Hyperbolic 4g-gons and Fuchsian representations

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This article is an expository summary (with Figures) of [O3].

Abstract. For any marked closed Riemann surface $S$ with genus $g \geq 2$, we can read a corresponding Fuchsian representation from its fundamental domain of hyperbolic 4g-gon, whose boundary consists of geodesic arcs representing generators of $\pi_1(S)$ with certain base point. Also, explicitly given is a conjugate transformation which moves such fundamental 4g-gon to a standard position. Consequently several applications to hyperbolic geometry on $S$ are obtained.

§0. Primitive questions

As is well-known, the hyperbolic regular 4g-gon ($g \geq 2$) in the Poincaré disk, with all the angles equal to $\pi/2g$, gives rise to a marked closed Riemann surface of genus $g$, whose marking is determined by the geodesic arcs in the boundary of the original 4g-gon. This marked Riemann surface is also characterized as the quotient of the Poincaré disk by the image of a faithful, discrete and "orientation preserving" $PSU(1,1)$-representation (we call this “Fuchsian” representation) of the genus $g$ surface group.

Questions. (1) How can we describe the Fuchsian representation (up to conjugacy) for the hyperbolic regular 4g-gon?
(2) How is the “positioning in the Riemann surface” of the base point which corresponds to the vertices of the above 4g-gon?

[Figure 1]

§1. Marked fundamental 4g-gon and its Fuchsian representations

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 2$, and fix a point $p \in \Sigma_g$. Take any hyperbolic metric $h$ on $\Sigma_g$. Then for any $\gamma \in \pi_1(\Sigma_g, p)$, there is a unique (not always simple) geodesic arc from $p$ to $p$, representing $\gamma$. Notice that this geodesic arc has a singularity at $p$ in general. Choose a generator system $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of $\pi_1(\Sigma_g, p)$ with the relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. Suppose that for these $\alpha_1, \ldots, \beta_g$, the corresponding geodesic arc representatives are all simple and have intersections only at $p$. Then cutting $(\Sigma_g, h)$ along
such simple geodesic arcs

\[
(*) \quad \alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1},
\]

we obtain a hyperbolic 4g-gon with boundary corresponding to (\(\ast\)). Hereafter we will assume that our generator systems of \(\pi_1(\Sigma_g, p)\) are chosen so that the order of (\(\ast\)) gives the clockwise orientation for the boundary.

**Definition.** Let \(l = (l_i) \in (R_+)^g, \bar{l} = (\bar{l}_i) \in (R_+)^g\) and \(\theta = (\theta_j) \in (0, 2\pi)^{4g}\). A marked fundamental 4g-gon \(X(l, \bar{l}; \theta)\) is a hyperbolic geodesic 4g-gon in the Poincare' disk with the clockwise namings (\(\ast\)) of its sides, having the following properties:

(i) length of \(\alpha_i = \text{length of } \alpha_i^{-1} = l_i\), length of \(\beta_i = \text{length of } \beta_i^{-1} = \bar{l}_i\) \((i = 1, \ldots, g)\).

(ii) angle between \(\alpha_1\) and \(\beta_1 = \theta_1\), angle between \(\beta_1\) and \(\alpha_1^{-1} = \theta_2, \ldots, \alpha_1^{-1}\) and \(\beta_1^{-1}\) and \(\alpha_1 = \theta_{4g}\) (clockwise order).

(iii) \(\sum_{j=1}^{4g} \theta_j = 2\pi\).

**Remarks.**

(1) From any marked fundamental 4g-gon, we have naturally a genus \(g\) Riemann surface with marking \((\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)\); topologically we will regard all these marked surfaces as those with the same marking \((\alpha_1, \ldots, \beta_g)\). Moreover \(\alpha_1, \cdots, \beta_g\) are specified as elements of \(\pi_1(\Sigma_g, p)\) for the point \(p\) corresponding to the vertices of the 4g-gon.

(2) For any marked Riemann surface of genus \(g\), due to L. Keen [K], there are choices of base point \(p_0\) and inner-automorphism of \(\pi_1(\Sigma_g, p_0)\), so that we can construct a strictly convex marked fundamental 4g-gon whose boundary gives the fixed generators \(\alpha_1, \cdots, \beta_g\) of \(\pi_1(\Sigma_g, p_0)\). Actually Keen's construction is as follows: For any closed curve \(\gamma\) in a Riemann surface, let \(\hat{\gamma}\) be the unique closed geodesic free-homotopic to \(\gamma\). Take \(p_0 = \hat{\alpha}_1 \cap \hat{\beta}_1\) and kill the ambiguity of inner-automorphisms of \(\pi_1(\Sigma_g, p_0)\) in the marking \((\alpha_1, \cdots, \beta_g)\) by specifying the generators \(\alpha_1 = \hat{\alpha}_1, \beta_1 = \hat{\beta}_1\). Then geodesic arcs from \(p_0\) to \(p_0\), corresponding to \(\alpha_1, \cdots, \beta_g\) are shown to be all simple and having intersections only at \(p_0\); thus we obtain a marked fundamental 4g-gon from this.

Now we read a Fuchsian representation from the data of a marked fundamental 4g-gon \(X(l, \bar{l}; \theta)\).

**Notation.** Denote \[
\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in PSU(1, 1) \quad \text{(i.e. } |a|^2 - |b|^2 = 1) \quad \text{by } [a; b].
\]
For \(x \in R/2\pi Z\) and \(y \in R\), let \(e(x) = [e^{ix^2}; 0]\) (rotation of angle \(x\) around 0 in the Poincaré disk) and \(eh(y) = [ch(y/2); sh(y/2)]\) (hyperbolic displacement of length \(y\) along the real axis).
**Theorem 1.** The following gives a corresponding Fuchsian representation \( \rho \) for any marked fundamental \( 4g \)-gon \( X(l, \tilde{l}; \theta) \):

\[
\rho(\alpha_i) = e(\theta_1 + \cdots + \theta_{4i-4}) eh(l_i) e(\pi - (\theta_{4i-2} + \theta_{4i-1})) e(-\theta_1 - \cdots - \theta_{4i-4}), \\
\rho(\beta_i) = e(\theta_1 + \cdots + \theta_{4i-1}) eh(\tilde{l}_i) e(-\pi + (\theta_{4i-3} + \theta_{4i-2})) e(-\theta_1 - \cdots - \theta_{4i-1}).
\]

**Proof.** First fix a position of \( X(l, \tilde{l}; \theta) \) in the Poincare' disk by lifting \( p \in \Sigma_g \) (this is the point corresponding to the vertices of \( X(l, \tilde{l}; \theta) \)) to the origin and also lifting \( \alpha_1 \subset \Sigma_g \) (geodesic arc from \( p \) to \( p \)) to the real axis. In this situation, we will read the corresponding holonomy representation, say, \( \rho \).

**Figure 2**

The lift \( \tilde{\alpha}_i \) of \( \alpha_i \), starting from the origin, is described as follows:

**Figure 3**

From the direction of the real axis, rotate by the angle \( \theta_1 + \cdots + \theta_{4i-4} \) and go straight by the length \( l_i \) (the reaching point will be denoted as \( \rho(\alpha_i) \cdot 0 \)). At the point \( p \), the angle from the incoming direction of \( \alpha_i \) (here \( p \) is the end point) to the outgoing direction of \( \alpha_i \) (here \( p \) is the starting point) is equal to \( \pi - (\theta_{4i-2} + \theta_{4i-1}) \). Thus by \( \rho(\alpha_i) \), the direction of \( \tilde{\alpha}_i \) at 0 is mapped to the direction of angle \( \pi - (\theta_{4i-2} + \theta_{4i-1}) \), measured from the direction of \( \tilde{\alpha}_i \) at \( \rho(\alpha_i) \cdot 0 \). Notice that the above data determine the element \( \rho(\alpha_i) \) of \( PSU(1,1) \). Now from Figure 4, we can see that the right-hand side of the formula for \( \alpha_i \) in the statement of Theorem 1 actually coincides with the element \( \rho(\alpha_i) \).

**Figure 4**

The case for \( \rho(\beta_i) \) is as well. □

**Remark.** Suppose that there is a hyperbolic \( 4g \)-gon \( X(l, \tilde{l}; \theta) \) with the conditions (i), (ii) and (iii)', \( \sum_{j=1}^{4g} \theta_j = \omega \), instead of (iii). Then by a direct calculation, we have, for the \( \rho \) in Theorem 1,

\[
[\rho(\alpha_1), \rho(\beta_1)] \cdots [\rho(\alpha_g), \rho(\beta_g)] e(\omega) = \\
\prod_{i=1}^{g} eh(l_i) e(-\pi + \theta_{4i-3}) eh(\tilde{l}_i) e(-\pi + \theta_{4i-2}) eh(l_i) e(-\pi + \theta_{4i-1}) eh(\tilde{l}_i) e(-\pi + \theta_{4i}),
\]
where $\prod_{i=1}^{g} A; mean A_1 \cdots A_g$. We can see that the right-hand side is equal to $I \in PSU(1, 1)$ (cf. \cite[Lemma]{O2}), which is equivalent to the condition that $X(l, \tilde{l}; \theta)$ is a hyperbolic 4g-gon. Thus the above $X(l, \tilde{l}; \theta)$ and $\rho$ give rise to a developing map of a genus $g$ hyperbolic cone manifold with one cone point $p$ of cone angle $\omega$.

§2. Moving a marked fundamental 4g-gon to the standard position

Suppose that we are given two marked fundamental 4g-gons $X = X(l, l; \theta)$ and $X' = X(l', \tilde{l}; \theta')$; by Theorem 1, we have the corresponding Fuchsian representations $\rho$ and $\rho'$. $X$ and $X'$ give the same marked Riemann surface $[(\Sigma_g, h), (\alpha_1, \cdots, \beta_g)]$ (i.e. the same element of the genus $g$ Teichmüller space $T_g)$ if and only if $\rho$ and $\rho'$ are conjugate to each other by an element of $PSU(1, 1)$. In this section, we will give a criterion for these. Of course, it is possible to choose more than $(6g-6)$ elements in $\pi_1(\Sigma_g)$, so that the geodesic lengths of these elements give a global coordinate system for $T_g$. In comparison, our method is more geometrical and direct one. We will construct conjugate transformations which move $X$ and $X'$ to standard positions (see below). Then applying such transformations to $\rho(\alpha_1), \cdots, \rho(\beta_g)$ and $\rho'(\alpha_1), \cdots, \rho'(\beta_g)$, we can answer whether $\rho$ is conjugate to $\rho'$ or not.

**Definition.** A marked fundamental 4g-gon in the Poincaré disk (or, its associated Fuchsian representation $\rho$ constructed in Theorem 1) is said to be in the standard position if the axes of $\rho(\alpha_1)$ and $\rho(\beta_1)$, denoted by $ax(\rho(\alpha_1))$ and $ax(\rho(\beta_1))$, satisfy that $ax(\rho(\alpha_1)) = \{0\}$. 

**Remark.** $ax(\rho(\alpha_1)), ax(\rho(\beta_1))$ are lifts of the closed geodesics $\hat{\alpha}_1, \hat{\beta}_1 \subset \Sigma_g$, respectively. These axes have a transverse intersection because there exists (see §1, Remarks 1, 2) a path $\epsilon \subset \Sigma_g$, from $p$ (the point corresponding to vertices of the 4-g gon) to $p_0 = \hat{\alpha}_1 \cap \hat{\beta}_1$, such that $c\hat{\alpha}_1 \epsilon^{-1} \simeq \alpha_1$ and $c\hat{\beta}_1 \epsilon^{-1} \simeq \beta_1$ in $\Sigma_g$ (lifts of $p$ and $\epsilon$ determine the point $ax(\rho(\alpha_1)) \cap ax(\rho(\beta_1))$).

**Theorem 2.1.** For any marked fundamental 4g-gon $X(l, \tilde{l}; \theta)$, we can explicitly give the conjugate transformation which moves its associated Fuchsian representation $\rho$ to the standard position.

**Proof.** The idea is to use the Iwasawa decomposition of $PSU(1, 1)$: Let $K = \{e^{i\varphi}; 0; \varphi \in$
$R/2πZ$, $N = \{[1 + ir; ir]; r \in R\}$ and $A = \{[ch(\lambda); sh(\lambda)]; \lambda \in R\}$. Then we have $PSU(1, 1) = ANK$ and we will determine the desired transformation, first for the components of $N$ and $K$, and second for the component of $A$.

**Step 1.** We will determine the element $P(\rho(\alpha_1)) = nk$ ($n \in N$ and $k \in K$) such that $P(\rho(\alpha_1)) \circ \rho(\alpha_1) \circ P(\rho(\alpha_1))^{-1} = [ch(L); sh(L)]$ for some $L > 0$.

Actually we can treat with this problem in a more general setting: Given $[p_1 + ip_2; q_1 + iq_2] \in PSU(1, 1)$ with $p_1 > 1$, we will solve the following equation for $n = [1 + ir; ir] \in N$ and $k = [e^{i\varphi}; 0] \in K$;

\[
(2.1) \quad nk[p_1 + ip_2; q_1 + iq_2](nk)^{-1} = [ch(L); sh(L)].
\]

By a direct calculation, we can see that (2.1) holds if and only if $p_1 = ch(L), q_2 \cos(2\varphi) + q_1 \sin(2\varphi) = p_2, q_1 \cos(2\varphi) - q_2 \sin(2\varphi) = sh(L)$ and $r = -p_2/2sh(L)$. We look for the solution with $L > 0$, so $sh(L) = (p_1^2 - 1)^{1/2}$. Now let $\Psi \in R/2\pi Z$ be the angle with $\cos\Psi = q_1/(q_1^2 + q_2^2)^{1/2}$ and $\sin\Psi = q_2/(q_1^2 + q_2^2)^{1/2}$. (Notice that if $q_1 = q_2 = 0$, then we have $p_2 = 0$ and thus $p_1 = 1$.) Then we have $\sin(2\varphi + \Psi) = p_2/(q_1^2 + q_2^2)^{1/2}$ and $\cos(2\varphi + \Psi) = (p_1^2 - 1)^{1/2}/(q_1^2 + q_2^2)^{1/2}$. (Notice that $|p_2/(q_1^2 + q_2^2)^{1/2}| \leq 1$ if and only if $p_1^2 \geq 1$.) These formulas determine $2\varphi + \Psi \in R/2\pi Z$, and thus determine $\varphi \in R/\pi Z$. In this way we can determine $r \in R$ and $\varphi \in R/\pi Z$ from (2.1). (In particular for $\rho(\gamma) = e(\psi_1)eh(s)e(\psi_2)$, let $\Phi = (\psi_1 + \psi_2)/2 \in R/2\pi Z$ with $\cos\Phi > 0$ and $\Psi = (\psi_1 - \psi_2)/2 \in R/2\pi Z$, so that $\Phi + \Psi = \psi_1$. Then $P(\rho(\gamma)) = [1 + ir; ir][e^{i\varphi}; 0]$ is determined by $e^{i(2\varphi + \Psi)} = ((\cos\Phi)^2ch(s/2)^2 - 1)^{1/2} + i\sin\Phi/th(s/2)$ and $r = -\tan(2\varphi + \Psi)/2$.)

**Step 2.** Because the group $A$ consists of hyperbolic displacements along the real axis and $ax(\rho(\alpha_1))$ and $ax(\rho(\beta_1))$ intersect transversely, there exist unique elements $eh(2\lambda), eh(2\tilde{\lambda}) \in A$ such that

\[
(2.2) \quad eh(2\lambda)P(\rho(\beta_1))(eh(2\lambda)P(\rho(\alpha_1)))^{-1} \cdot 0 = 0
\]

(here $\cdot$ means a fractional linear transformation; $[a; b] \cdot z = (az + b)/(\overline{b}z + \overline{a})$).

This $eh(2\lambda) \in A$ is exactly the one we want; $eh(2\lambda)P(\rho(\alpha_1))$ moves $\rho$ to the standard position. To get the formula for $\lambda$, we have to solve the equation (2.2) for $\lambda = \lambda(\rho)$ and
\[ \tilde{\lambda} = \tilde{\lambda}(\rho). \] Write \( P(\rho(\beta_1)) \circ P(\rho(\alpha_1))^{-1} = [a_1 + ia_2; b_1 + ib_2]. \) Then (2.2) is equivalent to 
\[-sh(\lambda - \tilde{\lambda}) a_1 + ch(\lambda - \tilde{\lambda}) b_1 = 0 \quad \text{and} \quad -sh(\lambda + \tilde{\lambda}) a_2 + ch(\lambda + \tilde{\lambda}) b_2 = 0.\] Notice that we have \( a_1 \neq 0 \) and \( a_2 \neq 0; \) otherwise the axes \( ax(\rho(\alpha_1)) \) and \( ax(\rho(\beta_1)) \) would coincide (orientation-preservingly or reversingly) with each other. Thus from \( th(\lambda - \tilde{\lambda}) = b_1/a_1 \) and \( th(\lambda + \tilde{\lambda}) = b_2/a_2 \), we can get the formula for \( \lambda: \)
\[ sh(\lambda) = \left| \frac{(a_1 + b_1)(a_2 + b_2)}{(a_1 - b_1)(a_2 - b_2)} \right|^{1/4} - \left| \frac{(a_1 - b_1)(a_2 - b_2)}{(a_1 + b_1)(a_2 + b_2)} \right|^{1/4} \]. □

**Remarks.**

(1) In the above, \( |a_1| > |b_1| \) and \( |a_2| > |b_2| \) must be satisfied because (2.2) has unique solutions \( \lambda \) and \( \tilde{\lambda} \).

(2) Step 1 and Step 2 can be automatically applied to two hyperbolic transformations \( H_1 \), \( H_2 \) with their axes having transverse intersections; we can get the explicit formula for the transformation which moves \( ax(H_1) \) to the real axis and \( ax(H_1) \cap ax(H_2) \) to 0.

As a summary of this section, we shall record the following

**Theorem 2.2.** For two marked fundamental 4g-gons \( X \) and \( X' \), let \( \rho \) and \( \rho' \) be their associated Fuchsian representations constructed in Theorem 1. Then \( \rho \) and \( \rho' \) are conjugate in \( PSU(1, 1) \) (i.e. give the same element of \( T_g \) if and only if \( eh(2\lambda) P(\rho(\alpha_1)) \circ \rho(\gamma) \circ (eh(2\lambda) P(\rho(\alpha_1)))^{-1} = eh(2\lambda') P(\rho'(\alpha_1)) \circ \rho'(\gamma) \circ (eh(2\lambda') P(\rho'(\alpha_1)))^{-1} \) for \( \gamma = \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \) where \( P(\ ) \) is given in Theorem 2.1, Step 1, and \( \lambda = \lambda(\rho) \) and \( \lambda' = \lambda(\rho') \) are given in Theorem 2.1, Step 2. □

**§3. Applications**

Once we know a Fuchsian representation (Theorem 1) and the standard position (Theorem 2.1) of a marked fundamental 4g-gon, we can investigate hyperbolic geometry of closed Riemann surfaces, in detail and in a direct way.

**Proposition 3.1.** For any marked fundamental 4g-gon \( X(l, \bar{l}; \theta) \), let \( \rho : \pi_1(\Sigma_g, p) \to PSU(1, 1) \) be its Fuchsian representation given in Theorem 1 (recall that, here \( p \) is corresponding to the vertices, 0 is a lift of \( p \) and the real axis is a lift of \( \alpha_1 \)). Let \( \delta \subset \Sigma_g \) be the geodesic arc from \( p_0 = \tilde{\alpha}_1 \cap \tilde{\beta}_1 \), to \( p \) such that \( \delta^{-1} \tilde{\alpha}_1 \delta \simeq \alpha_1 \) and \( \delta^{-1} \tilde{\beta}_1 \delta \simeq \beta_1 \). Then in the standard position of \( X(l, \bar{l}; \theta) \), we can write down the positioning of the lift \( \delta \) of \( \delta \), starting from 0.
Proof. We use the notation of Theorem 2.1. The end-point $w$ of $\tilde{\delta}$ is given by $w = e^{h(2\lambda)}P(\rho(\alpha_1)) \cdot 0$. Explicitly we have the following formula:

$$w = (e^{h(\lambda)}r \sin \varphi + sh(\lambda)(\cos \varphi - r \sin \varphi) + i(e^{h(\lambda)}r \cos \varphi - sh(\lambda)(r \cos \varphi + \sin \varphi))/$$

$$\left( e^{h(\lambda)}(\cos \varphi - r \sin \varphi) + sh(\lambda)r \sin \varphi - i(e^{h(\lambda)}(r \cos \varphi + \sin \varphi) - sh(\lambda)r \cos \varphi) \right).$$

\[ \square \]

Proposition 3.2. For any marked fundamental $4g$-gon $X(l, \bar{l}; \theta)$ and its associated Fuchsian representation $\rho$ constructed in Theorem 1, let $X(l^0, \bar{l}^0; \theta^0)$ and $\rho_0 : \pi_1(\Sigma_g, p_0) \to PSU(1,1)$ be the unique marked fundamental $4g$-gon and its associated Fuchsian representation such that $\alpha_1 = \hat{\alpha}_1, \beta_1 = \hat{\beta}_1 \cap \hat{\beta}_1 = \{p_0\}$ and $\rho_0$ is conjugate to $\rho$ in $PSU(1,1)$. Then we can write down these “canonical” parameters $l^0, \bar{l}^0$ and $\theta^0$ as functions of $l, \bar{l}$ and $\theta$.

Proof. By the construction of $\rho$ in Theorem 1, $\rho_0$ is by itself in the standard position. Thus we have $\rho_0(\gamma) = e^{h(2\lambda)}P(\rho(\alpha_1)) \circ \rho(\gamma) \circ (e^{h(2\lambda)}P(\rho(\alpha_1)))^{-1}$ (here $\gamma \in \pi_1(\Sigma_g, p_0)$ and $\gamma \in \pi_1(\Sigma_g, p)$ are identified by the path $\delta$ in Proposition 3.1). Let $\rho_0(\gamma) \cdot 0 = z(\gamma)$. Then $l^0$ and $\bar{l}^0$ are given by $l^0 = d_P(0, z(\alpha_i))$ and $\bar{l}^0 = d_P(0, z(\beta_i))$, where $d_P(0, z) = \log\{(1+|z|)/(1-|z|)\}$, the Poincare metric.

[Figure 7]

Let us deduce the formula for $\theta^0_i$, for example for $\theta^0_5$, the angle between the sides $\alpha_2$ and $\beta_2$ of $X(\ell_0, \bar{\ell}_0; \theta^0)$. In our orientation convention, $\theta^0_5$ is nothing but the angle from the vector $z(\alpha_2^{-1})$ to $z(\beta_2)$; thus we have $e^{i\theta^0_5} = (z(\beta_2)/|z(\beta_2)|)/(z(\alpha_2^{-1})/|z(\alpha_2^{-1})|)$. \[ \square \]

References.


