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UNIFORMLY PERFECT RIEMANN SURFACES AND ITS LENGTH SPECTRA
一様完全リーマン面と LENGTH SPECTRUM

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§1. UNIFORM PERFECTNESS OF THE HYPERBOLIC RIEMANN SURFACE

In this note, we shall consider only the hyperbolic Riemann surfaces R endowed with the Poincaré metric $\rho_R(z)|dz|$ of constant negative curvature -4. We denote by $D_R(p,r)$ the hyperbolic disk in R centered at $p \in R$ of radius r. We set $\sigma_R(p) = \sup\{r > 0; D_R(p,r)$ is simply connected $\}$ and $H_R = \inf_{p\in R} \sigma_R(p)$, called the *injectivity radius* of R.

According to [LM], R is called *uniformly perfect* if the injectivity radius H_R is positive (including infinity).

Let $\mathcal{R}_R$ be the set of essential ring domains in R, where ring domain R_0 is essential if the inclusion map $R_0 \hookrightarrow R$ is π_1-injective. The module $m(R_0)$ of $R_0 \in \mathcal{R}_R$ is defined by the number m such that R_0 is conformally equivalent to the annulus $\{z \in \mathbb{C}; 1 < |z| < e^m\}$. The core curve of $R_0 \in \mathcal{R}_R$, denoted by $\mathrm{Core}(R_0)$, is the unique simple closed geodesic of R_0 (with complete hyperbolic metric).

On R, another important continuous metric $\hat\rho_R$, called the *Hahn metric*, is defined by

$$\hat\rho_R(z)|dz| = \inf_G \rho_G(z)|dz|,$$

where G ranges over simply connected domains with $p \in G$ and z is a fixed local coodinate around $p \in R$. By the monotonicity of the Poincaré metric, $\hat\rho_R \geq \rho_R$.

We set $M_R = \sup_{R_0\in\mathcal{R}_R} m(R_0)$ and $K_R = \sup_{p\in R} \frac{\hat\rho_R}{\rho_R}(p)$. Now we have the following estimates. (The part (2) is due to Gotoh [G].)

Theorem 1.1.
(1) $2H_R \leq \pi^2/M_R \leq 2H_R e^{2H_R}$.
(2) $\frac{1}{4}\coth H_R \leq K_R \leq \coth H_R$.

Corollary 1.2.
The following conditions are mutually equivalent.
(1) R is uniformly perfect (i.e., $H_R > 0$),
(2) $M_R < +\infty$,
(3) $K_R < +\infty$.

We shall close this section by exhibiting a simple application of uniform perfectness. Let $A_2(R)$ and $B_2(R)$ be the complex Banach spaces of holomorphic quadratic differentials φ on R with norms $\|\varphi\|_1 = \iint_R |\varphi(z)|dxdy$ and $\|\varphi\|_\infty = \sup_R |\varphi|\rho_R{}^{-2}$, respectively. We set $\kappa_R = \sup\{\|\varphi\|_\infty; \varphi \in A_2(R), \|\varphi\|_1 = 1\}$.

Theorem 1.3. $\kappa_R \leq \frac{1}{\pi}\coth^2(H_R)$. *In particular, $A_2(R) \subset B_2(R)$, if R is uniformly perfect.*

Remark. Matsuzaki [M] proved this theorem in a sharper form, and with full generality. By our result, we see that $\kappa_R = O({H_R}^{-2})$ as $H_R \to 0$, but, in fact, $\kappa_R = O({H_R}^{-1})$ as $H_R \to 0$ by an argument using the Marden-Margulis constant (see [M]).

Proof of Theorem 1.3. Fix an arbitrary point p in R. Let $\pi : \Delta = \{|z| < 1\} \to R$ be a holomorphic universal covering map with $\pi(0) = p$. We denote by $\tilde{\varphi}$ the pull-back of $\varphi \in A_2(R)$ by π. Then $|\varphi {\rho_R}^{-2}|(p) = |\tilde{\varphi}(0)|$ by the conformal invariance of the differential forms. On the other hand, for $r = \tanh(\sigma(p))$, by the mean value property, we have

$$\tilde{\varphi}(0) = \frac{1}{\pi r^2}\iint_{|z|<r} \tilde{\varphi}(z)dxdy$$

Since π is injective in $D_\Delta(0, \sigma(p))$, we have

$$\begin{aligned} |\varphi {\rho_R}^{-2}|(p) = |\tilde{\varphi}(0)| &\leq \frac{1}{\pi r^2}\iint_{|z|<r} |\tilde{\varphi}(z)|dxdy \\ &\leq \frac{1}{\pi r^2}\iint_R |\varphi| = \frac{1}{\pi r^2}\|\varphi\|_1 \leq \frac{1}{\pi}\coth^2 H_R \cdot \|\varphi\|_1. \end{aligned}$$

Thus we have the assertion that $\|\varphi\|_\infty \leq \frac{1}{\pi}\coth^2 H_R \cdot \|\varphi\|_1$. □

§2. Hyperbolic and extremal lengths

We denote by $\mathcal{S}_R$ the set of all free homotopy classes of non-trivial simple closed loops in R. The hyperbolic length $\ell[\alpha]$ of $[\alpha] \in \mathcal{S}_R$ is defined by

$$\ell[\alpha] = \inf_{\alpha' \in [\alpha]} \int_{\alpha'} \rho_R(z)|dz|.$$

Let $\pi : \Delta \to R$ be a holomorphic universal covering map of R and Γ its covering transformation group. If an element $\gamma \in \Gamma$ covers $[\alpha] \in \mathcal{S}_R$, then we have $|\mathrm{tr}\gamma| = 2\cosh \ell[\alpha]$ (where $|\mathrm{tr}\gamma|$ denotes the absolute value of the trace of the element of $SL(2, \mathbb{R})$ representing γ).

Thus we can easily see that

$$H_R = \frac{1}{2}\inf_{[\alpha]\in\mathcal{S}_R} \ell[\alpha]$$

and

$$2\cosh(2H_R) = \inf_{\gamma\in\Gamma\setminus\{1\}} |\mathrm{tr}\gamma|.$$

In particular, the uniform perfectness of a Riemann surface R means that the bottom of its length spectrum is positive. Fernández [F] showed that the exponent of convergence of the Fuchsian group Γ is less than 1 for any uniformly perfect plane domain R. See also [A] and [Gon]. It is an interesting problem to extend Fernández' result to general Riemann surface case. We should remark that, at least, compact Riemann surfaces are always uniformly perfect, but with exponent 1.

The extremal length $E[\alpha]$ of $[\alpha] \in \mathcal{S}_R$ is defined by

$$E[\alpha] = \sup_{\tau} \frac{\left(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} \tau(z)|dz|\right)^2}{\iint_D \tau(z)^2 |dz|^2},$$

where the supremum is taken over all Borel measurable conformal metrics $\tau = \tau(z)|dz|$ on R. As for this, the following result due to Jenkins-Strebel is fundamental.

Theorem 2.1 (cf. [St]). *For any $[\alpha] \in \mathcal{S}_R$ with $E[\alpha] > 0$, there exists an integrable holomorphic quadratic differential φ_0 (Jenkins-Strebel differential) with closed trajectry homotopic to α, whose characteristic ring domain $R_0 \in \mathcal{R}_R$ satisfies the following conditions.*

(1) $E[\alpha] = \frac{(\inf_{\alpha' \in [\alpha]} \int_{\alpha'} |\varphi|^{1/2}|dz|)^2}{\iint_R |\varphi| dxdy}$,

(2) $m(R_0) = \frac{2\pi}{E[\alpha]}$,

(3) $m(R_1) \leq m(R_0)$ *for all* $R_1 \in \mathcal{R}_R$ *with* $\mathrm{Core}(R_1) \in [\alpha]$.

Corollary 2.2.

$$\inf_{[\alpha] \in \mathcal{S}_R} E[\alpha] = \frac{2\pi}{M_R}.$$

The following theorem connects amounts of the hyperbolic and extremal lengths, and from it we can directly deduce Theorem 1.1 (1).

Theorem 2.3.

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{\ell[\alpha]}{\arctan(\frac{1}{\sinh \ell[\alpha]})}.$$

By an elementary calculation, we know that $\frac{\pi}{2} < e^x \arctan(\frac{1}{\sinh x}) < 2$ for any $x > 0$, we have the next

Corollary 2.4.

$$\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \frac{2}{\pi}\ell[\alpha]e^{\ell[\alpha]}.$$

Remark. Maskit showed the similar result that $\frac{2}{\pi}\ell[\alpha] \leq E[\alpha] \leq \ell[\alpha]e^{\ell[\alpha]}$ in [Mas].

On the other hand, Matsuzaki [M] showed the next

Theorem 2.5.

$E[\alpha] \leq \kappa_R \ell[\alpha]^2$.

Proof. Let φ_0 be the holomorphic diffrential on R with $\|\varphi_0\|_1 = 1$ which gives an extremal metric $|\varphi_0|^{1/2}|dz|$ as in Theorem 2.1. Then, for $\alpha' \simeq \alpha$,

$$E[\alpha]^{1/2} \leq \int_{\alpha'} |\varphi_0|^{1/2}|dz| = \int_{\alpha'} |\varphi_0 \rho_R{}^{-2}|^{1/2} \cdot \rho_R |dz| \leq \|\varphi_0\|_\infty{}^{1/2} \int_{\alpha'} \rho_R |dz|.$$

Since α' is arbitrary, we obtain that $E[\alpha] \leq \|\varphi_0\|_\infty \ell[\alpha]^2$. □

By combining Theorem 1.3, we have the next

Corollary 2.6. $E[\alpha] \le \frac{1}{\pi}\coth^2 H_R \cdot \ell[\alpha]^2$.

Remark. Of course, by a refined result of Matsuzaki [M], we shall have a better estimate than the above.

Corollary 2.7.

$$\pi^2/M_R \le 2H_R^2 \coth^2 H_R$$

By this we can see that $1/M_R = O(H_R^2)$ as $H_R \to \infty$. Thus, the above estimate is much bettar than one in Theorem 1.1 (1) as H_R tends to ∞. The author does not know whether the exponent 2 is best possible or not.

Finally, we refer to the quasi-invariance of these amounts. Let $f : R \to R'$ be K-quasiconformal homeomorphism, and set $\alpha' = f(\alpha)$. Then, it is clear that $E[\alpha]/K \le E[\alpha'] \le KE[\alpha]$. Moreover it also holds that $\ell[\alpha]/K \le \ell[\alpha'] \le K\ell[\alpha]$ (see Wolpert [W]).

§3. Uniformly perfect plane domains

As we have seen in the previous sections, the uniform perfectness can be defined by the intrinsic hyperbolic geometry of the surface. But, the uniform perfectness seems to have its most importance in plane domains. The various equivalent definitions of uniform perfectness for plane domains tell us the richness of this notion.

In the sequel, let D be subdomain of $\widehat{\mathbb{C}}$ with $\#(\widehat{\mathbb{C}} \setminus D) \ge 3$. And, let $\pi : \Delta \to D$ be a holomorophic universal covering map. We set $N_D = \|S_\pi\|_\Delta := \sup_{z\in\Delta} |S_\pi(z)|(1-|z|^2)^2$, where $S_\pi = (\pi''/\pi')' - \frac{1}{2}(\pi''/\pi')^2$ is the Schwarzian derivative of π. Note that N_D does not depend on particular choice of π.

By the Nehari-Kraus theorem, we know that $N_D \le 6$ if D is simply connected. Now we state the supplementary result concerning with N_D.

Theorem 3.1 (Minda [Mi]). *If D is not simply connected, we have*

$$\frac{\pi^2}{2H_D} + 2 \le N_D \le 6\coth^2 H_D.$$

Let $\mathcal{A}_D$ denote a subclass of $\mathcal{R}_R$ consisting of all round annuli, and set $A_D := \sup_{R_0 \in \mathcal{A}_D} m(R_0) (\le M_D)$. Then, we can show the following result.

Theorem 3.2 (cf. McMullen [Mc]). *If $D \subset \mathbb{C}$, it holds that $M_D \le A_D + 5\log 2$.*

In case of $\infty \in D$, we have the next auxiliary result.

Theorem 3.3. *If $L \in$ Möb, $\frac{1}{2}A_{L(D)} - \log 4/3 \le A_D$.*

If $D \subset \mathbb{C}$, further we define the domain constant

$$c_D = \inf_{z\in D} \delta_D(z)\rho_D(z),$$

where $\delta_D(z) = \mathrm{dist}(z, \partial D) = \inf_{a\in\partial D} |z-a|$.

That is, c_D is the infimum of the ratio of the Poincaré metric $\rho_D(z)|dz|$ to the quasi-hyperbolic one $|dz|/\delta_D(z)$. We should note that $\delta_D(z)\rho_D(z) \le 1$ for any $z \in D$, thus $c_D \le 1$. Concerning this, the similar result as Theorem 1.1 (2) is verified.

Theorem 3.4 (Minda [Mi]).

$$\frac{\tanh H_D}{4} \le c_D \le \frac{2\sqrt{3}}{\pi} H_D.$$

Remark. The assumption that $\infty \notin D$ is essential for c_D. In fact, if $D = \Delta^* = \widehat{\mathbb{C}} \setminus \overline{\Delta}$, we have $\delta_{\Delta^*}(z) = |z| - 1$ and $\rho_{\Delta^*}(z) = \frac{1}{|z|^2-1}$, therefore $\delta_{\Delta^*}(z)\rho_{\Delta^*}(z) = \frac{1}{|z|+1} \to 0$ as $z \to \infty$.

Finally, we summalize our results.

Theorem 3.4. *Let D be a plane domain of hyperbolic type. Then the following conditions are mutually equivalent.*

(1) $H_D > 0$,
(2) $M_D < \infty$,
(3) $A_D < \infty$,
(4) $N_D < \infty$,
(5) $c_D > 0$ *(if* $D \subset \mathbb{C}$*).*

The other features of uniformly perfect domains can be seen in [Pom1] and [Pom2].

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