathbb{R}^N$ such that
\[
(x, y) = \left(\int_0^t s \nabla H(a + bs)ds, \int_0^t \nabla H(a + bs)ds\right),
\]
and the mapping $(x, y) \to (a, b)$ is $C_{loc}^{1,1}$. So, we define
\[
\hat{\alpha}(s) = \nabla H(a + bs), \quad s \in [0, t],
\]
and easily we have
\[
\nabla H^*(\hat{\alpha}(s)) = \nabla H^*(\nabla H(a + bs)) = a + bs, \quad s \in [0, t].
\]
Therefore, $\hat{\alpha}(\cdot)$ satisfies (25) and (27) holds. From the regularity assumption on $H(p)$, we conclude the proof.

Theorem 5 leads the local semi-concavity regularizing effect of (1).

**Corollary 1** Let the Hamiltonian $H(p)$ satisfy the assumptions in Theorem 5. Then, for any continuous function $u_0(x, y)$ in $\mathbb{R}^N \times \mathbb{R}^N$, for any $R > 0$,
\[
u(t, x, y) = \inf_{(x', y') \in \mathbb{R}^N \times \mathbb{R}^N} \left\{ u_0(x', y') + L(t, x - x' - ty', y - y') \right\} \quad (3)
\]
\[t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
is locally semi-concave.

**Proof**

It is a direct result from the $C_{loc}^{1,1}$ regularity of $L(t, x, y)$ proved in Theorem 5.

The local semi-concavity regularizing effect of (1) leads the $C_{loc}^{1,1}$ regularizing effect with the convex initial functions.
Theorem 6 Let the Hamiltonian $H(p)$ satisfy the assumptions in Theorem 5. Then, for any bounded from below, continuous, and convex function $u_0(x, y)$ in $R^N \times R^N$,

$$u(t, x, y) = \inf_{(x', y') \in R^N \times R^N} \{u_0(x', y') + L(t, x - x' - ty', y - y')\}$$

for $t > 0$, $(x, y) \in R^N \times R^N$,

belongs to $C^{1,1}_{loc}(R^N \times R^N)$.

Proof

First, we remark that $u(t, x, y)$ is convex in $(x, y) \in R^N \times R^N$ for any $t > 0$. To see this, let $(x, y), (\hat{x}, \hat{y}) \in R^N \times R^N$, $0 \leq k \leq 1$ be arbitrary. Let $\alpha(s), \beta(s)$ be measurable functions from $[0, t]$ to $R^N$. Let $(x_\alpha(s), y_\alpha(s))$, $0 \leq s \leq t, (\hat{x}_\beta(s), \hat{y}_\beta(s))$, $0 \leq s \leq t$ be the solutions of the O.D.E. $(8)$, with the initial values $(x_\alpha(0), y_\alpha(0)) = (x, y)$, $(\hat{x}_\beta(0), \hat{y}_\beta(0)) = (\hat{x}, \hat{y})$ respectively.

We denote $(x_\alpha(t), y_\alpha(t)) = (x', y')$, $(\hat{x}_\beta(t), \hat{y}_\beta(t)) = (\hat{x}', \hat{y}')$, put $\gamma(s) = k\alpha(s) + (1 - k)\beta(s)$, $0 \leq s \leq t$, and solve

$$\frac{d}{ds}(x_\gamma(s), y_\gamma(s)) = (-y_\gamma(s), -\gamma(s)), \quad 0 \leq s \leq t,$$

$$(x_\gamma(0), y_\gamma(0)) = (kx + (1 - k)\hat{x}, ky + (1 - k)\hat{y}).$$

Then since,

$$(x_\gamma(t), y_\gamma(t)) = (kx' + (1 - k)\hat{x}', ky' + (1 - k)\hat{y}'),$$

by the convexity of $u_0$ and $H^*(p)$,

$$u(t, kx + (1 - k)\hat{x}, ky + (1 - k)\hat{y}) \leq u_0(kx' + (1 - k)\hat{x}', ky' + (1 - k)\hat{y}') + \int_0^t H^*(\gamma(s))ds$$

$$\leq ku_0(x' y') + (1 - k)u_0(\hat{x}', \hat{y}') + k \int_0^t H^*(\alpha(s))ds + (1 - k) \int_0^t H^*(\beta(s))ds$$

$$\leq k\{u_0(x', y') + \int_0^t H^*(\alpha(s))ds\} + (1 - k)\{u_0(\hat{x}', \hat{y}') + \int_0^t H^*(\beta(s))ds\},$$

and

$$u(t, kx + (1 - k)\hat{x}, ky + (1 - k)\hat{y}) \leq ku(t, x, y) + (1 - k)u(t, \hat{x}, \hat{y}),$$

for $\alpha(\cdot), \beta(\cdot)$ are arbitrary.

Therefore, from Corollary 1, there is a number $C_R > 0$ such that

$$u(t, x, y) + C_R(|x|^2 + |y|^2) \quad \text{is convex in} \quad B_R(0, 0),$$

$$u(t, x, y) - C_R(|x|^2 + |y|^2) \quad \text{is concave in} \quad B_R(0, 0).$$
These relationships yield the $C^1$ differentiability of $u(t, x, y)$, from standard results of convex analysis theory. Next, by using the same techniques as in [4], [6], since $L \in C_{loc}^{1,1}(R^N \times R^N)$, we have $u(t, x, y) \in C_{loc}^{1,1}(R^N \times R^N)$ for $t > 0$.

Example 1 Let $N=1$, $H(p) = |p|^2$. Then, by using the argument in the proof of Theorem 5, we can explicitly compute

$$L(t, x, y) = \frac{3x^2 - 3xyt + y^2t^2}{t^3},$$

and we can see directly that $u(t, x, y)$ given by the inf-convolution formula is semiconcave for any continuous initial condition $u_0(x, y)$, and that if we assume moreover that $u_0(x, y)$ is bounded from below and convex, $u(t, x, y)$ belongs to $C_{loc}^{1,1}(R \times R)$ for any $t > 0$ as in [4].

5 Uniqueness result and characterization of the kernel

In this section, we give the uniqueness result for the solution of (1), with possibly unbounded, continuous initial condition $u_0(x, y)$ in the framework of the continuous, positive solution in $(0, \infty) \times R^N \times R^N$. That is, the inf-convolution formula (3) gives a unique solution of (1) in this framework. This result is stated in Theorem 7. Next, we deduce from this fact a characterization of the kernel $L(t, x, y)$ given in (10) in terms of the partial differential equation (1) with a singular-valued initial condition. This will be shown in Theorem 8.

Theorem 7 Let $u(t, x, y)$ be a continuous solution of

$$\frac{\partial u}{\partial t} + y \cdot \nabla_x u + H(\nabla_y u) = 0, \quad t > 0, \quad (x, y) \in R^N \times R^N,$$

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in R^N \times R^N. \quad (29)$$

Then, the following holds.

(i) For any arbitrary number $R > 0$,

$$u(t, x, y) = \inf_{\alpha(\cdot)} \left\{ \int_0^{\tau_R} H^*(\alpha(s)) ds + u(t - \tau, x_\alpha(\tau), y_\alpha(\tau))1_{(\tau \leq t)} + u_0(x_\alpha(t), y_\alpha(t))1_{(\tau > t)} \right\}, \quad t > 0, \quad (x, y) \in B_R(0, 0), \quad (30)$$

where for each $\alpha(\cdot)$, $\tau_R = \inf\{t \geq 0 \mid (x_\alpha(t), y_\alpha(t)) \in \overline{B_R(0, 0)}\}$. 


(ii) If
\[
u(t, x, y) \geq 0, \quad t \geq 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
then
\[
u(t, x, y) = \inf_{(x', y') \in \mathbb{R}^N \times \mathbb{R}^N} \{u_0(x', y') + L(t, x - x' - ty', y - y')\}
\]
\[
t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.
\]

Proof

We do not give the proof of this Theorem here. See our paper to appear.

Theorem 8 Let \(L(t, x, y) \in C^{0,1}_{loc}((0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N)\) be a solution of
\[
\frac{\partial \hat{L}}{\partial t} + y \cdot \nabla_x \hat{L} + H(\nabla_y \hat{L}) = 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
\[
\lim_{t \downarrow 0} \hat{L}(t, x, y) = 0, \quad (x, y) = (0, 0); \quad = \infty, \quad (x, y) \neq (0, 0),
\]
and assume that \(\hat{L}(t, x, y) \geq 0, \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.\) Then,
\[
L(t, x, y) = \hat{L}(t, x, y), \quad t > 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N
\]

Proof

By using Theorem 7,
\[
\hat{L}(h + t, x, y) = \inf_{(x', y') \in \mathbb{R}^N \times \mathbb{R}^N} \{\hat{L}(h, x', y') + L(t, x - x' - ty', y - y')\}, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
for any \(h, t > 0.\) Therefore, clearly
\[
\hat{L}(h + t, x, y) \leq L(t, x, y), \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
holds for any \(h, t > 0\) which leads to \(\hat{L} \leq L.\) Next, from (34), for any small \(\varepsilon > 0, \quad h > 0,\) there exists \((x_h^*, y_h^*)\) such that
\[
\hat{L}(h + t, x, y) + \varepsilon \geq \hat{L}(h, x_h^*, y_h^*) + L(t, x - x_h^* - ty_h^*, y - y_h^*)
\]
\[
\geq L(t, x - x_h^* - ty_h^*, y - y_h^*),
\]
here we used \(\hat{L} \geq 0.\) We see from (33), (34) that \((x_h^*, y_h^*) \to (0, 0)\) as \(h \downarrow 0.\) Therefore, the right-hand side of the above inequality tends to \(L(t, x, y)\) as \(h \downarrow 0,\) and we have
\[
\hat{L}(t, x, y) \geq L(t, x, y), \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.
\]
References


