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The automorphism group of the Klein curve in the mapping class group of genus 3

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1 The main result and its proof

Let $R$ be a compact Riemann surface of genus $g \geq 2$. Then $\text{Aut}(R)$, the automorphism group of $R$, can be embedded into the mapping class group (for its definition, see [Bir, Ch. 4]) or the Teichmüller group $\Gamma_g$ of genus $g$;

$$\iota: \text{Aut}(R) \rightarrow \Gamma_g \simeq \text{Out}^+(\pi_1(R)) = \text{Aut}^+(\pi_1(R))/\text{Int}(\pi_1(R)).$$

Here, $\text{Aut}^+(\pi_1(R))$ consists of the automorphisms of $\pi_1(R)$ inducing the trivial action on $H_2(\pi_1(R), \mathbb{Z}) \simeq \mathbb{Z}$.

Recall the Hurwitz theorem, which states that

$$\#\text{Aut}(R) \leq 84(g - 1).$$  

If the equality holds in (1.2), then $R$ is called a Hurwitz Riemann surface and $\text{Aut}(R)$ is called a Hurwitz group.

Let $X$ be the Klein curve of genus 3 defined by the equation

$$x^3y + y^3z + z^3x = 0.$$  

It is well known that $X$ is a Hurwitz Riemann surface; $G := \text{Aut}(X)$ is isomorphic to $\text{PSL}_2(\mathbb{F}_7)$ and has order 168.

Now let us forget about the Klein curve, and consider an orientable compact $C^\infty$ surface $X$ of genus 3. We define the canonical generators of $\pi_1(X, b)$ with base point $b$ as in the figure below;

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They satisfy the fundamental relation

\[(a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1})(b_3 a_3 b_3^{-1} a_3^{-1}) = 1.\]

Let $\tilde{\varphi}_2, \tilde{\varphi}_3, \tilde{\varphi}_7$ be the elements of $\text{Aut}^+(\pi_1(X))$ defined by

\[
\begin{align*}
\tilde{\varphi}_2(a_1) &= a_2 b_2^{-1} a_2^{-1} a_1^{-1} b_3^{-1} b_2 \\
\tilde{\varphi}_2(b_1) &= b_2^{-1} b_3 b_1^{-1} a_2 b_2 a_2^{-1} \\
\tilde{\varphi}_2(a_2) &= b_3^{-1} a_2^{-1} \\
\tilde{\varphi}_2(b_2) &= a_2 b_3 b_2^{-1} a_2^{-1} \\
\tilde{\varphi}_2(a_3) &= a_2 b_2^{-1} a_2^{-1} b_1^{-1} a_1^{-1} a_3 a_2^{-1} \\
\tilde{\varphi}_2(b_3) &= a_2 b_3 a_2^{-1},
\end{align*}
\]

\[
\begin{align*}
\tilde{\varphi}_3(a_1) &= a_3 b_3 a_3^{-1} a_1 a_2 b_2 a_2^{-1} \\
\tilde{\varphi}_3(b_1) &= a_2 b_2^{-1} a_2^{-1} a_1^{-1} a_3 a_1 a_2 b_2 a_2^{-1} \\
\tilde{\varphi}_3(a_2) &= a_3^{-1} a_1 b_1 a_1^{-1} \\
\tilde{\varphi}_3(b_2) &= a_1 b_1^{-1} a_1^{-1} a_3 a_2 b_2^{-1} a_2^{-1} b_1 a_1^{-1} \\
\tilde{\varphi}_3(a_3) &= a_2 b_2 a_2^{-1} a_2^{-1} b_1 \\
\tilde{\varphi}_3(b_3) &= a_1 b_1^{-1} a_1^{-1} a_3 a_2 b_2^{-1} a_2^{-1} b_1,
\end{align*}
\]

\[
\begin{align*}
\tilde{\varphi}_7(a_1) &= b_1^{-1} a_1^{-1} a_3 b_3^{-1} a_2^{-1} \\
\tilde{\varphi}_7(b_1) &= a_2 b_3 a_3^{-1} a_1 a_2 b_2 a_2^{-1} b_3^{-1} a_2^{-1} \\
\tilde{\varphi}_7(a_2) &= a_2 b_2^{-1} a_2^{-1} a_1^{-1} \\
\tilde{\varphi}_7(b_2) &= a_1 a_2 b_2 b_3 a_3^{-1} \\
\tilde{\varphi}_7(a_3) &= b_1^{-1} a_2 b_2^{-1} a_2^{-1} a_3^{-1} a_1 b_1 a_1^{-1} \\
\tilde{\varphi}_7(b_3) &= a_1 a_2 b_2 a_3^{-1} a_1 b_1 a_1^{-1}.
\end{align*}
\]

Then, we have the following:
Theorem 1.1. (1) The classes $\varphi_i$ of $\tilde{\varphi}_i$ in $\text{Out}^+(\pi_1(X))$ generate a subgroup $H$ of $\Gamma_3$, which is isomorphic to $PSL_2(\mathbb{F}_7)$.

(2) Moreover, if $X$ is the Klein curve, then $H$ is conjugate to the image of $\imath$.

Proof. (1) First note that $H \neq \{1\}$, because the action of $H$ on the homology group $H_1(X, \mathbb{Z})$ is not trivial. By direct computation using (1.3), we have (1.4)

$$\varphi_2^2 = \varphi_3^3 = \varphi_7^7 = 1, \quad \varphi_2 \varphi_3 \varphi_7 = 1,$$

$$(\varphi_7 \varphi_3 \varphi_2)^4 = \text{[conjugation by } a_2 b_2^{-1} a_2^{-1} b_1].$$

For example,

$$\varphi_2^2 \cdot b_3 = (a_2^{-1} b_2 a_2 a_1 a_3 a_1^{-1} a_2^{-1} b_2^{-1} a_2)(a_3 a_1^{-1} b_1^{-1} a_1)$$
\begin{equation*}
\times (a_1^{-1} b_1 a_1 a_3^{-1} a_2^{-1} b_2 a_2^{-1} a_1)(a_1^{-1} b_1 a_1 a_3^{-1} a_2^{-1} b_2^{-1} a_2)
\times (a_2^{-1} b_2 a_2 a_1 a_3^{-1} a_1^{-1} a_2^{-1} b_2^{-1} a_2)(a_2^{-1} b_2 a_2 a_1 a_3^{-1} b_2 a_2)
\end{equation*}
\begin{equation*}
= a_2^{-1} b_2 a_2 a_1 a_3^{-1} a_2 b_2 b_3^{-1} a_3^{-1} b_3 a_2
\end{equation*}
\begin{equation*}
= a_2^{-1} b_2 a_2 b_1 a_3 a_1^{-1} a_2,
\end{equation*}

hence

$$\varphi_3^3 \cdot b_3 = (a_3 a_1^{-1} b_1^{-1} a_1)(a_1^{-1} b_1 a_2^{-1} b_2^{-1} a_2 a_3 a_1^{-1} b_1^{-1} a_1)$$
\begin{equation*}
\times (a_1^{-1} b_1 a_1 a_3^{-1})(a_2^{-1} b_2 a_2 a_1 a_3^{-1} a_1^{-1} b_2^{-1} a_2)
\times (a_2^{-1} b_2 a_2 a_1 a_3^{-1} b_3 a_2)(a_2^{-1} b_2 a_2 a_3 a_1^{-1} b_2 a_1)(a_1^{-1} b_1 a_1 a_3^{-1} a_2^{-1} b_2^{-1} a_2)
\end{equation*}
\begin{equation*}
= b_3.
\end{equation*}

From (1.4) we obtain

$$\varphi_2^2 = \varphi_3^3 = \varphi_7^7 = \varphi_2 \varphi_3 \varphi_7 = (\varphi_7 \varphi_3 \varphi_2)^4 = 1$$

in $\text{Out}^+(\pi_1(X))$. Since (1.5) is the presentation of $PSL_2(\mathbb{F}_7)$ (see [CM, p. 96]), there is a surjective map

$$PSL_2(\mathbb{F}_7) \rightarrow H.$$

The group $PSL_2(\mathbb{F}_7)$ is simple, and the map is an isomorphism.

(2) To see that $H$ is the automorphism group of a Riemann surface, it is enough to recall the Nielsen realization problem, which was positively solved in [Ker]:

$$\varphi_2^2 = \varphi_3^3 = \varphi_7^7 = \varphi_2 \varphi_3 \varphi_7 = (\varphi_7 \varphi_3 \varphi_2)^4 = 1$$
Theorem of Kerckhoff. For any finite subgroup $G$ of $\Gamma_g$, there is a compact Riemann surface $R$ of genus $g$ such that

$$G \subset \text{Aut}(R) \subset \Gamma_g.$$ 

This theorem shows that there exists a Riemann surface $R$ of genus $3$ such that $G \subset \text{Aut}(R) \subset \Gamma_g$. Consequently $H = \text{Aut}(R)$. It is classically known that the Klein curve is the unique compact Riemann surface of genus $3$ such that $\text{Aut}(R) \simeq \text{PSL}_2(\mathbb{F}_7)$. Thus we have proved Theorem 1.1.

2 $\pi_1(X)$ as a subgroup of the triangle group of type $(2,3,7)$

In this section, we give a more elementary proof of Theorem 1.1. The outline is as follows: Let $T$ be the triangle group with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$ defined below, and $N$ its normal subgroup. Then, $T$ (resp. $N$) has a fundamental domain $\Delta$ (resp. $\Lambda$) in the Poincaré unit disk. As was shown in [Kle], the Klein curve $X$ can be realized by gluing the boundaries of $\Lambda$. The elements of $T$ act on $\Lambda$, hence on $X$. This action induces an isomorphism $T/N \simeq \text{Aut}(X)$. Moreover, $N$ is isomorphic to $\pi_1(X)$. Because $T$ acts on $N$ by conjugation, $T/N$ can be embedded in $\text{Out}^+(\pi_1(X))$. In this way, we obtain the map $\iota$ in (1.1). First, we compute the elements of $N$ corresponding to the generators of $\pi_1(X)$. Using this identification, we show that $\iota(T/N) = H$, which is equivalent to Theorem 1.1.

Let $S = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), T = \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)$ be the generators of $\text{PSL}_2(\mathbb{F}_7)$. Then $S^2 = T^7 = (ST^{-1})^3 = 1$. For the triangle group

$$T := \langle \gamma_2, \gamma_3, \gamma_7 | \gamma_2^2 = \gamma_3 = \gamma_7^2 = 1, \gamma_2 \gamma_3 \gamma_7 = 1 \rangle,$$

we define a group homomorphism

$$\varphi: T \to \text{PSL}_2(\mathbb{F}_7)$$

by $\varphi(\gamma_2) = S, \varphi(\gamma_3) = ST^{-1}, \varphi(\gamma_7) = T$. Clearly $\varphi$ is surjective. The map $\varphi$ gives an exact sequence

$$1 \to N \to T \to \text{PSL}_2(\mathbb{F}_7) \to 1,$$  

(2.1)
where $N := \ker \varphi \simeq \pi_1(X)$ is the kernel of $\varphi$. Hence we have $G \simeq T/N$. For any element $\hat{\alpha}$ of $N$, we shall denote by $\alpha$ the loop with base point $b$ representing $\hat{\alpha}$. First, we give the elements of $N$ corresponding to the canonical generators of $\pi_1(X, b)$. Note that, for two elements $\hat{\alpha}, \hat{\beta} \in N$, their product $\hat{\alpha} \hat{\beta} \in N$ corresponds to the loop $\beta \alpha$.

**Proposition 2.1.** Define $\hat{a}_i, \hat{b}_i \in N, i = 1, 2, 3$ by

\[
\hat{a}_1 = \gamma_7 \gamma_3^{-1} \gamma_7^3 \gamma_2 \gamma_7^2 (\gamma_3 \gamma_2 \gamma_7)^4 \gamma_7^{-2} \gamma_2 \gamma_7^3 \gamma_3 \gamma_7^{-1}
\]

\[
\hat{b}_1 = \gamma_7 \gamma_3^{-1} \gamma_7^3 \gamma_2 \gamma_7^2 (\gamma_2 \gamma_7 \gamma_3)^4 \gamma_7^{-2} \gamma_2 \gamma_7^3 \gamma_3 \gamma_7^1
\]

\[
\hat{a}_2 = \gamma_2 \gamma_7^{-4} \gamma_2 \gamma_7^4 \gamma_2 \gamma_7^2 (\gamma_3 \gamma_2 \gamma_7)^4 \gamma_7^{-2} \gamma_2 \gamma_7^4 \gamma_2 \gamma_7^4 \gamma_2
\]

\[
\hat{b}_2 = \gamma_3 \gamma_7^{-4} \gamma_2 \gamma_7^4 \gamma_3 (\gamma_7 \gamma_3)^4 \gamma_7^{-2} \gamma_2 \gamma_7^4 \gamma_2 \gamma_7^4 \gamma_2
\]

\[
\hat{a}_3 = \gamma_3 \gamma_7^{-2} \gamma_2 \gamma_7^4 \gamma_3 (\gamma_7 \gamma_3)^4 \gamma_7^{-1} \gamma_7 \gamma_7^2 \gamma_7^1
\]

\[
\hat{b}_3 = \gamma_3 \gamma_7^{-2} \gamma_2 \gamma_7^4 \gamma_3 (\gamma_2 \gamma_7 \gamma_3)^4 \gamma_7^{-1} \gamma_7 \gamma_7^2 \gamma_7^1.
\]

Set $\hat{a}_3' = \hat{a}_3 \hat{b}_3, \hat{b}_3' = \hat{a}_3$. Then the elements $\hat{a}_1, \hat{a}_2, \hat{a}_3', \hat{b}_1, \hat{b}_2, \hat{b}_3'$ are identified with the canonical generators of $\pi_1(X)$ and they satisfy the equation $[\hat{a}_3', \hat{b}_3'] [\hat{a}_2, \hat{b}_2] [\hat{a}_1, \hat{b}_1] = 1$. Here $[\alpha, \beta] := \beta^{-1} \alpha^{-1} \beta \alpha$.

**Proof.** Let $\Delta$ (resp. $\Lambda$) be the fundamental domain of $T$ (resp. $N$). Figure 2 below illustrates that $\Delta$ is a hyperbolic triangle with angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{7}$ and $\Lambda$ is the union of 168 copies of $\Delta$. By tracing paths, we can easily see that the elements $\hat{a}_i, \hat{b}_i$ in the figure can be written as above.

By gluing corresponding edges, we obtain the Riemann surface $X$. The elements $\hat{a}_i, \hat{b}_i$ are represented by the loops $a_i, b_i$ in Figure 1. We can also check the fundamental relation by computation.

The conjugation gives the canonical map

\[
\tilde{\iota} : T \to \text{Aut}^+(N).
\]

This induces the map $\iota$ in (1.1). We take $\tilde{\iota}(\gamma_2), \tilde{\iota}(\gamma_3), \tilde{\iota}(\gamma_7)$ as the generators of $\iota(T/N)$.

The following proposition finishes the direct proof of Theorem 1.1.

**Proposition 2.2.** $\iota(\gamma_j)$ can be identified with $\tilde{\varphi}_j$ for $j = 1, 2, 3$. 
Proof. Set $\gamma_j \cdot \alpha := \iota(\gamma_j)(\alpha) = \gamma_j \alpha \gamma_j^{-1}$ for $\alpha \in N$. Then, for $\hat{a}_i, \hat{b}_i$ in Proposition 2.1, we can describe $\gamma_j \cdot \hat{a}_i, \gamma_j \cdot \hat{b}_i \in \Lambda$ as in the Figure 3.

We shall show that $\tilde{\varphi}_7(a_1)$ represents $\iota(\gamma_7)(\hat{a}_1)$. By gluing the edges of $\Lambda$, we get the following loop $\ell$ representing $\gamma_7 \cdot \hat{a}_1$.

We can check that $\ell$ is homotopic to the loop below, which is the loop $b_1^{-1}a_1^{-1}a_3b_3^{-1}a_2^{-1}$.

The proofs for the other cases are similar and omitted.

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References


Figure 2: Fundamental domain $\Lambda$ of $\mathcal{N}$ ([Kle, p. 126])

Glue $1=6$, $7=12$, $2=11$, $3=8$, $5=10$, $4=13$, $9=14$ in this order. Each loop is connected to the base point $b$ by dotted path.
Figure 3-(i): Action of $\gamma_2$
Figure 3-(ii): Action of $\gamma_3$
Figure 3-(iii): Action of $\gamma_7$