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Kyoto University
UNIFORMLY PERFECT RIEMANN SURFACES AND ITS LENGTH SPECTRA

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§1. Uniform perfectness of the hyperbolic Riemann surface

In this note, we shall consider only the hyperbolic Riemann surfaces $R$ endowed with the Poincaré metric $\rho_R(z)dz$ of constant negative curvature $-4$. We denote by $D_R(p, r)$ the hyperbolic disk in $R$ centered at $p \in R$ of radius $r$. We set $\sigma_R(p) = \sup\{r > 0; D_R(p, r) \text{ is simply connected}\}$ and $H_R = \inf_{p \in R} \sigma_R(p)$, called the injectivity radius of $R$.

According to [LM], $R$ is called uniformly perfect if the injectivity radius $H_R$ is positive (including infinity).

Let $\mathcal{R}_R$ be the set of essential ring domains in $R$, where ring domain $R_0$ is essential if the inclusion map $R_0 \hookrightarrow R$ is $\pi_1$-injective. The module $m(R_0)$ of $R_0 \in \mathcal{R}_R$ is defined by the number $m$ such that $R_0$ is conformally equivalent to the annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$. The core curve of $R_0 \in \mathcal{R}_R$, denoted by $\text{core}(R_0)$, is the unique simple closed geodesic of $R_0$ (with complete hyperbolic metric).

On $R$, another important continuous metric $\hat{\rho}_R$, called the Hahn metric, is defined by

$$\hat{\rho}_R(z)|dz| = \inf_G \rho_G(z)|dz|,$$

where $G$ ranges over simply connected domains with $p \in G$ and $z$ is a fixed local coordinate around $p \in R$. By the monotonicity of the Poincaré metric, $\hat{\rho}_R \geq \rho_R$.

We set $M_R = \sup_{R_0 \in \mathcal{R}_R} m(R_0)$ and $K_R = \sup_{p \in R} \hat{\rho}_R(p)$. Now we have the following estimates. (The part (2) is due to Gotoh [G].)

**Theorem 1.1.**

(1) $2H_R \leq \pi^2/M_R \leq 2H Re^{2H_R}$.

(2) $\frac{1}{4} \coth H_R \leq K_R \leq \coth H_R$.

**Corollary 1.2.**

The following conditions are mutually equivalent.

(1) $R$ is uniformly perfect (i.e., $H_R > 0$),

(2) $M_R < +\infty$,

(3) $K_R < +\infty$.

We shall close this section by exhibiting a simple application of uniform perfectness. Let $A_2(R)$ and $B_2(R)$ be the complex Banach spaces of holomorphic quadratic differentials $\varphi$ on $R$ with norms $\|\varphi\|_1 = \iint_R |\varphi(x)|dxdy$ and $\|\varphi\|_\infty = \sup_{R} |\varphi|\rho_R^{-2}$, respectively. We set $\kappa_R = \sup\{\|\varphi\|_\infty; \varphi \in A_2(R), \|\varphi\|_1 = 1\}$. 


Theorem 1.3. \( \kappa_R \leq \frac{1}{\pi} \coth^2(H_R) \). In particular, \( A_2(R) \subset B_2(R) \), if \( R \) is uniformly perfect.

Remark. Matsuzaki [M] proved this theorem in a sharper form, and with full generality. By our result, we see that \( \kappa_R = O(H_R^{-2}) \) as \( H_R \to 0 \), but, in fact, \( \kappa_R = O(H_R^{-1}) \) as \( H_R \to 0 \) by an argument using the Marden-Margulis constant (see [M]).

Proof of Theorem 1.3. Fix an arbitrary point \( p \) in \( R \). Let \( \pi : \Delta = \{ |z| < 1 \} \to R \) be a holomorphic universal covering map with \( \pi(0) = p \). We denote by \( \tilde{\varphi} \) the pull-back of \( \varphi \in A_2(R) \) by \( \pi \). Then \( |\varphi \rho_R^{-2}(p)| = |\tilde{\varphi}(0)| \) by the conformal invariance of the differential forms. On the other hand, for \( r = \tanh(\sigma(p)) \), by the mean value property, we have

\[
\tilde{\varphi}(0) = \frac{1}{\pi r^2} \iint_{|z| < r} \tilde{\varphi}(z) dxdy
\]

Since \( \pi \) is injective in \( D_{\Delta}(\text{O}, \sigma(p)) \), we have

\[
|\varphi \rho_R^{-2}|(p) = |\tilde{\varphi}(0)| \leq \frac{1}{\pi r^2} \iint_{|z| < r} |\tilde{\varphi}(z)| dxdy
\]

\[
\leq \frac{1}{\pi r^2} \int_R |\varphi| = \frac{1}{\pi r^2} \|\varphi\|_1 \leq \frac{1}{\pi} \coth^2 H_R \cdot \|\varphi\|_1.
\]

Thus we have the assertion that \( \|\varphi\|_\infty \leq \frac{1}{\pi} \coth^2 H_R \cdot \|\varphi\|_1 \). \( \square \)

§2. HYPERBOLIC AND EXTREMAL LENGTHS

We denote by \( S_R \) the set of all free homotopy classes of non-trivial simple closed loops in \( R \). The hyperbolic length \( \ell[\alpha] \) of \( [\alpha] \in S_R \) is defined by

\[
\ell[\alpha] = \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \rho_R(z) |dz|.
\]

Let \( \pi : \Delta \to R \) be a holomorphic universal covering map of \( R \) and \( \Gamma \) its covering transformation group. If an element \( \gamma \in \Gamma \) covers \( [\alpha] \in S_R \), then we have \( |\text{tr}\gamma| = 2 \cosh \ell[\alpha] \) (where \( |\text{tr}\gamma| \) denotes the absolute value of the trace of the element of \( SL(2, \mathbb{R}) \) representing \( \gamma \)).

Thus we can easily see that

\[
H_R = \frac{1}{2} \inf_{[\alpha] \in S_R} \ell[\alpha]
\]

and

\[
2 \cosh(2H_R) = \inf_{\gamma \in \Gamma \setminus \{1\}} |\text{tr}\gamma|.
\]

In particular, the uniform perfectness of a Riemann surface \( R \) means that the bottom of its length spectrum is positive. Fernández [F] showed that the exponent of convergence of the Fuchsian group \( \Gamma \) is less than 1 for any uniformly perfect plane domain \( R \). See also [A] and [Gon]. It is an interesting problem to extend Fernández'
result to general Riemann surface case. We should remark that, at least, compact Riemann surfaces are always uniformly perfect, but with exponent 1.

The extremal length $E[\alpha]$ of $[\alpha] \in S_R$ is defined by

$$E[\alpha] = \sup_{\tau} \frac{\left( \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \tau(z) |dz| \right)^2}{\int_{D} \tau(z)^2 |dz|^2},$$

where the supremum is taken over all Borel measurable conformal metrics $\tau = \tau(z)|dz|$ on $R$. As for this, the following result due to Jenkins-Strebel is fundamental.

**Theorem 2.1 (cf. [St]).** For any $[\alpha] \in S_R$ with $E[\alpha] > 0$, there exists an integrable holomorphic quadratic differential $\varphi_0$ (Jenkins-Strebel differential) with closed trajectory homotopic to $\alpha$, whose characteristic ring domain $R_0 \in \mathcal{R}_R$ satisfies the following conditions.

1. $E[\alpha] = \frac{\left( \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \varphi^{1/2} |dz| \right)^2}{\int_{R} |\varphi| |dz|^2}$,
2. $m(R_0) = \frac{2\pi}{E[\alpha]}$,
3. $m(R_1) \leq m(R_0)$ for all $R_1 \in \mathcal{R}_R$ with Core$(R_1) \in [\alpha]$.

**Corollary 2.2.**

$$\inf_{[\alpha] \in S_R} E[\alpha] = \frac{2\pi}{M_R}.$$

The following theorem connects amounts of the hyperbolic and extremal lengths, and from it we can directly deduce Theorem 1.1 (1).

**Theorem 2.3.**

$$\frac{2}{\pi} \ell[\alpha] \leq E[\alpha] \leq \frac{\ell[\alpha]}{\arctan \left( \frac{1}{\sinh \ell[\alpha]} \right)}.$$

By an elementary calculation, we know that $\frac{\pi}{2} < e^x \arctan \left( \frac{1}{\sinh x} \right) < 2$ for any $x > 0$, we have the next

**Corollary 2.4.**

$$\frac{2}{\pi} \ell[\alpha] \leq E[\alpha] \leq \frac{2}{\pi} \ell[\alpha] e^{\ell[\alpha]}.$$

**Remark.** Maskit showed the similar result that $\frac{2}{\pi} \ell[\alpha] \leq E[\alpha] \leq \ell[\alpha] e^{\ell[\alpha]}$ in [Mas]. On the other hand, Matsuzaki [M] showed the next

**Theorem 2.5.**

$$E[\alpha] \leq \kappa_R \ell[\alpha]^2.$$

**Proof.** Let $\varphi_0$ be the holomorphic differential on $R$ with $\|\varphi_0\|_1 = 1$ which gives an extremal metric $|\varphi_0|^{1/2} |dz|$ as in Theorem 2.1. Then, for $\alpha' \simeq \alpha$,

$$E[\alpha]^{1/2} \leq \int_{\alpha'} |\varphi_0|^{1/2} |dz| = \int_{\alpha'} |\varphi_0 \rho_R^{-2}|^{1/2} \cdot |\rho_R| dz| \leq \|\varphi_0\|_{\infty}^{1/2} \int_{\alpha'} |\rho_R| dz|.$$

Since $\alpha'$ is arbitrary, we obtain that $E[\alpha] \leq \|\varphi_0\|_{\infty} \ell[\alpha]^2$. □

By combining Theorem 1.3, we have the next
Corollary 2.6. $E[\alpha] \leq \frac{1}{\pi} \coth^2 H_R \cdot \ell[\alpha]^2$.

Remark. Of course, by a refined result of Matsuzaki [M], we shall have a better estimate than the above.

Corollary 2.7.

$$\frac{\pi^2}{M_R} \leq 2H_R^2 \coth^2 H_R$$

By this we can see that $1/M_R = O(H_R^2)$ as $H_R \to \infty$. Thus, the above estimate is much better than one in Theorem 1.1 (1) as $H_R$ tends to $\infty$. The author does not know whether the exponent 2 is best possible or not.

Finally, we refer to the quasi-invariance of these amounts. Let $f : R \to R'$ be $K$-quasiconformal homeomorphism, and set $\alpha' = f(\alpha)$. Then, it is clear that $E[\alpha]/K \leq E[\alpha'] \leq KE[\alpha]$. Moreover it also holds that $\ell[\alpha]/K \leq \ell[\alpha'] \leq K\ell[\alpha]$ (see Wolpert [W]).

§3. UNIFORMLY PERFECT PLANE DOMAINS

As we have seen in the previous sections, the uniform perfectness can be defined by the intrinsic hyperbolic geometry of the surface. But, the uniform perfectness seems to have its most importance in plane domains. The various equivalent definitions of uniform perfectness for plane domains tell us the richness of this notion.

In the sequel, let $D$ be subdomain of $\hat{\mathbb{C}}$ with $\#(\hat{\mathbb{C}} \setminus D) \geq 3$. And, let $\pi : \Delta \to D$ be a holomorphic universal covering map. We set $N_D = \|S_\pi\|_\Delta := \sup_{z \in \Delta} |S_\pi(z)|(1-|z|^2)^2$, where $S_\pi = (\pi''/\pi')' - \frac{1}{2}(\pi''/\pi')^2$ is the Schwarzian derivative of $\pi$. Note that $N_D$ does not depend on particular choice of $\pi$.

By the Nehari-Kraus theorem, we know that $N_D \leq 6$ if $D$ is simply connected. Now we state the supplementary result concerning with $N_D$.

Theorem 3.1 (Minda [Mi]). If $D$ is not simply connected, we have

$$\frac{\pi^2}{2H_D} + 2 \leq N_D \leq 6 \coth^2 H_D.$$  

Let $A_D$ denote a subclass of $\mathcal{R}_D$ consisting of all round annuli, and set $A_D := \sup_{R_0 \in A_D} m(R_0)(\leq M_D)$. Then, we can show the following result.

Theorem 3.2 (cf. McMullen [Mc]). If $D \subset \mathbb{C}$, it holds that $M_D \leq A_D + 5\log 2$.

In case of $\infty \in D$, we have the next auxiliary result.

Theorem 3.3. If $L \in \text{M"ob}$, $\frac{1}{2}A_L(D) - \log 4/3 \leq A_D$.

If $D \subset \mathbb{C}$, further we define the domain constant

$$c_D = \inf_{z \in D} \delta_D(z)\rho_D(z),$$

where $\delta_D(z) = \text{dist}(z, \partial D) = \inf_{a \in \partial D} |z - a|$.

That is, $c_D$ is the infimum of the ratio of the Poincaré metric $\rho_D(z) |dz|$ to the quasi-hyperbolic one $|dz|/\delta_D(z)$. We should note that $\delta_D(z)\rho_D(z) \leq 1$ for any $z \in D$, thus $c_D \leq 1$. Concerning this, the similar result as Theorem 1.1 (2) is verified.
Theorem 3.4 (Minda [Mi]).

\[ \frac{\tanh H_D}{4} \leq c_D \leq \frac{2\sqrt{3}}{\pi} H_D. \]

Remark. The assumption that \( \infty \notin D \) is essential for \( c_D \). In fact, if \( D = \Delta^* = \hat{\mathbb{C}} \setminus \overline{\Delta} \), we have \( \delta_{\Delta^*}(z) = |z| - 1 \) and \( \rho_{\Delta^*}(z) = \frac{1}{|z|^2 - 1} \), therefore \( \delta_{\Delta^*}(z) \rho_{\Delta^*}(z) = \frac{1}{|z| + 1} \to 0 \) as \( z \to \infty \).

Finally, we summarize our results.

Theorem 3.4. Let \( D \) be a plane domain of hyperbolic type. Then the following conditions are mutually equivalent.

1. \( H_D > 0 \),
2. \( M_D < \infty \),
3. \( A_D < \infty \),
4. \( N_D < \infty \),
5. \( c_D > 0 \) (if \( D \subset \mathbb{C} \)).

The other features of uniformly perfect domains can be seen in [Pom1] and [Pom2].

REFERENCES


