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# Hausdorff dimension of the limit sets of classical Schottky groups

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## 0. Introduction

1.1. In this paper we will consider the following problem : Given any number $t$ satisfying $0 < t < 1$, does there exist a finitely generated Kleinian group $G$ with the limit set $\Lambda(G)$ having infinite $t$ - dimensional Hausdorff measure ?

In 1971, Beardon[1] gave an affirmative answer by using Hecke groups for this problem. The method of Beadon depends on a close, direct analysis of the action of the group $G$. Furthermore, in 1985 Phillips and Sarnak [5] showed by using the bottom of the spectrum for the Laplacian $\Delta$ (the smallest eigenvalue of $\Delta$) that there is a Hecke group having the desired property. Here we will consider the problem by studying Fuchsian Schottky groups (Sato [7]).

We will state the method of Beardon in §1 and the method of Phillips-Sarnak in §2. In §3 we will state some results on Fuchsian Schottky groups.

## 1. The method of Beardon

In this section we will state the proof of the following theorem due to Beardon

THEOREM A (Beardon [1]). *Given any number $t$ satisfying $t < 1$, there exists a finitely generated Fuchsian group $G$ of the second kind with $\infty$ an ordinary point of $G$ and with the limit set $\Lambda(G)$ having infinite $t$-dimensional Hausdorff measure.*

DEFINITION 1.1. Let $E$ be any set and $t$ a positive number. Define

$$m_{t,\delta}(E) = \inf \sum_i |I_i|^t,$$

where the infimum is taken over all coverings of $E$ by sequences $\{I_i\}$ of sets $I_i$ with diameter $|I_i|$ less than $\delta$. Furthermore, we define

$$m_t(E) = \sup\{m_{t,\delta}(E) | \delta > 0\}$$

and we call $m_t(E)$ the *$t$-dimensional Hausdorff measure* of $E$.

Set $d(E) = \inf\{t | m_t(E) = 0\}$. We call $d_t(E)$ the *Hausdorff dimension* of $E$.

DEFINITION 1.2. A set $E$ is said to be a *spherical Cantor set* if and only if it can be expressed in the form

$$E = \cap_{n=1}^{\infty} \cup_{i_1,\ldots,i_n=1}^{K} \Delta(i_1, \ldots, i_n)$$

where $K \geq 2$ is an integer and where the $\Delta_{i_1,\ldots,i_n}$ are closed spheres of radius $r(i_1, \ldots, i_n)$ satisfying

(1) $\Delta(i_1, \ldots, i_n) \supset \Delta(i_1, \ldots, i_n, i_{n+1})$,

(2) $\Delta(1), \cdots, \Delta(K)$ are mutually disjoint,

(3) there exists a constamt $A$ $(0 < A < 1)$ such that

$$r(i_1, \ldots, i_n, i_{n+1}) \geq Ar(i_1, \ldots, i_n) \qquad (i_{n+1} = 1, 2, \ldots, K),$$

(4) there exists a constant $B$ $(0 < B < 1)$ such that

$$\rho(\Delta(i_1, \ldots, i_n, j), \Delta(i_1, \ldots, i_n, k)) \geq Br(i_1, \ldots, i_n)$$

$$(j, k = 1, 2, \ldots, K, j \neq k)$$

where

$$\rho(S, T) = \inf\{|s - t||s \in S, t \in T\}$$

DEFINITION 1.3. Let $P(z) = z + 2(1+\varepsilon)$ and $E(z) = -1/z$. We call the group $G[\varepsilon]$ generated by $P(z)$ and $E(z)$ a *Hecke group.*

1.2 Since the point $\infty$ is a limit point of $G[\varepsilon]$, we conjugate $G[\varepsilon]$ by $A \in$ Möb such that $\infty$ is an ordinary point of $AG[\varepsilon]A^{-1}$. We denote by $\Lambda(G)$ the limit set of a group $G$. For simplicity, we denote by $\Lambda_\varepsilon$ the limit set of a Hecke group $G[\varepsilon]$.

LEMMA 1.1.

$$d(A(\Lambda_\varepsilon)) \geq d(A(\Lambda_\varepsilon \cap [-1, 1])) \geq d(\Lambda_\varepsilon \cap [-1, 1]).$$

REDUCTION 1. It suffices to show that for sufficiently small $\varepsilon$, $d(\Lambda_\varepsilon \cap [-1, 1]) \geq t$.

NOTATION.

$Q := \{z||z| \leq 1\}$ ,

$V_n(z) := EP^n(z) \quad (n \neq 0)$,

$V(n_1, n_2, \ldots, n_k)(z) := V_{n_1}V_{n_2} \cdots V_{n_k}(z) \quad (n_j \neq 0)$,

$Q(n_1, n_2, \ldots, n_k) := V(n_1, n_2, \ldots, n_k)(Q)$

$L_1 := \cap_{k=1}^{\infty} \cup_{V \in G_k} V(Q)$,

where $G_k = \{V(n_1, n_2, \ldots, n_k)|n_j = 1, 2, \ldots, K\}$.

LEMMA 1.2. *$L_1$ is a subset of* $\Lambda_\varepsilon \cap [-1,1]$.

1.3. Let $\epsilon$ be a positive number and $N$ an integer satisfying $N \geq 2$. Let $\Gamma_1$ be the set consisting of the following elements (1) and (2):

(1) (A) $V_2, V_{-2}, \ldots, V_N, V_{-N}$

(2) $V(n_1, n_2, \ldots, n_k, m)$ with

(B) $1 \leq k \leq N, n_1 = \cdots = n_k = 1$ and $2 \leq |m| \leq N$,

(B') $1 \leq k \leq N, n_1 = \cdots = n_k = -1$ and $2 \leq |m| \leq N$,

(C) $1 \leq k \leq N, n_1 = \cdots = n_k = 1$ and $m = -1$ ,

(C') $1 \leq k \leq N, n_1 = \cdots = n_k = -1$ and $m = 1$ .

We set

$$\Gamma_{n+1} := \{UV | U \in \Gamma_n, V \in \Gamma_1\}$$

and

$$L_2 := \cap_{k=1}^{\infty} \cup_{V \in \Gamma_n} V(Q).$$

Then we have $L_2 \subset \Lambda_\epsilon \cap [-1,1]$. Hence

$$d(L_2) \leq d(L_1) \leq d(\Lambda_\epsilon \cap [-1,1]) \leq 1.$$

REDUCTION 2. It suffices to show that

$$\lim_{\epsilon \to \infty} \limsup_{N \to \infty} d(L_2) = 1$$

1.4. Set $\Gamma := \cup_n \Gamma_n$. We denote by $|\Delta|$ the diameter of a disc $\Delta$.

LEMMA 1.3. *Let $J = [-1,1]$, let $I$ be any sub-interval of $J$ and let $U \in \Gamma$.* Then

(1) $\frac{1}{5}|I| \le \frac{|U(I)|}{|U(J)|} \le \frac{5}{4}|I|$

(2) *If $V \in \Gamma_1$, then* $|UV(J)| \le \frac{5}{6}|U(J)|$.

LEMMA 1.4. *The set $L_2$ is a spherical Cantor set constructed from the discs $\{U(Q) | U \in \Gamma_n, n \ge 1\}$.*

LEMMA 1.5. *If $\theta$ satisfies $0 \le \theta \le 1$ and if*

$$\sum_{V \in \Gamma_1} |UV(Q)|^\theta \ge |U(Q)|^\theta$$

*for all $U \in \Gamma$, then $d(L_2) \ge \theta$.*

LEMMA 1.6. *Let $k > 1$ be any integer and let the positive numbers $\delta_1, \ldots, \delta_k, \delta$ and $s$ satisfy $0 \le \delta_j \le \delta < 1$ and $0 \le s \le \delta_1 + \cdots + \delta_k < 1$. Then*

$$\delta_1^\theta + \cdots + \delta_k^\theta \ge 1,$$

*where $\theta = 1 - (1-s)(1-\delta)^{-1}$.*

1.5. We set $F := (-1,1) - \cup_{V \in \Gamma_1} V(J)$. Then

$$|U(J)| = m_1(U(F)) + \sum_{V \in \Gamma_1} |U(V(J))|.$$

By Lemma 3 we have

$$m_1((U)F)) \le \frac{5}{4} m_1(F)|U(J)|$$

and

$$\begin{aligned} \delta_1 + \cdots + \delta_k \; &:= \sum_{V \in \Gamma_1} \frac{|V(J)|}{|U(J)|} = 1 - \frac{m_1(U(F))}{|U(J)|} \\ &\ge \; 1 - \frac{5}{4} m_1(F). \end{aligned}$$

We take $s$ in Lemma 1.6 $s = 1 - \frac{5}{4}m_1(F)$ , and we take $\delta = \frac{6}{5}$ by Lemma 1.3. Then

$$\theta = 1 - \frac{1-s}{1-\delta} = 1 - \frac{15}{2}m_1(F).$$

By Lemma 1.6 we have $\sum \delta_j^\theta \geq 1$, that is,

$$\sum_{V\in\Gamma_1} \frac{|UV(J)|^\theta}{|U(J)|^\theta} \geq 1.$$

Hence

$$\sum |UV(J)|^\theta \geq |U(J)|^\theta.$$

By Lemma 1.5 $d(L_2) \geq \theta$ and so

$$d(L_2) \geq 1 - \frac{15}{2}m_1(F) > 1 - 8m_1(F).$$

REDUCTION 3. It suffices to show that

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} m_1(F) = 0.$$

1.6. PROOF of Theorem A. We set $T := (-1.1) - \cup_{|n|=1}^N V(n)(J)$. For convenience, define $u_n = 1$ and $v_n = -1$ for each positive integer $n$. Then we have

$$\begin{aligned} F - T &= \cup_{n=-1,1} V(n)(J) - \cup_{V\in\Gamma_1} V(J) \\ &= [F \cap V(1)(J)] \cup [F \cap V(-1)(J)] \end{aligned}$$

Hence

$$m_1(F) = m_1(T) + m_1[F \cap V(1)(J)] + m_1[F \cap V(-1)(J)].$$

After calculations, we have

$$m_1[F \cap V(1)(J)]$$
$$= \sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] + m_1[V(u_1, \ldots, u_{N+1})(J)]$$

Noting that both $T$ and $J$ are symmetrical with the imaginary axix, we have

$$m_1(F)$$
$$= m_1(T) + 2\sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] + 2m_1[V(u_1, \ldots, u_{N+1})(J)].$$

We estimate three terms on the right hand side.

(1) The first term: If we set $\mu = 2 + 2\varepsilon$ , then

$$m_1(T) = \frac{2}{N\mu + 1} + 2\sum_{r=0}^{N}[\frac{1}{\mu r + 1} - \frac{1}{\mu(r+1) - 1}] < \frac{1}{N} + 6\varepsilon.$$

(2) The second term:

$$\sum_{r=1}^{N} m_1[V(u_1, \ldots, u_r)(T)] \leq \frac{3m_1(T)}{\sqrt{\varepsilon}} < \frac{3}{\sqrt{\varepsilon}}(\frac{1}{N} + 6\varepsilon).$$

(3) The last term :

$$m_1[V(u_1, \ldots, u_{N+1})(J)] \leq (\frac{p-q}{p-1})^2 \frac{m_1(J)}{p^{2(N+1)}},$$

where $p = (\mu + \sqrt{\mu^2 - 4})/2$ and $q = (\mu - \sqrt{\mu^2 - 4})/2$. Hence we have

$$m_1(F) \leq (\frac{1}{N} + 6\epsilon)(1 + \frac{6}{\sqrt{\epsilon}}) + 4(\frac{p-q}{p-1})^2 \frac{1}{p^{2(N+1)}}.$$

Therefore

$$\limsup_{N \to \infty} m_1(F) \leq 6\epsilon + 36\sqrt{\epsilon}$$

and so

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} m_1(F) = 0,$$

which is the desired result.

## 2. The method of Phillips-Sarnak

2.1. In this section we will state the proof of the following theorem due to Phillips-Sarnak. Let $G[\varepsilon]$ be the Hecke group defined in §1. We denote by $G_\mu$ the Hecke group $G[\varepsilon]$ with $\mu = 2 + 2\epsilon$.

THEOREM B (Phillips-Sarnak [5]). *Let $\lambda_0(G_\mu)$ be the smallest eigenvalue of the Laplacian $\Delta$ for a Hecke group $G_\mu$. As $\mu$ ranges from 2 to $\infty$, $\lambda_0(G_\mu)$ increases continuously and strictly monotonically from* 0 *to* 1/4.

COROLLARY. *Let $d(G_\mu)$ be the Hausdorff dimension of the limit set of a Hecke group $G_\mu$. As $\mu$ ranges from* 2 *to $\infty$, $d(G_\mu)$ decreases continuously and strictly monotonically from* 1 *to* 1/2.

This corolarry follows from Theorem B and Pattrson-Sullivan's theorem below.

2.2. Let $H = \{(x,y)|x \in R, y > 0\}$ be the upper half plane with the line element $ds^2 = (dx^2 + dy^2)/y^2$. We denote by $\Delta, \nabla$ and $dV$ the Laplacian, gradient and volume element, respectively, with respect to the hyperbolic metric. Let $\Omega$ be an open connected subset of $H$. We denote by $W^1(\Omega)$ the space of functions

$$W^1(\Omega) = \{f \in L^2(\Omega)|\nabla f \in L^2(\Omega)\}.$$

The quadratic forms $H$ and $D$ on $W^1(\Omega)$ are defined as

$$H(f,g) := \int_| \Omega f\bar{g}dV$$

and

$$D(f,g) := \int_\Omega < \nabla f, \bar{\nabla} g > dV.$$

Here we are interested in the selfadjoint Laplasian $\Delta$ defined on $L^2(\Omega)$ with Neumann boundary condition. This means that the domain of this operator consists of the set of all functions $u \in W^1(\Omega)$ with square integrable satisfying the condition $H(\Delta u, v) = D(u,v)$ (which is equivalent to $\partial u/\partial n = 0$ , where $\partial/\partial n$ is the unit outer normal derivative). We denote by $\lambda_0(\Omega)$ the bottom of the spectrum for $\Delta$ on $L^2(\Omega)$. $\lambda_0(\Omega)$ can be described variationally as

$$\lambda_0(\Omega) = \inf\{D(u) | u \in W^1(\Omega), H(u) = 1\}.$$

DEFINITION 2.1. We call a domain $\Omega$ *free* if $\lambda_0(\Omega) = 1/4$.

We remark that $\Omega$ is free if and only of the spectrum for $\Delta$ on $L^2(\Omega)$ have no discrete spectrum.

Let $G$ be a discrete group acting on the upper half plane $H$. We set

$$\delta(G) := \inf\{s | \sum_{\gamma \in G} \exp(-s(\rho(z, \gamma w)) < +\infty\},$$

where $\rho(z, \gamma w)$ is the hyperbolic distance from $z$ to $\gamma w$. We call $\delta(G)$ the *exponent of convergence* of $G$.

**Patterson-Sullivan's theorem** (Patterson [4], Sullivan [8]).

(1) $\delta(G) \geq 1/2$ then $\lambda_0(G) = \delta(G)(1 - \delta(G))$.

(2) If $G$ is geometrically finite, then $\delta(G) = d(\Lambda(G))$.

COROLLARY. *$G$ is a geometrically finite group with $\lambda_0(G) = 0$, then $d(\Lambda(G)) = 1$.*

DEFINITION 2.3. If a domain $\Omega$ is bounded by nonoverlapping circles, then we call $\Omega$ a *Schottky domain.* We call a discrete group $G$ a

*Schottky group in the sense of Phillips-Sarnak* or a *P-S Schottky group* if it has a fundamental domain which is a Schottky domain.

REMARK. A Hecke group $G_\mu$ is both a Fuchsian group of the second kind (resp.a Fuchsian group of the first kind) and a symmetric P-S Schottky group if $\mu = 2 + 2\epsilon > 2$ (resp. $\mu = 2$), where a domain $\Omega$ is symmetric if $\Omega$ is symmetric with respect to the imaginary axix.

2.5. Let $G$ be a discrete group. We denote by $\lambda_0(G) < \lambda_1(G) \leq \cdots$ the discrete eigenvalue of $\Delta$ on the Hilbert space of $G$ automorphic functions. We note that $\lambda_j(\Omega) \leq \lambda_j(G)$ if $\Omega$ is a fundamental domain for $G$.

LEMMA 2.1. *If $G$ is a symmetric P-S Schottky group, then $\lambda_j(G) = \lambda_j(\Omega^+)$ , where $\Omega^+$ is the part of the right side of $\Omega \cap H$ with respect to the imaginary axis.*

COROLLARY. *If $G_\mu$ is a Hecke group, then $\lambda_0(G_\mu) = \lambda_0(F_\mu^+)$, where $F_\mu^+$ is the part of the right side of the symmetric fundamental domain $F_\mu$ for $G_\mu$ with respect to the imaginary axis.*

2.6. LEMMA 2.2. *Suppose $\Omega' \to \Omega$ in $H$.*

(1) *If no cusp is broken in going from $\Omega$ to $\Omega'$ , then*

$$\lim \lambda_j(\Omega') = \lambda_j(\Omega).$$

(2) *If a cusp is broken and if $\Omega' \supset \Omega$ and $\Omega' \backslash \Omega$ is free, then* $\lim \lambda_0(\Omega') = \lambda_0(\Omega)$.

COROLLARY. *If $G_\mu$ is a Hecke group, then $\lambda_0(G_\mu)$ is continuous in $\mu (2 \leq \mu < \infty)$.*

LEMMA 2.3. *Let $G_\mu$ is a Hecke group. Then $\lambda_0(G_\mu) = 0$ for $\mu = 2$, that is, $d(\Lambda(G_\mu)) = 1$ for $\mu = 2$.*

LEMMA 2.4. *Suppose $\Omega_0$ and $\Omega_1$ are two domains with $\bar{\Omega}_1 \subset \Omega_0$ and set $\Omega_2 = \Omega_0 \backslash \bar{\Omega}_1$.*

(1) *If $\Omega_2$ is free, then $\lambda_j(\Omega_0) \geq \lambda_j(\Omega_1)$ for all $j$.*

(2) *Furthermore, if $\Omega_0$ has the finite geometric property and $\Omega_1$ is not*

*free, then* $\lambda_0(\Omega_0) > \lambda_0(\Omega_1)$.

COROLLARY. *If* $G_\mu$ *is a Hecke group, then* $\lambda_0(G_\mu)$ *increase strictly monotonically in* $\mu$ $(2 \leq \mu < \infty)$

LEMMA 2.5. *For Schottky domains, if* $\Omega' \to \Omega$ , *then* $\lambda_0(\Omega_k) \to \lambda_0(\Omega)$.

COROLLARY. *If* $G_\mu$ *is a Hecke group, then* $\lambda(G_\mu) \to \lambda_0(G_\infty)$ as $\mu \to \infty$.

LEMMA 2.6. *If* $\Omega$ *is a domain in* $H$ *with at most* $[(n+4)/2]$ *sides, then* $\Omega$ *is free.*

COROLLARY. *If* $G_\mu$ *is a Hecke group, then* $\lambda_0(G_\infty) = 1/4$.

Theorem B follows from the above lemmas and corollaries.

## 3. Some results

3.1. In this section we will consider the problem stated in the introduction by using Fuchsian Schottky groups. Let $G =< A_1, A_2 >$ be a Schottky group generated by Möbius transformations $A_1$ and $A_2$. We define $t_j$ $(0 < |t_j| < 1)$ in such a way that $1/t_j$ is the multiplier of $A_j$ $(j = 1, 2)$. Let $p_j$ and $q_j$ be the repelling and the attracting fixed points of $A_j$ $(j = 1, 2)$. We define $\rho \in \mathbf{C} - \{0, 1\}$ by setting $(0, \infty, 1, \rho) = (p_1, q_1, p_2, q_2)$ , where $(z_1, z_2, z_3, z_4)$ is the cross ratio of $z_1, z_2, z_3$ and $z_4$. We say $< A_1, A_2 >$ represents $(t_1, t_2, \rho)$, or $(t_1, t_2, \rho)$ corresponds to $< A_1, A_2 >$. There are eight kinds of classical Schottky groups of real type of genus two (see Sato [6] for detail). Here we will consider the Hausdorff dimension of the limit sets of two kinds of classical Schottky groups, that is, Fuchsian Schottky groups.

DEFINITION 3.1. Let $(t_1, t_2, \rho)$ be the point corresponding to a Schottky group $G =< A_1, A_2 >$.

(1) $G$ is the first kind if $t_1 > 0, t_2 > 0$ and $\rho > 0$.

(2) $G$ is the fourth kind if $t_1 > 0, t_2 > 0$ and $\rho < 0$.

We call the above Schottky group (1) or (2) a *Fuchsian Schottky group* of genus two.

We denote by $R_I\mathfrak{S}_2^0$ ($resp. R_{IV}\mathfrak{S}_2^0$) the space of all classical Schottky groups of type I (resp. type IV).

3.2. PROPOSITION 3.1. *Let $G =< A_1, A_2 >$ be a Fuchsian Schottky group of type I, and let $(t_1, t_2, \rho)$ be the point representing $G$. Let $d(G)$ be the Hausdorff dimension of the limit set of $G$. If $t_1 = t_2$ with $0 < t_1 < \sqrt{5} - 2$ and $\rho = -1/3$ , then*

$$\frac{\log 3}{\frac{2r(1-r)}{1-2r} - \log \frac{r^2}{5+4\sqrt{1+r^2}+3r^2}} \leq d(G) \leq \frac{\log 3}{\log(1-r) - \log r},$$

where $r = 2\sqrt{t}/(1-t)$.

EXAMPLE. If $\rho = -1/3, t_1 = t_2 = 33 - 8\sqrt{17}$, then

$$0.2797 < d(E) \leq 0.5.$$

**Bishop-Jones' theorem** [2]. If $\{G_n\}$ is a sequence of $N$-generated Kleinian groups which converges algebraically to $G$, then

$$d(\Lambda(G)) \leq \liminf d(\Lambda(G_n)).$$

It suffices to consider the Hausdorff dimension of the limit sets of Schottky groups in a fundamental regions for the Schottky modular group acting on $R_I\mathfrak{S}_2^0$ and $R_{IV}\mathfrak{S}_2^0$ (see Sato [6]). By Proposition 3.1 and Bishop-Jones' theorem we have the following.

THEOREM 1.

(1) $\sup\{d(G)|G \in R_{IV}\mathfrak{S}_2^0\} = 1,$

(2) $\inf\{d(G)|G \in R_{IV}\mathfrak{S}_2^0\} = 0.$

THEOREM 2.

(1) $\sup\{d(G)|G \in R_I\mathfrak{S}_2^0\} \geq 1/2.$

(2) $\inf\{d(G)|G \in R_I\mathfrak{S}_2^0\} = 0.$

3.3. We end this paper by presenting some problems.

PROBLEM.

1. Given $(t_1, t_2, \rho)$ corresponding to a classical Schottky group $G = < A_1, A_2 >$, represent the Hausdorff dimension of the limit set of $G$ in terms of $t_1, t_2$ and $\rho$.

2. Find the best upper bound of the Hausdorff dimension of the limit set of classical Schottky groups of genus two (cf. Doyle [3]).

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