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Commutators and Iterated Commutators in Kleinian Groups

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1. Introduction.

Let M denote the group of all Möbius transformations of the extended complex plane $\hat{C} = C \cup \{\infty\}$. We associate with each

$$f = \frac{az+b}{cz+d} \in M, ad - bc = 1,$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C).$$

We set $tr(f) = tr(A)$ where $tr(A)$ denotes the trace of A. Next for each f and g in M we let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. We define the three complex numbers

$$(1) \qquad \beta(f) = tr^2(f) - 4, \quad \beta(g) = tr^2(g) - 4, \quad \gamma(f, g) = tr([f, g]) - 2$$

the parameters of the two generator subgroup $< f, g >$.

The following inequality [5][7] gives an important necessary condition for a two generator group $G =< f, g >$ to be nonelementary and discrete.

Lemma 1[5][7]. *If $< f, g >$ is nonelementary and discrete, then*

$$(2) \qquad |\gamma(f, g)| + |\beta(f)| \geq 1 \quad \textit{and} \quad |\gamma(f, g) - \beta(f)| + |\beta(f)| \geq 1.$$

If $< f, g >$ is a nonelementary discrete Fuchsian subgroup of M, then

$$(3) \qquad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

See,[10].

When $< f, g >$ is not Fuchsian, we can not obtain lower bounds for $|\gamma(f, g)|$ in a nonelementary discrete group $< f, g >$. For example, Jϕrgensen has shown that if $1 < a < \infty$, then the transformations

$$f(z) = -a^2 z, \quad g(z) = \frac{(a^2 + a^{-2})z - 2}{2z - (a^2 + a^{-2})}$$

generate a nonelementary discrete subgroup of M with

$$\beta(f) = -(a + a^{-1})^2, \quad \beta(g) = -4, \quad \gamma(f, g) = 4(a - a^{-1})^{-2}.$$

Since $\gamma(f,g) \to 0$ as $a \to \infty$,(3) fails if $< f,g >$ is not Fuchsian. But if we require some restrictive conditions for elements of generators,then we have the lower bounds for $|\gamma(f,g)|$. For example,if f or g is parabolic, then,

$$|\gamma(f,g)| \geq 1. \tag{4}$$

And further, Jϕrgensen proved in [8] that

$$|\gamma(f,g)| \geq \frac{1}{8}, \tag{5}$$

if $\beta(f) = \beta(g)$

Gehring and Martin showed the following lemma 2.

Lemma 2[4]. *Suppose that* $< f,g >$ *is a discrete subgroup of* M *with* $\gamma(f,g) \neq 0, \beta(f) = \beta(g) \neq -4$. *Then*

$$|\gamma(f,g)| > 0.193. \tag{6}$$

The constant 0.193 is not sharp. Gehring and Martin considered the subgroup $< [f,g], f[f,g]f^{-1} >$of $< f,g >$ and showed $|\gamma([f,g], f[f,g]f^{-1})| + |\beta[f,g]| < 1$ under the condition $|\gamma(f,g)| \leq 0.193$.

Lemma 3[4]. *Suppose that* $< f,g >$ *is a discrete subgroup of* M *with* $\gamma(f,g) \neq 0, \beta(f) = \beta(g) \neq -4$, *and*

$\min\{|\beta(f)|, |\beta(fg)|, |\beta(fg^{-1})|\} \geq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. *Then*

$$|\gamma(f,g)| \geq 2 - 2\cos(\pi/7). \tag{7}$$

Proof. If A represents f, then by replacing A by $-A$ if necessary, we may assume that $tr(f) = tr(g)$. Let $a = tr(fg) - 2$, $b = tr(fg^{-1}) - 2$, $c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1)$. Next by replacing g by g^{-1} we may aassume that $|a| \leq |b|$. Then $\beta(f) = a + b, \beta(fg) = a^2 + 4a, \gamma(f,g) = -ab$ by Fricke's formula. Hence

$$(|a| + 2)^2 \geq |\beta(fg)| + 4 \geq c + 4 = (2\cos(2\pi/7) + 1)^2$$

by the assumption of Lemma 3 and $|a| \geq d$ where $d = 2\cos(2\pi/7) - 1$.

If $|a| \geq 0.5$, then $|\gamma(f,g)| = |ab| \geq |a|^2 \geq 0.25$ while if $|a| < 0.5$, then $|a| < c/2$ and

$$|\gamma(f,g)| = |a||\beta(f) - a| \geq d(c - d) = 2 - 2\cos(\pi/7)$$

by the asuumption. This completes the proof of Lemma 3.

Remark 1. To show that (7) is sharp,let $< \phi, \psi >$ be the $(2,3,7)$ triangle group with $\phi^2 = \psi^3 = (\phi\psi)^7 = id.$ and set $f = [\phi, \psi], h = \phi\psi, g = hf^{-1}h^{-1}$.Then $\beta(f) = \beta(g) = c$

where $c = 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$, $\beta(fg) = \beta(f,h)\{\beta(f,h) + 4\} = c$, $\beta(fg^{-1}) = \beta(fhfh^{-1}) = \{\beta(f) - \gamma(f,h)\}\{\beta(f) - \gamma(f,h) + 4\} \geq c$ and $\gamma(f,g) = 2\cos(\pi/7) - 2$.And also, we have $\gamma(f,g) = \gamma(fg,g) = \gamma(fg^{-1},g)$.

Lemma 4. *Suppose that* $< f,g >$ *is a discrete subgroup of* M *with* $\gamma(f,g) \neq 0, \beta(g) \neq -4$, *and* $|\beta(f)| \leq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. *Then*

$$|\gamma(f,g)| \geq 2 - 2\cos(\pi/7) \quad or \quad |\gamma(f,g) - \beta(f)| > 1. \tag{8}$$

The following Lemma 5 is a direct consequence of Lemma 4.

Lemma 5. *Suppose that* $< f,g >$ *is a discrete subgroup of* M *with* $\gamma(f,g) \neq 0, \beta(g) \neq -4$, *and* $|\beta(f)| \leq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. *Then*

$$|\gamma(f,g)| \geq 2 - 2\cos(\pi/7) \quad or \quad |\gamma(f,gfg^{-1})| \geq 2 - 2\cos(\pi/7) \tag{9}$$

Proof. Suppose $|\gamma(f,g)| < 2 - 2\cos(\pi/7)$, then $|\beta(f) - \gamma(f,g)| > 1$ from Lemma 4. Next, $< [f,g], f >$ is a discrete subgroup of $< f,g >$ with$\gamma' = \gamma([f,g],f) = \gamma(\gamma - \beta) \neq 0$, $\beta' = \beta[f,g] = \gamma(\gamma + 4) \neq -4$ and $|\beta'| \leq 0.9 < c$ where $\gamma = \gamma(f,g)$ and $\beta = \beta(f)$. Then we have $|\gamma'| \geq 2 - 2\cos(\pi/7)$ or $|\beta' - \gamma'| > 1$ from Lemma 4. But the second case does not yield since $|\beta' - \gamma'| = |\gamma||\beta + 4| < 1$. This complete the proof.

It is easily seen that $\gamma(f,g) \neq 0$ if and only if $fix(f) \cap fix(g) = \phi$ where $fix(f)$ denotes the fixed point set of f in $\hat{C}$.

The triple $\beta(f), \beta(g), \gamma(f,g)$ with $\gamma(f,g) \neq 0$ and $\beta(f) \neq 0$ determine the two generator group $< f,g >$ up to conjugacy. Since $\gamma(f,g) \neq 0$,f and g have no fixed points in common and there exists $\phi \in M$ such that $\phi f \phi^{-1}(0) = 0$, $\phi f \phi^{-1}(\infty) = \infty$ and $\phi g \phi^{-1}(1) = 1$. Then

$$\phi f \phi^{-1}(z) = kz, \qquad \phi g \phi^{-1}(z) = \frac{az + b}{cz + d},$$

where $ad - bc = 1, a + b = c + d, bc \neq 0$ and $\arg(a + d) \in [0, \pi)$ if $a + d \neq 0$. Next by replacing ϕ by $\psi\phi$ where $\psi(z) = \frac{1}{z}$, we may arrange that $|\frac{b}{c}| \geq 1$ with $\mathrm{Im}(\frac{b}{c}) \geq 0$ if $|\frac{b}{c}| = 1$. Next by replacing f by f^{-1} and g by g^{-1} if necessary, we may also assume that $|k| \geq 1$ with $\mathrm{Im}(k) \geq 1$ if $|k| = 1$ and that $\arg(a - d) \in [0, \pi)$ or $\arg(b + c) \in [0, \pi)$ if $a = d$. Then the equations $\beta = k - 2 + \frac{1}{k}, \beta' = (a + d)^2 - 4 = (a - d)^2 + 4bc = (b + c)^2, \gamma = -bc\beta, \frac{\beta\beta'}{\gamma} = -(\frac{b}{c} + 2 + \frac{c}{b})$ determine $k, a, d, \frac{b}{c}, b + c$ and therefore b,c uniquely.

2. Preliminary.

We state two lemmas needed to establish theorems. These statements are easily obtained by operating matrices.

Lemma 6[4]. *If* f *and* g *are in* M *with* $\gamma(f,g) = \gamma$ *and* $\beta(f) = \beta$, *then*

$$\gamma(f, gfg^{-1}) = \gamma(f, [f,g]) = \gamma(\gamma - \beta) \quad and \quad \beta([f,g]) = \gamma(\gamma + 4). \tag{10}$$

Lemma 7[4]. *If* $< f, g >$ *is an elementary discrete subgroup of* M *with* $\gamma(f,g) \neq 0$, $\beta(f) = \beta(g) \neq -4$, *then*

$$(11) \qquad |\gamma(f,g)| \geq (3-\sqrt{5})/2.$$

3. A lower bound for the commutator.

The following lemma gives a key tool to show Theorem 10.

Lemma 8[4]. *If* $< f, g >$ *is a discrete subgroup of* M *with* $\gamma(f,g) \neq 0$, *then*

$$(12) \qquad |\beta(f)-1| \leq |\gamma(f,g)|^2/(1-|\gamma(f,g)|) \quad \textit{or} \quad |\beta(f)-1| \geq 1-|\gamma(f,g)|.$$

Proof. Suppose (12) does not hold and let $\gamma = \gamma(f,g)$ and $\beta = \beta(f)$. Then $\frac{|\gamma|^2}{1-|\gamma|} < |\beta - 1| < 1 - |\gamma|$ and hence $p(\gamma) = \{\gamma^2 - (\beta-1)\gamma - (\beta-1)\}^2 \neq 0$ since otherwise we would have

$$|\beta - 1| = |\frac{\gamma^2}{\gamma+1}| \leq \frac{|\gamma|^2}{1-|\gamma|}$$

contradicting the first statement.

Let $\gamma_j = \gamma(f, g_j)$ where

$$g_{j+1} = g_j f^{-1} g_j^{-1} f g_j f g_j^{-1} f^{-1} g_j \quad (j = 1, 2, \ldots), \quad g_1 = g.$$

Then $\gamma_1 = \gamma$,$|\gamma_{j+1}| = |\gamma_j||p(\gamma_j)|$ and

$$|\gamma_j| + |\beta - 1| < 1, \quad p(\gamma_j) \neq 0$$

implies that

$$0 < |p(\gamma_j)| = |\gamma_j{}^2 - (\beta-1)\gamma_j - (\beta-1)|^2 < (|\gamma_j| + |\beta-1|)^2 < 1.$$

Thus we see by induction that $0 < |\gamma_j| \leq |\gamma_1|(|\gamma_j| + |\beta - 1|)^{2j} \to 0$ as $j \to \infty$ contradicting Lemma 12. This completes the proof.

Lemma 9. *If* $< f, g >$*is a non-elementary discrete subgroup of* M *with* $\gamma(f,g) \neq 0$, *and* $|\gamma(f,g)| < 2 - 2\cos(\pi/7)$, *then*

$$(13) \qquad |\beta(f) - 1| \geq 1 - |\gamma(f,g)|.$$

Proof. Suppose that $|\beta(f)-1| \leq |\gamma(f,g)|^2/(1-|\gamma(f,g)|)$, and we lead the contradiction. Let $\delta = \inf\{|\gamma(f,g)|; \gamma \neq 0, \beta(f) = \beta(g), |\beta(f)| \leq c, < f, g > \text{discrete}\}$ where $c = 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. It is easily seen $0.193 < \delta < d$. For sufficiently small $\epsilon > 0$ with $0 < \epsilon < (d-\delta)/2 < d$, we have a discrete group $G =< f, g >$ such that $|\gamma(f,g)| \leq \delta + \epsilon$,$|\beta(f)| \leq c$ and $\beta \neq -4$. If $\beta(f) - \gamma(f,g) = 1/\{\gamma(f,g)+1\}$ for the given discrete group G, we consider the subgroup $< A_1, A_0A_1A_0^{-1} >$ of G where $A_0 = [f, g]$ and $A_1 = [A_0, f]$. Then we have

$$\delta \leq |\gamma(A_1, A_0A_1A_0^{-1})| = |\gamma^2(\gamma-\beta)^2\gamma(\beta+4)(\gamma+2)^2| \leq |\gamma|^2|1+\frac{1}{\gamma+1}|^2.$$

Since the function $|z||1+\frac{1}{z+1}|^2$ is subharmonic in the closed disk $\{z; |z| \leq d\}$, then we have the maximum value 1 of $|z||1+\frac{1}{z+1}|^2$ at $z = -d$. Therefore $0 < \max_{0.193 \leq |\gamma| \leq \delta+(d-\delta)/2} |z||1+\frac{1}{z+1}|^2 = k < 1$ and also we have $\delta \leq |\gamma(A_1, A_0A_1A_0^{-1})| \leq |\gamma|k \leq (\delta+\epsilon)k$. We remark that $|\beta(A_0A_1A_0^{-1})| = |\beta(A_1)| = |\gamma(\gamma-\beta)||\gamma(\gamma-\beta)+4| < c$. Thus if we select $\epsilon < \frac{(1-k)\delta}{k}$, this leads the contradiction. Therefore we have $\beta(f) - \gamma(f,g) \neq 1/\{\gamma(f,g)+1\}$ for the given group G. Next we consider the subgroup $< f, hfh^{-1} >$ where $h = gf^{-1}g^{-1}fgfg^{-1}f^{-1}g$, then

$$\gamma(f, hfh^{-1}) = \gamma\{\gamma^2 - (\beta-1)\gamma - (\beta-1)\}^2[\gamma\{\gamma^2 - (\beta-1)\gamma - (\beta-1)\}^2 - \beta].$$

The above argument states $\gamma^2 - (\beta-1)\gamma - (\beta-1) \neq 0$. The assumption $|\gamma| < d$ and Jø rgensen's inequality shows $|\beta| \geq 1-d > 0.8$, then $|\gamma\{\gamma^2-(\beta-1)\gamma-(\beta-1)\}^2-\beta| > 0.7$. Therefore we have $\gamma(f, hfh^{-1}) \neq 0$ but $|\gamma\{\gamma^2 - (\beta-1)\gamma - (\beta-1)\}^2| < 0.0125$ shows $0 < |\gamma(f, hfh^{-1})| < 0.013$, this contradicts to Lemma 2.

The following theorem is essentially based on Lemma 9 and we consider the subgroup $< f, hfh^{-1} >$ of $< f, g >$. By the analysis with the help of computer , we have the following theorem.

Theorem 10. *Suppose that $< f, g >$ is a discrete subgroup of M with $\gamma(f,g) \neq 0, \beta(f) = \beta(g) \neq -4$, then*

$$|\gamma(f,g)| \geq 2 - 2\cos(\pi/7). \tag{14}$$

The following Lemma 11 shows that Theorem 10 is applicable to the collar lemma. Let A_f be the axis of $f(\in M)$ in H^3 connecting with two fixed points in ∂H^3.

Lemma 11[2][6]. *Let f and g be nonparabolic elements in M and let μ be the complex distance between A_f and A_g. If $\gamma = tr(fgf^{-1}g^{-1}) - 2 \neq 0$, then*

$$4\gamma = \beta(f)\beta(g)\sinh^2(\mu). \tag{15}$$

If we consider the subgroup $< f, gfg^{-1} >$ of $< f, g >$ and suppose $\gamma(f, gfg^{-1}) \neq 0$, then $\gamma(f, gfg^{-1}) = \beta(f)^2 \sinh^2 \mu\ (A_f, g(A_f))$.

4. Iterative commutators.

Let β be a complex number. Let $R(z) = z(z-\beta)$,R^n denote the n-th iterate of

R and $R^0 = id.$. We define $g_0 = g, g_1 = gfg^{-1}$and $g_{n+1} = g_n f g_n^{-1}$,inductively and consider a subgroup $< f, g_n f g_n^{-1} >$ of a Kleinian group $< f, g >$.Then we find $R^{n+1}(\gamma) = \gamma(f, g_n)\{\gamma(f, g_n) - \beta(f)\} = \gamma(f, g_{n+1})$ for any positive integer n,inductively.

Lemma 12[4]. *If f and g_j are elements of a Kleinian group G with $\gamma(f, g_j) \neq 0$ for $j = 1, 2, \ldots$ and if f is not of order 2, then*

(16)
$$\liminf_{n\to\infty} |\gamma(f, g_j)| > 0$$

Proof. Let $\gamma_j = \gamma(f, g_j)$. If $\liminf_{n\to\infty} |\gamma(f, g_j)| = 0$, then we can choose j so that $2|\gamma_j| < \min\left[\frac{1}{|\beta|+1}, \frac{1}{|\gamma_j|+4}\right]$ and $\gamma_j \neq \beta$. Then $\gamma'_j = \gamma([f, g_j], f) = \gamma_j(\gamma_j - \beta)$ and $0 < |\gamma'_j| < 1/2$ and $\beta'_j = \beta([f, g_j]) = \gamma_j(\gamma_j + 4)$ and $0 < |\beta'_j| < 1/2$. Now $\gamma'_j \neq 0$ and $\gamma'_j \neq \beta'_j$ and the pypothesis that $\beta \neq -4$. Since $< f, [f, g_j] >$ is a Kleinian subgroup of $< f, g >$, $|\gamma'_j| + |\beta'_j| \geq 1$ and we have a contradiction.

Corollary 13. *Let $< f, g_{n-1} f g_{n-1}^{-1} >$ be a subgroup of a Kleinian group $< f, g >$, $R^n(\gamma) \neq 0, \beta(f) \neq -4$, then*

(17)
$$|R^n(\gamma)| \geq 2 - 2\cos(\pi/7).$$

Lemma 14[9]. *Let $< f, g >$ is Kleinian with $\gamma \neq 0$ and $R(\gamma) = 0$, then either f is elliptic of order 2, 3, 4 or 6 or g is elliptic of order 2.*

Let $< f, g >$ be Kleinian with $|\beta(\beta + 2)| \leq 1$, and if $|2\gamma - \beta| < 1 + \sqrt{1 - |\beta(\beta + 2)|}$, then we have $|2R(\gamma) - \beta| < 1 + \sqrt{1 - |\beta(\beta + 2)|}$. Let D_r be the open disk centered at $\beta/2$ with the radius $r = \{1 + \sqrt{1 - |\beta(\beta + 2)|}\}/2$. Then, we have the following lemma.

Lemma 15. *Suppose that $< f, g >$ is Kleinian with $|\beta(\beta + 2)| \leq 1$. If γ is in the open disk D_r stated above, then*

(18)
$$R^n(\gamma) \in D_r.$$

Lemma 16[5]. *Suppose that $< f, g >$ is a non-elementary Kleinian group. If $|\beta(\beta + 2)| \leq 1$, $\gamma \neq \beta + 1$ and $\gamma \neq -1$, then*

(19)
$$|2\gamma - \beta| \geq 1 + \sqrt{1 - |\beta(\beta + 2)|}.$$

Proof. Suppose that $< f, g >$ is non-elementary, Kleinian with $\gamma \neq \beta + 1$ and $\gamma \neq -1$. Suppose that $\gamma \in D_r$, and let $\gamma_j = \gamma(f, g_j)$ where $g_1 = g$ and $g_{j+1} = g_j f g_j^{-1}$ for $j = 1, 2, \ldots$. If $z \in D_r$, then $|2R(z) - \beta| \leq r$. Thus $R^j(D_r) \subset D_r$,D_r lies in the Fatou set for $R(z)$ and $\gamma_{j+1} = R^j(\gamma) \in D_r$ for $j \geq 0$. If $\beta = 0$, then D_r is the unit disk, f is parabolic and

$|\gamma| \geq 1$,hence $\gamma \notin D_r$. Next if $\beta = -1$, then $D_r = \{z; |2z+1| < 1\}$,f is of order 6 , $\gamma_1 \neq 0$ and $\gamma_{j+1} = \gamma_j(\gamma_j - \beta - 1)^2 = \gamma_j^3 = \gamma^{3^j}$. Therefore $|\gamma| \geq 1$ by Lemma 12. Finally if $\beta = -2$,then $D_r = \{z; |z+1| < 1\}$, $0 < |\gamma_{j+1}+1| = |R(\gamma_j)+1| = |\gamma_j+1|^2 = |\gamma+1|^{2^j}$ for $j \geq 0$,and $\gamma \notin D_r$ by Lemma 12. We have that $\beta \notin \{0,-1,-2\}$. Suppose that $|\beta| < 1$. Then 0 is an attracting fixed point for $R(z)$, $0 \in D_r$ and $\gamma_j \to 0$ as $j \to \infty$. Hence $\gamma_j = \gamma(f, g_j) = 0$ for some j by Lemma 12. Let k be the smallest such integer j. Since $\gamma \notin \{0,\beta\}$,$\gamma_1 = \gamma \neq 0$ and $\gamma_2 = R(\gamma) \neq 0$,therefore $k \geq 3$. Then $R(\gamma_{k-1}) = 0, \gamma_{k-1} \neq 0$ and $\gamma(f, g_{k-1}) = \gamma_{k-1} = \beta(g_{k-1}) \neq 0$. Thus f and g_{k-1} are elliptic of order $2,3,4$, or 6 by Lemma 13 and $|\beta| \geq 1$, thus we have a contradiction. Finally suppose that $|\beta| \geq 1$. Then $|\beta+2| < 1$ by the assumption and the fact that $\beta \notin \{-1,-2\}$, $\beta+1$ is an attracting fixed point for $R(z)$ and $\beta \in D_r$. Thus as above, $\gamma_j = \beta+1$ for some j by Lemma 12. Let k be the smallest such integer j. Because $\gamma_1 = \gamma \notin \{-1,\beta+1\}$, $\gamma_1 \neq \beta+1$ and $\gamma_2 = R(\gamma) \neq \beta+1$,thus $k \geq 3$. Thus $\gamma_{k-2} \in R^{-1}(-1)$ and $|2\gamma_{k-2} - \beta| = \sqrt{|\beta^2-4|} > 1 + \sqrt{1 - |\beta(\beta+2)|}$. In paticular, $\gamma_{k-2} \notin D_r$ which contradict $\gamma_{j+1} \in D_r$. This completes the proof of Lemma 16.

Lemma 17. *Suppose that $< f,g >$ is non-elementary,Kleinian. If $|\beta(\beta+2)| \leq 1$ and $\gamma \in D_r$, then*

$$\gamma = \beta + 1 \quad or \quad \gamma = -1. \tag{20}$$

Lemma 18. *Suppose that $< f,g >$ is non-elementary,Kleinian. If f is elliptic of order $n \geq 6$, then*

$$|2\gamma - \beta| \geq 1 + \sqrt{1 - |\beta(\beta+2)|}. \tag{21}$$

Theorem 19. *Suppose that $< f,g >$ is Kleinian with $\gamma(f,g) \neq 0$ and $|\gamma - \beta/2| > |\beta^2/4 + \beta/2| + 1$, then $R^n(\gamma) \to \infty$ as $n \to \infty$.*

Proof. Let $G(z) = z^2 + c$. If $|z| > |c|+1$, then the orbit of z under $G(z)$ is not bounded. Since $|z^2+c| \geq |z|||z| - |\frac{c}{z}|| > |z|||c|+1 - \frac{|c|}{|c|+1}|$ for $|z| > |c|+1$, we have $|G(z)| > r|z|$ where $r = 1 + |c|^2/(|c|+1) > 1$. Let $H(z) = z - \beta/2$, then $HRH^{-1}(z) = z^2 - \beta^2/4 - \beta/2$. Therefore we have $|R^n(\gamma) - \beta/2| \geq r^n|\gamma - \beta/2|$ where $r > 1$, $n \geq 1$. Thus we have Theorem 19.

If $< f,g >$ is non-elementary Kleinian group, then Jørgensen showed the inequalities $|\gamma| + |\beta| \geq 1$ and $|\gamma - \beta| + |\beta| \geq 1$. We say that a non-elementary Kleinian group $< f,g >$ is extremal if one of these inequalities holds with equality. There exists many groups which are extremal for these inequalities; for example, the triangle groups with signatures $(2,3,n)$ with $n \geq 7$ have this property.

Let $< f,g >$ be a non-elementary Kleinian group where f is not parabolic. Then $|\beta| > 0$ and $|\gamma| > 0$. Let $U = \{z; |z| \leq 1 - |\beta|\}$, $U' = \{z; |z - \beta| \leq 1 - |\beta|\}$ and

$V = \{z; |z-\beta| \le 1\}$. It is easy to show that $R(U) \subset U, R(U') \subset U, U \subset D_\beta$ where $D_\beta = \{z \in C; \sup|R^n(z)| < \infty\}$. Suppose that $< f, g >$ is extremal in the inequality $|\gamma| + |\beta| \ge 1$. Then $|\beta| < 1$, $\gamma \in \partial U$, $R^n(\gamma) \in U$ and $\gamma \notin \mathrm{int}(D_\beta)$ because Gehring and Martin proved in [6] that if $< f, g >$ is non-elementary, Kleinian and $0 < |\beta| < 1$, then $\gamma \notin D_\beta$. The complete invariance of $\mathrm{int}(D_\beta)$ under R implies that $R^n(\gamma) \in \partial U$ for $n \ge 0$. Therefore we have

$$1 - |\beta| = |R^n(\gamma)| = |R^{n-1}(\gamma)||R^{n-1}(\gamma) - \beta| = (1 - |\beta|)|R^{n-1}(\gamma) - \beta|,$$

for $n \ge 1$. and we have $\gamma, R(\gamma) \in \partial U \cap \partial V$. Now $U \subset V, U \neq V$ and hence the circles ∂U, ∂V meet in just one point. Thus $R(\gamma) = \gamma$ and $\gamma = \beta + 1$. Suppose next that $< f, g >$ is extremal in the sense $|\gamma - \beta| + |\beta| = 1$. Then $\gamma \in \partial U', R^n(\gamma) \in U$, $\gamma \notin \mathrm{int}(D_\beta)$ and $R^n(\gamma) \in \partial U$ for $n \ge 1$. Therefore $R(\gamma), R^2(\gamma) \in \partial U \cap \partial V$. Thus $R(\gamma) = \beta + 1$ and $\gamma = -1$ since $|\gamma - \beta| < 1$. Finally we have the followings.

Lemma 20[6]. *Suppose that $< f, g >$ is non-elementary, Kleinian and f is not parabolic. If $< f, g >$ satisfies the condition $|\gamma| + |\beta| = 1$,then $\gamma = \beta + 1$. And also if $< f, g >$ satisfies $|\gamma - \beta| + |\beta| = 1$,then $\gamma = -1$.*

Therefore, if a non-elementery Kleinian group $< f, g >$ is extemal, then $R(\gamma) = \gamma$ or $R^2(\gamma) = R(\gamma)$.

Remark 2. Let $< \phi, \psi >$ be the $(2, 3, 7)$ triangle group stated in Remak 1. And also set $f = [\phi, \psi], h = \phi\psi, g = hf^{-1}h^{-1}$, then $\gamma(f, g) = 2\cos(\pi/7) - 2, R(\gamma) = 2\cos(2\pi/7) - 1$ and we have $R^2(\gamma) = \gamma$ for $< f, g >$.

5. An arithmetic condition

Let $< f, g >$ be a non-elementary,Kleinian and set $C = \{\gamma; \gamma(f', g') \neq 0, f', g' \in < f, g >\}$. Applying chapter 4 to this, we have $R^n(\gamma) \in C$ if $\gamma \in C$. We eatablish here a description of the elements h in $< f, g >$ for which $\gamma(f, h)$ can be expressed as a polynomial in $\gamma(f, g)$ and $\beta(f)$ with integer coefficients.

Lemma 21. *Suppose that r is an integer and $u \in C \setminus \{0\}$. Then*

$$u^{2r} + u^{-2r} = p_r((u - u^{-1})^2), \tag{22}$$

$$u^{2r+1} + u^{-(2r+1)} = (u + u^{-1})q_r((u - u^{-1})^2), \tag{23}$$

where p_r and q_r are polynomials with integer coefficients and where $p_r(0) = 2$ and $q_r(0) = 1$.

Theorem 22. *Let f and g are in M and that*

$$h = f^{r_1}g^{s_1} \cdots f^{r_n}g^{s_n}f^{r_{n+1}} \tag{24}$$

Then $tr(h) = (tr(f))^{\epsilon} p(\gamma(f,g), \beta(f))$ *where* p *is a polynomial in both variables with integer coefficents and where* $\epsilon = 0$ *or* 1 *depending on whether* $r_1 + r_2 + \ldots + r_{n+1}$ *is even or odd.*

Proof. If $r_1' = r_1 + r_{n+1}$, then

$$tr(h) = tr(f^{r_1'} g^{s_1} \cdots f^{r_n} g^{s_n}) = tr(f^{r_n} g^{s_n} f^{r_1'} g^{s_1} \cdots f^{r_{n-1}} g^{s_{n-1}}).$$

Hence we may assume without loss of generality that $r_{n+1} = 0$ and that $s_j = (-1)^{j+1}$ for $j = 1, 2, \cdots, n$. Suppose first that $n = 2$ and that f and g are represented by matrices

$$A = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{SL}(2, C)$ where we consider first f is nonparabolic. If f is parabolic,then we have a similar result so we omit that case. Then $tr(f) = (u + u^{-1}), \beta = (u - u^{-1})^2, \gamma = -bc(u - u^{-1})^2$ and we see that

$$tr(h) = (1 + bc)[u^{r_1+r_2} + u^{-(r_1+r_2)} - bc[u^{r_1-r_2} + u^{r_2-r_1}].$$

If $2r = r_1 + r_2$ is even, then so is $2s = r_1 - r_2$ and with above arguments we have

$$tr(h) = (1 + bc)p_r(\beta) - bcp_s(\beta) = p_r(\beta) - \gamma\frac{p_r(\beta) - p_s(\beta)}{\beta} = p(\gamma, \beta).$$

Then $p_r(0) - p_s(0) = 0$,hence p is a polynomial in γ and β with integer coefficents. If $2r + 1 = r_1 + r_2$ is odd, then so is $2s = r_1 - r_2$ thus we have

$$tr(h) = tr(f)[q_r(\beta) - \gamma\frac{q_r(\beta) - q_s(\beta)}{\beta}] = tr(f)p(\gamma, \beta).$$

Thus we have the proof of theorem for the case where $n = 2$. It is easy to show generally by inductive argument but we do not discuss here.

Lemma 23. *If* f *and* g *are in* M *and*

$$h = f^{r_1} g^{s_1} \cdots\ f^{r_n} g^{s_n} f^{r_{n+1}} \tag{25}$$

where $s_1 = \pm 1$ *and* $s_j = (-1)^{j+1} s_1$ *for* $j = 1, 2, \ldots, n$. *Then*

$$\gamma(f) = p(\gamma(f,g), \beta(f)) \tag{26}$$

where p *is a polynomial in both variables with integer coefficents. If,in addition,* n *is even, then* $\beta(f) = q(\gamma(f,g), \beta(f)))$ *where* q *is a polynomial in both variables with integer coefficients.*

Recently, Prof. Gehring announced that Chun Cao proved Theorem 10. We expect that his proof is not depend on computer and also this result has many applications.

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